CHARACTERIZATION OF SHADOWING FOR LINEAR AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Jaroslav Kurzweil

Abstract. A well-known shadowing theorem for ordinary differential equations is generalized to delay differential equations. It is shown that a linear autonomous delay differential equation is shadowable if and only if its characteristic equation has no root on the imaginary axis. The proof is based on the decomposition theory of linear delay differential equations.

Keywords: delay differential equation; linear autonomous equation; shadowing

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1. Introduction and the main result

Roughly speaking, a differential equation is *shadowable* if around its approximate solutions we can always find a true solution (for the precise notion, see Definition 1.2 below). Shadowing and the similar concept of Hyers-Ulam stability has become an important part of the qualitative theory of differential and difference equations, see e.g. [3], [6]. In this note, we are interested in the shadowing of a general class of linear autonomous delay differential equations which, among others, includes the equation

$$x'(t) = \sum_{j=1}^{N} A_j x(t - r_j) + \int_{-r}^{0} A(\theta)x(t+\theta)\,\mathrm{d}\theta, \quad t \geqslant 0, \tag{1.1}$$

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where $0 \leqslant r_j \leqslant r$, $A_j \in \mathbb{C}^{n\times n}$, $1 \leqslant j \leqslant N$, and $A\colon [-r,0] \to \mathbb{C}^{n\times n}$ is a continuous matrix-valued function. Solutions of (1.1) are generated by initial data $x(t) = \varphi(t)$ for $t \in [-r,0]$, where $\varphi \in C := C([-r,0],\mathbb{C}^n)$ is a given initial function. As usual, the symbol $C = C([-r,0],\mathbb{C}^n)$ denotes the Banach space of continuous functions from $[-r,0]$ into $\mathbb{C}^n$ equipped with the supremum norm

$$\|\varphi\| := \sup_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)|, \quad \varphi \in C,$$

where $|\cdot|$ is any norm on $\mathbb{C}^n$. The characteristic values of (1.1) are the complex roots of its characteristic equation

$$\det\Bigl(zI - \sum_{j=1}^{N} A_j \mathrm{e}^{-zr_j} - \int_{-r}^{0} A(\theta)\mathrm{e}^{z\theta}\,\mathrm{d}\theta\Bigr) = 0, \tag{1.2}$$

where I is the $n\times n$ identity matrix. In the special case of a linear ordinary differential equation

$$x' = Bx \tag{1.3}$$

with a constant coefficient $B \in \mathbb{C}^{n\times n}$, it is well-known that both the shadowing property and Hyers-Ulam stability are equivalent to the nonexistence of a characteristic value with zero real part, see e.g. [4], Theorem 2.1. To the best of our knowledge, the analogue of this result for delay differential equations is not available in the literature. It is therefore natural to ask whether a similar conclusion is true for delay differential equations. In this note, we shall give a positive answer to this question for the most general class of linear autonomous delay differential equations of the form

$$x'(t) = L(x_t), \quad t \geqslant 0, \tag{1.4}$$

where $x_t \in C$ is defined by $x_t(\theta) := x(t+\theta)$ for $\theta \in [-r,0]$ and $L\colon C \to \mathbb{C}^n$ is a bounded linear functional. In contrast with the ordinary differential equation (1.3), the phase space C for (1.4) is infinite dimensional, therefore the proof requires different arguments. Our proof will be based on the decomposition theory of linear autonomous delay differential equations, see [5]. Before we formulate our main result, recall that by a *solution* of (1.4) we mean a continuous function $x\colon [-r,\infty) \to \mathbb{C}^n$ which is differentiable on $[0,\infty)$ and satisfies (1.4) for all $t \geqslant 0$. (By the derivative at $t = 0$, we mean the right-hand side derivative.) It is well-known (see [5], Chapter 6) that for every $\varphi \in C$, equation (1.4) has a unique solution with initial value $x_0 = \varphi$. According to the Riesz representation theorem, L can be written in the form

