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ON ALMOST PERIODICITY DEFINED VIA NON-ABSOLUTELY CONVERGENT INTEGRALS

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In memory of Professor Jaroslav Kurzweil

Abstract. We investigate some properties of the normed space of almost periodic functions which are defined via the Denjoy-Perron (or equivalently, Henstock-Kurzweil) integral. In particular, we prove that this space is barrelled while it is not complete. We also prove that a linear differential equation with the non-homogenous term being an almost periodic function of such type, possesses a solution in the class under consideration.

Keywords: almost periodic function in view of the Lebesgue measure; barrelled space; Bohr almost periodic function; Denjoy-Bochner almost periodic function; Denjoy-Perron integral; Henstock-Kurzweil integral; linear differential equation

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1. Introduction

The first author of this article had a great pleasure to meet Professor Jaroslav Kurzweil in person during the Prague Mathematical Conference 1996 in honor of the 70th birthdays of Ivo Babuška, Miroslav Fiedler, Jaroslav Kurzweil and Vlastimil Pták. He presented there the talk entitled "On the structure of solution sets of differential and integral equations, and the Perron integral", see [6]. In particular, the following mathematicians (in the alphabetic order) listened to that talk: Ralph Henstock, Jaroslav Kurzweil, Jean Mawhin and Štefan Schwabik. Therefore, it was a big challenge for a young mathematician to present a talk in front of such a great audience.

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The theory of the Denjoy-Perron integral (see [22] for more details) allows to integrate the Newton, Riemann and Lebesgue integrable functions. In particular, it means that it allows to integrate an arbitrary derivative, that is, for any differentiable function $f\colon [a,b] \to \mathbb{R}$, the following formula holds:

$$(\mathrm{DP}) \int_a^b f'(s)\,\mathrm{d}s = f(b) - f(a),$$

where the sign "$(\mathrm{DP}) \int_a^b f'(s)\,\mathrm{d}s$" stands for the Denjoy-Perron integral of the function f' over the interval $[a,b]$. Kurzweil in [16], in 1957 (independently Henstock in [13] in 1961) used the original Riemann definition of the integral to define the new integral, which is equivalent to the Denjoy-Perron integral. That integral has found many applications, for example, in the theory of nonlinear differential and integral equations, see e.g. [3], [7], [10], [12], [17], [16] and [23].

On the other hand, the theory of almost periodic functions initiated by Bohr nearly a hundred years ago has been widely developed, which is connected, in particular, with the fact that such functions have found applications in many areas. A very important example among those applications are quasicrystals which can be described as almost periodic patterns corresponding to almost periodic measures, see [18]. For the introduction into the theory of Bohr almost periodic functions the interested reader is referred e.g. to the recently published monograph, see [4].

There are many various classes of almost periodic functions, see e.g. [2]. Among those generalizations of Bohr almost periodic functions let us indicate the so-called Stepanov almost periodic functions, the definition of which is based on (see e.g. [24] for more details) the integral metric defined on some subspace of locally integrable functions.

Burkill in [11] extended the concept of almost periodicity in the Stepanov sense to functions integrable in the Denjoy sense, defining the so-called D-a.p. functions. Recall that D-a.p. functions form a linear space. Moreover, for such functions there exists the mean value and they possess Fourier series, see [11] for the proofs and further properties of such functions.

The goal of this article is to examine some properties of almost periodic functions which are defined via the Denjoy-Perron integral. First, we compare this type of almost periodicity with the almost periodicity in view of the Lebesgue measure. Next, we establish that the space of Denjoy-Perron almost periodic functions is a non-complete barreled space. Finally, we consider a linear differential equation and we prove that such an equation possesses a Denjoy-Perron almost periodic solution provided the non-homogeneous term of the equation under consideration is a function of such type.