$$L(\varphi) = \int_{-r}^{0} \mathrm{d}[\eta(\theta)]\varphi(\theta), \quad \varphi \in C, \tag{1.5}$$

where $\eta\colon [-r,0] \to \mathbb{C}^{n\times n}$ is a matrix function of bounded variation normalized so that η is left continuous on $(-r,0)$ and $\eta(0)=0$. The *characteristic equation* of (1.4) has the form

$$\det\Delta(z) = 0, \quad \Delta(z) = zI - \int_{-r}^{0} \mathrm{e}^{z\theta}\,\mathrm{d}\eta(\theta). \tag{1.6}$$

Definition 1.1. Given $\delta > 0$, by a *δ-pseudosolution* (*δ-approximate solution*) of (1.4), we mean a continuous function $y\colon [-r,\infty) \to \mathbb{C}^n$ which is continuously differentiable on $[0,\infty)$ and satisfies

$$|y'(t) - L(y_t)| \leqslant \delta, \quad t \geqslant 0. \tag{1.7}$$

Definition 1.2 ([1])**.** Equation (1.4) is said to be *shadowable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudosolution y of (1.4) there exists a solution x of (1.4) satisfying

$$\|x_t - y_t\| \leqslant \varepsilon, \quad t \geqslant 0. \tag{1.8}$$

Our main result is the following generalization of a classic shadowing theorem for ordinary differential equations to delay differential equations.

Theorem 1.1. *Equation* (1.4) *is shadowable if and only if its characteristic equation* (1.6) *has no root with zero real part.*

The proof of Theorem 1.1 will be given in Section 3 after summarizing some facts from the spectral theory of linear autonomous delay differential equations in Section 2.

2. Preliminaries

It is known (see [5], Chapter 7, Lemma 1.2) that (1.4) generates a C_0-semigroup $(T(t))_{t\geqslant 0}$ on C defined by

$$T(t)\varphi = x_t(\varphi), \quad t \geqslant 0,\ \varphi \in C,$$

where $x(\varphi)$ is the unique solution of (1.4) with initial value $x_0 = \varphi$. The infinitesimal generator $(A, \mathcal{D}(A))$ of the solution semigroup is given by

$$A(\varphi) = \varphi', \quad \mathcal{D}(A) = \{\varphi \in C\colon\ \varphi' \in C,\ \varphi'(0) = L(\varphi)\}.$$

The spectrum of A, denoted by $\sigma(A)$, is a point spectrum and consists of the roots of characteristic equation (1.6), see [5], Chapter 7, Lemma 2.1.

Let $\lambda \in \sigma(A)$. If k_λ is the order of λ as a pole of Δ^{-1}, then the generalized eigenspace $\mathcal{M}_\lambda(A)$ of A corresponding to λ is given by

$$\mathcal{M}_\lambda(A) = \mathcal{N}((\lambda I - A)^{k_\lambda}),$$

the null space of the operator $(\lambda I - A)^{k_\lambda}$, see [5], Chapter 7, Theorem 4.2. The phase space C can be decomposed into the direct sum

$$C = \mathcal{M}_\lambda(A) \oplus Q_\lambda(A),$$

where $Q_\lambda(A) = \mathcal{R}((\lambda I - A)^{k_\lambda})$, the range of $(\lambda I - A)^{k_\lambda}$. This means that each $\varphi \in C$ can be written in a unique way as

$$\varphi = \varphi^{\mathcal{M}_\lambda(A)} + \varphi^{\mathcal{Q}_\lambda(A)}, \tag{2.1}$$

where $\varphi^{\mathcal{M}_\lambda(A)} \in \mathcal{M}_\lambda(A)$ and $\varphi^{\mathcal{Q}_\lambda(A)} \in \mathcal{Q}_\lambda(A)$, see [5], Chapter 7, Lemma 2.1.

Define $C' = C([0,r], \mathbb{C}^{n*})$, where $\mathbb{C}^{n*}$ denotes the space of row n-vectors with complex entries. If $y\colon (-\infty, r] \to \mathbb{C}^{n*}$ is a continuous function, then for $s \geqslant 0$ the symbol y^s designates the element of C' defined by $y^s(\xi) = y(-s+\xi)$ for $\xi \in [0,r]$. With (1.4), we can associate its transpose equation

$$y'(s) = -\int_{-r}^{0} y(s-\xi)\,\mathrm{d}[\eta(\xi)], \quad s \leqslant 0. \tag{2.2}$$

For every $\psi \in C'$, the transpose equation (2.2) has a unique (backward) solution $y = y(\psi)$ with initial value $y^0 = \psi$. The transposed semigroup $(T^\top(s))_{s\geqslant 0}$ on C' is defined by

$$T^\top(s)\psi = y^s(\psi), \quad s \geqslant 0, \quad \psi \in C'.$$

Its infinitesimal generator $(A^\top, \mathcal{D}(A^\top))$ is given by

$$A^\top(\psi) = -\psi', \quad \mathcal{D}(A^\top) = \{\psi \in C'\colon\ \psi' \in C',\ \psi'(0) = L'(\psi)\},$$

where

$$L'(\psi) = -\int_{-r}^{0} \psi(-\xi)\,\mathrm{d}[\eta(\xi)], \quad \psi \in C',$$

see [5], Chapter 7, Lemma 1.4. For $\varphi \in C$ and $\psi \in C'$, define the bilinear form

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(\theta-\tau)\,\mathrm{d}[\eta(\tau)]\varphi(\theta)\,\mathrm{d}\theta. \tag{2.3}$$

The spectra $\sigma(A)$ and $\sigma(A^\top)$ coincide and $\mathcal{M}_\lambda(A^\top) = \mathcal{N}((\lambda I - A^\top)^{k_\lambda})$. It is known that

$$\dim \mathcal{M}_\lambda(A) = \dim \mathcal{M}_\lambda(A^\top) = m_\lambda,$$

where m_λ is the multiplicity of λ as a root of the entire function $\det \Delta$, see [5], Chapter 7, Theorem 4.2 and Lemma 5.2. Let $\Phi_\lambda = (\varphi_1, \ldots, \varphi_{m_\lambda})$ and $\Psi_\lambda =$

$\operatorname{col}(\psi_1, \ldots, \psi_{m_\lambda})$ be bases for $\mathcal{M}_\lambda(A)$ and $\mathcal{M}_\lambda(A^\top)$, respectively, and let $(\Psi_\lambda, \Phi_\lambda) = ((\psi_i, \varphi_j))_{i,j=1,\ldots,m_\lambda}$. Then $(\Psi_\lambda, \Phi_\lambda)$ is nonsingular and thus we may (and do) choose Ψ_λ such that $(\Psi_\lambda, \Phi_\lambda) = I$. The decomposition in (2.1) can be given explicitly by

$$\varphi^{\mathcal{M}_\lambda(A)} = \Phi_\lambda(\Psi_\lambda, \varphi) \tag{2.4}$$

and $\varphi^{\mathcal{Q}_\lambda(A)} = \varphi - \varphi^{\mathcal{M}_\lambda(A)}$, see [5], Chapter 7, Lemma 5.2. It is known (see [5], Chapter 7, Lemma 2.2 and the remark below Lemma 5.2 in [5]) that there exists an $m_\lambda \times m_\lambda$ matrix B_λ with complex entries such that $\sigma(B_\lambda) = \{\lambda\}$, $A\Phi_\lambda = \Phi_\lambda B_\lambda$ and $A^\top \Psi_\lambda = B_\lambda \Psi_\lambda$ so that

$$\Phi_\lambda(\theta) = \Phi_\lambda(0)\mathrm{e}^{B_\lambda \theta}, \quad -r \leqslant \theta \leqslant 0, \tag{2.5}$$

and

$$\Psi_\lambda(\xi) = \mathrm{e}^{-B_\lambda \xi}\Psi_\lambda(0), \quad 0 \leqslant \xi \leqslant r. \tag{2.6}$$