2. Preliminaries

In this section we collect basic definitions and facts which will be needed in the sequel. Let us denote by $L^p_{\rm loc}(\mathbb{R})$, $p \geqslant 1$ ($D^*_{\rm loc}$), the class of all real-valued functions defined on the real line $\mathbb{R}$ and integrable in the Lebesgue sense (integrable in the Denjoy-Perron or equivalently Henstock-Kurzweil sense) on each compact interval in $\mathbb{R}$. For any $f \in D^*_{\rm loc}$, let us introduce the quantity

$$\|f\|_{D^*} = \sup_{x\in\mathbb{R}}\left\{\sup_{0\leqslant h\leqslant 1}\left|({\rm DP})\int_x^{x+h} f(t)\,{\rm d}t\right|\right\}. \tag{2.1}$$

Let us notice that the quantity defined above may be finite or infinite. Moreover, let us recall that the quantity

$$\|f\|_{A,x} = \sup_{0\leqslant h\leqslant 1}\left|({\rm DP})\int_x^{x+h} f(t)\,{\rm d}t\right| \tag{2.2}$$

is the well-known Alexiewicz norm (see [1]) of f on the interval $[x, x+1]$. Such a norm is equivalent to the norm

$$\|f\|'_{A,x} = \sup_{I\subset[x,x+1]}\left|({\rm DP})\int_I f(t)\,{\rm d}t\right|$$

(where I denotes an interval), because we have

$$\|f\|_{A,x} \leqslant \|f\|'_{A,x} \leqslant 2\|f\|_{A,x}, \quad \text{see [26].}$$

However,

$$\sup_{x\in\mathbb{R}}\|f\|_{A,x} = \sup_{x\in\mathbb{R}}\|f\|'_{A,x}, \tag{2.3}$$

because every interval $[a,b] \subset [x, x+1]$ is taken into account evaluating $\|f\|_{A,x}$. Moreover, by the definition, we have

$$\|f\|_{D^*} = \sup_{x\in\mathbb{R}}\|f\|_{A,x}.$$

Therefore, the quantity $\|\cdot\|_{D^*}$ is a norm on the set of functions

$$\{f \in D^*_{\rm loc}\colon\ \|f\|_{D^*} < \infty\},$$

because on each fix interval $[x, x+1]$ the quantity $\|\cdot\|_{A,x}$ is a norm.

For a function $f\colon\ \mathbb{R}\to\mathbb{R}$ and $\tau\in\mathbb{R}$, let $f_\tau(x) = f(x+\tau)$ for $x\in\mathbb{R}$. Now we are going to recall some basic definitions from the theory of almost periodic functions.

Definition 2.1. A nonempty set $E \subset \mathbb{R}$ is called *relatively dense* if there exists a positive number ω such that in each open interval $(a, a+\omega)$, $a\in\mathbb{R}$, there exists at least one element of the set E.

Based on the above definition one can state the definition of a uniformly almost periodic function (or a Bohr almost periodic function).

Definition 2.2. A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be uniformly almost periodic if for every $\varepsilon > 0$ the set

$$\Big\{\tau \in \mathbb{R}\colon \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leqslant \varepsilon\Big\}$$

is relatively dense.

Now we are going to define the class of Stepanov almost periodic functions.

Definition 2.3. A function $f \in L^p_{\mathrm{loc}}(\mathbb{R})$, $p \geqslant 1$ is said to be S^p-almost periodic or Stepanov almost periodic (briefly S^p-a.p.) if for every $\varepsilon > 0$ the set

$$\left\{\tau \in \mathbb{R}\colon \sup_{x\in\mathbb{R}} \left(\int_x^{x+1} |f(t+\tau) - f(t)|^p \,\mathrm{d}t\right)^{1/p} \leqslant \varepsilon\right\}$$

is relatively dense.

If $p < p'$, then every $S^{p'}$-almost periodic function is S^p-almost periodic, so the space of S^1-almost periodic functions is the largest class in the sense of inclusion, see [2]. Therefore, in the context of Stepanov almost periodic functions, we assume in advance that $p = 1$.

Now, let us recall the definition of a D^*-almost periodic (briefly D^*-a.p.) function, cf. [11].

Definition 2.4. A function $f \in D^*_{\mathrm{loc}}$ is said to be D^*-a.p. (or Denjoy-Perron almost periodic) if, given $\varepsilon > 0$, there exists a relatively dense set (in the sense of Bohr) of its $(D^*;\varepsilon)$ almost periods, that is, the set of such $\tau \in \mathbb{R}$ for which the following inequality holds:

$$\|f_\tau - f\|_{D^*} < \varepsilon. \tag{2.4}$$

Denote by D^* the set of all D^*-a.p. functions.

Remark 2.5. Since in the above definition, ε is any positive number, we can also use a weak inequality. Then inequality (2.4) can be rewritten as

$$\left|(\mathrm{DP})\int_x^{x+h} (f(t+\tau) - f(t))\,\mathrm{d}t\right| \leqslant \varepsilon \tag{2.5}$$

for all $x \in \mathbb{R}$, $0 \leqslant h \leqslant 1$, where τ is a $(D^*;\varepsilon)$ almost period of the function f. Hence, it is clear that the function

$$x \to (\mathrm{DP})\int_x^{x+h} f(t)\,\mathrm{d}t \tag{2.6}$$

is almost periodic in the sense of Bohr (or uniformly almost periodic) for every $0 \leqslant h \leqslant 1$, and therefore, in particular, it is bounded and uniformly continuous.

Remark 2.6. It is well-known that the set D^* is a vector space, cf. [19]. Moreover, it can be proved that for any $f \in D^*$, it holds that

$$\|f\|_{D^*} < \infty, \tag{2.7}$$

see [19], Theorem 2.6. Therefore, the vector space D^* is a normed space.

Let $L^0(\mathbb{R})$ denote the set of all functions $\mathbb{R} \to \mathbb{R}$ measurable in the Lebesgue sense. For $\eta > 0$ and $f, g \in L^0(\mathbb{R})$ let $D(\eta; f, g)\colon = \sup_{u\in\mathbb{R}} \mu(\{t \in [u, u+1]\colon\ |f(t) - g(t)| \geqslant \eta\})$.

Now, let us recall the definition of an almost periodic function in view of the Lebesgue measure. At the beginning of the next section we are going to compare the types of almost periodicity under consideration.

Definition 2.7. A function $f \in L^0(\mathbb{R})$ is said to be *almost periodic in view of the Lebesgue measure* μ (or μ*-almost periodic*) if for arbitrary numbers $\varepsilon, \eta > 0$ the set of all (ε, η)-almost periods of f, defined as $E\{\varepsilon, \eta, f\} := \{\tau \in \mathbb{R}\colon\ D(\eta; f_\tau, f) \leqslant \varepsilon\}$, is relatively dense.

The following lemma will be useful in the sequel.

Lemma 2.8 ([15])**.** *If $f \in L^0(\mathbb{R})$ is μ-almost periodic, then for each $\varepsilon, \eta > 0$ the set*

$$E\{\varepsilon, \eta, f\} \cap \mathbb{Z}$$

is relatively dense.

Let us emphasize that μ-a.p. functions possess more complex nature than classical Stepanov almost periodic functions. More details concerning such functions can be found e.g. in papers [8], [9] and [25].

Now, let us recall the definition of a barrelled space and one of its characterization.

Definition 2.9. Let E be a locally convex space. An absorbing, balanced, convex and closed subset of E is said to be a barrel. The space E is said to be barrelled if every barrel in E is a neighborhood of zero.

Proposition 2.10 ([14], page 212)**.** *A locally convex space E is barrelled if and only if every subset of its dual E' which is bounded for the* weak* *topology $\sigma(E', E)$ is also equicontinuous.*

Finally, for the convenience of the reader we will quote three results which will be used in the proof of the result that the space of all D^*-a.p. functions is barrelled.

Denote by $\mathrm{HK}([a,b])$ the space of all functions $[a,b] \to \mathbb{R}$ which are integrable in the Henstock-Kurzweil sense.