If x is a solution of the nonhomogeneous equation

$$x'(t) = L(x_t) + f(t), \quad t \geqslant 0, \tag{2.7}$$

where $f\colon [0,\infty) \to \mathbb{C}^n$ is a continuous function, then (2.1) and (2.4) imply that for all $t \geqslant 0$,

$$x_t = x_t^{\mathcal{M}_\lambda(A)} + x_t^{\mathcal{Q}_\lambda(A)},$$

where

$$x_t^{\mathcal{M}_\lambda(A)} = \Phi_\lambda u(t), \quad u(t) = (\Psi_\lambda, x_t),$$

and $x_t^{\mathcal{Q}_\lambda(A)} = x_t - x_t^{\mathcal{M}_\lambda(A)}$. Moreover, according to [5], Chapter 7, Theorem 9.1, the function $u\colon [0,\infty) \to \mathbb{C}^{m_\lambda}$ satisfies ordinary differential equation

$$u'(t) = B_\lambda u(t) + \Psi_\lambda(0) f(t), \quad t \geqslant 0. \tag{2.8}$$

3. Proof of the main result

Now we are in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. The "if" part is a simple consequence of a recent result on nonautonomous delay differential equations (see [2], Theorem 2.3) as we now demonstrate. Suppose that (1.4) has no characteristic root with zero real part. Then (1.4) has an exponential dichotomy on $\mathbb{R}$ (see, e.g. [5], Section 10.1, page 303), which is a special case of the shifted exponential dichotomy with splitting at $\gamma = 0$ as defined in [2]. By the application of Theorem 2.3 in [2] with $\gamma = 0$ and $f = 0$, we conclude that (1.4) is shadowable.

Now we prove the “only if“ part. Suppose, for the sake of contradiction, that (1.4) is shadowable and there exists a characteristic value $\lambda \in \sigma(A)$ such that $\operatorname{Re}\lambda = 0$. Taking $\varepsilon = 1$, the definition of the shadowing implies the existence of $\delta > 0$ such that for every δ-pseudosolution y of (1.4) there exists a solution x of (1.4) satisfying

$$\|x_t - y_t\| \leqslant 1, \quad t \geqslant 0. \tag{3.1}$$

Suppose that Φ_λ and Ψ_λ are bases for $\mathcal{M}_\lambda(A)$ and $\mathcal{M}_\lambda(A^\top)$, respectively, such that $(\Psi_\lambda, \Phi_\lambda) = I$. As noted in Section 2, there exists an $m_\lambda \times m_\lambda$ matrix B_λ such that the only eigenvalue of B_λ is λ and (2.5) and (2.6) are satisfied. Choose a nonzero vector $v_\lambda \in \mathbb{C}^{m_\lambda *}$ such that $v_\lambda B_\lambda = \lambda v_\lambda$. Without loss of generality, we may (and do) assume that

$$|[v_\lambda \Psi_\lambda(0)]^*| \leqslant \delta, \tag{3.2}$$

where the superscript * indicates the conjugate transpose. Otherwise, we replace v_λ with ηv_λ, where $\eta > 0$ is sufficiently small. It follows by induction that $v_\lambda B_\lambda^k = \lambda^k v_\lambda$ for $k = 0, 1, 2, \ldots$ This, combined with the definition of the matrix exponential, implies that

$$v_\lambda \mathrm{e}^{B_\lambda t} = \mathrm{e}^{\lambda t} v_\lambda, \quad t \in \mathbb{R}. \tag{3.3}$$

We will show that

$$v_\lambda \Psi_\lambda(0) \neq 0. \tag{3.4}$$

Suppose, for the sake of contradiction, that $v_\lambda \Psi_\lambda(0) = 0$. This, together with (2.6) and (3.3), implies that

$$v_\lambda \Psi_\lambda(\xi) = v_\lambda \mathrm{e}^{-B_\lambda \xi} \Psi_\lambda(0) = \mathrm{e}^{-\lambda \xi} v_\lambda \Psi_\lambda(0) = 0 \quad \text{for all } \xi \in [0, r]. \tag{3.5}$$

On the other hand, $v_\lambda \neq 0$ implies that $v_\lambda \Psi_\lambda$ is a nontrivial linear combination of the basis functions $\psi_1, \psi_2, \ldots, \psi_{m_\lambda}$ of $\mathcal{M}_\lambda(A^\top)$ which is necessarily a nonzero element of $\mathcal{M}_\lambda(A^\top)$. Thus, $v_\lambda \Psi_\lambda$ cannot be identically zero on $[0, r]$. This contradicts (3.5) and hence (3.4) holds. Define $f\colon\ [0, \infty) \to \mathbb{C}^{m_\lambda}$ by

$$f(t) = \mathrm{e}^{\lambda t} [v_\lambda \Psi_\lambda(0)]^*, \quad t \geqslant 0. \tag{3.6}$$

Let y be the unique solution of the nonhomogeneous equation (2.7) with initial value $y_0 = 0$ and f as in (3.6). From (3.2), taking into account that

$$|\mathrm{e}^{\lambda t}| = \mathrm{e}^{t \operatorname{Re}\lambda} = \mathrm{e}^0 = 1, \quad t \geqslant 0, \tag{3.7}$$

we conclude that

$$|y'(t) - L(y_t)| = |f(t)| \leqslant |e^{\lambda t}||[v_\lambda \Psi_\lambda(0)]^*| \leqslant \delta, \quad t \geqslant 0.$$

Thus, y is δ-pseudosolution of (1.4) and therefore (1.4) has a solution x satisfying (3.1). Define

$$z(t) = y(t) - x(t), \quad t \geqslant -r.$$

By (3.1), we have that

$$\sup_{t\geqslant 0} \|z_t\| \leqslant 1. \tag{3.8}$$

Since x is a solution of (1.4), z and y satisfy the same nonhomogeneous equation (2.7) with f given by (3.6). As noted in Section 2, the function u defined by

$$u(t) = (\Psi_\lambda, z_t), \quad t \geqslant 0,$$

is a solution of the ordinary differential equation (2.8). Using the definition of the bilinear form in (2.3) and the fact that η is of bounded variation on $[-r, 0]$, it is easy to show that there exists $K > 0$ such that

$$|(\Psi_\lambda, \varphi)| \leqslant K\|\varphi\|, \quad \varphi \in C.$$

Hence,

$$|u(t)| = |(\Psi_\lambda, z_t)| \leqslant K\|z_t\| \leqslant K, \quad t \geqslant 0, \tag{3.9}$$

the last inequality being a consequence of (3.8). This implies that the scalar function $w\colon [0,\infty) \to \mathbb{C}$ defined by

$$w(t) = v_\lambda u(t), \quad t \geqslant 0,$$

is bounded on $[0,\infty)$. Multiplying (2.8) by v_λ from left, using (3.6) and the relation $v_\lambda B_\lambda = \lambda v_\lambda$, we find that

$$w'(t) = \lambda w(t) + c e^{\lambda t}, \quad t \geqslant 0, \tag{3.10}$$

where

$$c = [v_\lambda \Psi_\lambda(0)][v_\lambda \Psi_\lambda(0)]^* = |[v_\lambda \Psi_\lambda(0)]^*|_2^2 > 0.$$

The symbol $|\cdot|_2$ denotes the l_2-norm on $\mathbb{C}^{m_\lambda}$ and the positivity of c follows from (3.4). From (3.10), by the variation of constants formula, we obtain

$$w(t) = e^{\lambda t}(w(0) + ct), \quad t \geqslant 0.$$

From this and (3.7), we conclude that

$$|w(t)| = |w(0) + ct| \to \infty, \quad t \to \infty,$$

which contradicts the boundedness of w. □

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