Theorem 2.11 ([26]). *Let $h_k \in \mathrm{HK}([a,b])$, $k \in \mathbb{N}$ and (J_k) be a pairwise disjoint sequence of open subintervals of $[a,b]$. Let us denote $X = [a,b] \setminus \bigcup_{k=1}^{\infty} J_k$. Assume that the boundary of X, ∂X, is countable with $\partial X = \{r_k\colon k \in \mathbb{N}\}$. Assume also that $H(J) = \sum_{k=1}^{\infty} \int_J h_k \chi_{J_k}$ exists for every subinterval $J \subset [a,b]$ and $\lim_{|J| \to 0} H(J) = 0$. If $h = \sum_{k=1}^{\infty} h_k \chi_{J_k}$ (pointwise), then $h \in \mathrm{HK}([a,b])$ and $\int_J h = H(J)$ for every $J \subset [a,b]$.*

Proposition 2.12 ([26]). *Let $f_k \in \mathrm{HK}([a,b])$ with $\sum_{k=1}^{\infty} \|f_k\|_{A,a} < \infty$ and let (J_k) be a pairwise disjoint sequence of open subintervals of $[a,b]$. Set $f = \sum_{k=1}^{\infty} f_k \chi_{J_k}$ (pointwise).*

(i) *$H(J) = \sum_{k=1}^{\infty} \int_J f_k \chi_{J_k}$ converges uniformly for $J \subset [a,b]$,*

(ii) *$\lim_{|J| \to 0} H(J) = 0$,*

(iii) *if $f \in \mathrm{HK}([a,b])$, then $\Big\| f - \sum_{k=1}^{n} f_k \chi_{J_k} \Big\|_{A,a} \to 0$ (i.e., $\int_J f = \sum_{k=1}^{\infty} \int_J f_k \chi_{J_k}$ uniformly for $J \subset [a,b]$.)*

Theorem 2.13 ([26]). *Let $a_{jk} \in \mathbb{R}$ for $j,k \in \mathbb{N}$ and $M = [a_{jk}]$. Suppose that:*

(i) *$\lim_j a_{jk} = 0$ for every k;*

(ii) *for every increasing sequence of positive integers $\{n_k\}$ there exists a subsequence $\{m_k\}$ of $\{n_k\}$ such that $\lim_j \sum_{k=1}^{\infty} a_{jm_k}$ exists.*

Then $\lim_j a_{jk} = 0$ uniformly for $k \in \mathbb{N}$. In particular, $\lim_j a_{jj} = 0$.

3. Some properties of the space of D^*-a.p. functions

It is well known that every uniformly almost periodic function is Stepanov almost periodic. Moreover, every Stepanov almost periodic function is D^*-almost periodic. This is the consequence of inequalities

$$\sup_{x\in\mathbb{R}} \sup_{0<h\leqslant 1} \left| \int_x^{x+h} f(t)\,\mathrm{d}t \right| \leqslant \sup_{x\in\mathbb{R}} \sup_{0<h\leqslant 1} \int_x^{x+h} |f(t)|\,\mathrm{d}t \leqslant \sup_{x\in\mathbb{R}} \int_x^{x+1} |f(t)|\,\mathrm{d}t,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is any S^1-almost periodic function. It can be also easily shown that every Stepanov almost periodic function is μ-almost periodic.

At the beginning of this section we are going to provide an example of continuous and bounded D^*-almost periodic function g which is not μ-a.p.

Example 3.1. Let us define the functions $g, g_k, G_k\colon \mathbb{R} \to \mathbb{R}$ for $k \in \mathbb{N}$ as follows:

$$g(x) = \begin{cases} \sin(2^{n+1}\pi x) & \text{for } x \in [2^{n-1}, 2^{n-1}+1) + 2^n\mathbb{Z},\ n\in \mathbb{N}, \\ 0 & \text{for other } x\in \mathbb{R}, \end{cases}$$

$$g_k(x) = \begin{cases} \sin(2^{n+1}\pi x) & \text{for } x \in [2^{n-1}, 2^{n-1}+1) + 2^n\mathbb{Z},\ n > k, \\ 0 & \text{for other } x\in \mathbb{R}, \end{cases}$$

$$G_k(x) = \begin{cases} \dfrac{1}{2^{n+1}}(1-\cos(2^{n+1}\pi x)) & \text{for } x \in [2^{n-1}, 2^{n-1}+1) + 2^n\mathbb{Z},\ n > k, \\ 0 & \text{for other } x\in \mathbb{R}. \end{cases}$$

The functions g, g_k, G are well defined because for $n \neq m$ we have

$$(2^{n-1} + 2^n\mathbb{Z}) \cap (2^{m-1} + 2^m\mathbb{Z}) = \emptyset.$$

Such sets also satisfy the equality

$$\bigcup_{n=1}^{\infty} (2^{n-1} + 2^n\mathbb{Z}) = \mathbb{Z} \setminus \{0\}. \tag{3.1}$$

The continuity of the function g (and the functions g_k, G_k as well) follows from the fact that the values of these functions at integer points are equal to zero.

For $k\in \mathbb{N}$ and $x\in \mathbb{R}$ we have $G_k'(x) = g_k(x)$. This is clear on the sets $(z, z+1)$, $z \in \mathbb{Z}$. For integer points it suffices to consider the left and right derivative of the functions G_k. Moreover, for $\tau \in 2^k\mathbb{Z}$, $x\in \mathbb{R}$ we have

$$g(x+\tau) - g(x) = g_k(x+\tau) - g_k(x). \tag{3.2}$$

Indeed, for $z \in \mathbb{Z}$ and $n \leqslant k$ we have

$$z \in 2^{n-1} + 2^n\mathbb{Z} \Leftrightarrow z + \tau \in 2^{n-1} + 2^n\mathbb{Z}.$$

Therefore, if $z \in 2^{n-1} + 2^n\mathbb{Z}$ for $n \leqslant k$, then for $x \in [z, z+1]$ we have $g(x+\tau) = g(x)$. Moreover, $g_k(x+\tau) = g_k(x) = 0$. If $z \notin \bigcup_{n=1}^{k} (2^{n-1} + 2^n\mathbb{Z})$, then according to the above observation $z + \tau \notin \bigcup_{n=1}^{k} (2^{n-1} + 2^n\mathbb{Z})$ and then for $x \in [z, z+1]$ we have $g(x) = g_k(x)$ and $g(x+\tau) = g_k(x+\tau)$. In each case we get (3.2).

Let us observe that

$$0 \leqslant G_k(x) \leqslant \frac{1}{2^{k+1}} \quad \text{for } x\in \mathbb{R}.$$

Let us fix $\varepsilon > 0$. We choose $k \in \mathbb{N}$ such that $4/2^{k+1} < \varepsilon$. Then for every $0 < h \leqslant 1$, $\tau \in 2^k\mathbb{Z}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned}
&\left|\int_x^{x+h} g(t+\tau)\,\mathrm{d}t - \int_x^{x+h} g(t)\,\mathrm{d}t\right| \\
&\qquad = \left|\int_x^{x+h} g_k(t+\tau)\,\mathrm{d}t - \int_x^{x+h} g_k(t)\,\mathrm{d}t\right| \\
&\qquad = |G_k(x+\tau+h) - G_k(x+\tau) - G_k(x+h) + G_k(x)| \leqslant \frac{4}{2^{k+1}} < \varepsilon.
\end{aligned}$$

This means that

$$2^k\mathbb{Z} \subset \{\tau \in \mathbb{R}\colon\ \|g_\tau - g\|_{D^*} < \varepsilon\}.$$

Since the set $2^k\mathbb{Z}$ is relatively dense, the set $\{\tau \in \mathbb{R}\colon\ \|g_\tau - g\|_{D^*} < \varepsilon\}$ is also relatively dense for each $\varepsilon > 0$, which means that g is D^*- almost periodic.

Now we are going to prove that g is not μ-a.p. Since

$$\mu\Big(\Big\{x \in [0, 2^{n+1}\pi]\colon\ |\sin(x)| \geqslant \frac{1}{2}\Big\}\Big) = 2^n \cdot \frac{4}{3}\pi,$$

thus,

$$\begin{aligned}
\mu\Big(\Big\{x \in [0,1]\colon\ |\sin(2^{n+1}\pi x)| \geqslant \frac{1}{2}\Big\}\Big) &= \frac{1}{2^{n+1}\pi}\mu\Big(\Big\{x \in [0, 2^{n+1}\pi]\colon\ |\sin(x)| \geqslant \frac{1}{2}\Big\}\Big) \\
&= \frac{1}{2^{n+1}\pi} 2^n \cdot \frac{4}{3}\pi = \frac{2}{3}.
\end{aligned}$$

According to (3.1) for $x \in [0,1]$ we have $g(x) = 0$. By (3.1) for each nonzero $\tau \in \mathbb{Z}$ there exists $n \in \mathbb{N}$ such that $\tau \in 2^{n-1} + 2^n\mathbb{Z}$. Therefore,

$$\begin{aligned}
\mu\Big(\Big\{x \in [0,1]\colon\ |g(x+\tau) - g(x)| \geqslant \frac{1}{2}\Big\}\Big) &= \mu\Big(\Big\{x \in [0,1]\colon\ |g(x+\tau)| \geqslant \frac{1}{2}\Big\}\Big) \\
&= \mu\Big(\Big\{x \in [\tau, \tau+1]\colon\ |g(x)| \geqslant \frac{1}{2}\Big\}\Big) \\
&= \mu\Big(\Big\{x \in [\tau, \tau+1]\colon\ |\sin(2^{n+1}\pi x)| \geqslant \frac{1}{2}\Big\}\Big) \\
&= \mu\Big(\Big\{x \in [0,1]\colon\ |\sin(2^{n+1}\pi x)| \geqslant \frac{1}{2}\Big\}\Big) = \frac{2}{3}.
\end{aligned}$$

The function g is not μ-p.o. because

$$\Big\{\tau \in \mathbb{R}\colon\ \sup_{u \in \mathbb{R}} \mu\Big(\Big\{x \in [u, u+1]\colon\ |g(x+\tau) - g(x)| \geqslant \frac{1}{2}\Big\}\Big) \leqslant \frac{1}{3}\Big\} \cap \mathbb{Z} = \{0\}$$

and according to Lemma 2.8 the above set should be relatively dense.

The next example is a continuous μ-almost periodic function, which is not D^*-almost periodic.

Example 3.2. Let

$$f(x) = \frac{1}{2 + \cos x + \cos(x\sqrt{2})}$$

for $x \in \mathbb{R}$. It was shown in [25] that f is μ-a.p. Moreover, we know that

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} \frac{1}{2 + \cos t + \cos(t\sqrt{2})}\,\mathrm{d}t = \infty,$$

see [9] for more details. Therefore, $\|f\|_{D^*} = \infty$, so according to Remark 2.6, f cannot be D^*-almost periodic. Let us add that a bounded μ-almost periodic function is S^p-almost periodic, see [25]. In particular, it means that it is D^*-almost periodic.

Now we will establish that the space D^*-almost periodic functions is not complete. For that we will expand, see [26], Example 1, page 73.

Example 3.3. Let $p\colon [0,1] \to \mathbb{R}$ be a continuous and nowhere differentiable function with $p(0) = 0$. Pick a sequence of polynomials $\{p_k\}$ such that $p_k\colon [0,1] \to \mathbb{R}$, $p_k \to p$ uniformly and $p_k(0) = 0$. Then we have $p_k(x) = \int_0^x p_k'(s)\,\mathrm{d}s$ for every $x \in [0,1]$, so $\{p_k'\}$ is a Cauchy sequence in $\mathrm{HK}([0,1])$ with respect to the Alexiewicz norm.

Let us observe that for each $f \in \mathrm{HK}([0,1])$ if we consider the 1-periodic function $\overline{f}$ such that

$$\overline{f}(x) = f(x) \quad \text{for } x \in [0,1),$$

then we have $\|\overline{f}\|_{D^*} \leqslant 2\|f\|_{A,0}$. Indeed, if for $x \in \mathbb{R}$, $0 < h \leqslant 1$ we have $x+h \leqslant [x]+1$, then

$$\int_x^{x+h} \overline{f}(t)\,\mathrm{d}t = \int_{x-[x]}^{x-[x]+h} f(t)\,\mathrm{d}t.$$

If $x + h > [x] + 1$, then

$$\left|\int_x^{x+h} \overline{f}(t)\,\mathrm{d}t\right| = \left|\int_I \overline{f}(t)\,\mathrm{d}t + \int_J \overline{f}(t)\,\mathrm{d}t\right| = \left|\int_{I-[x]} f(t)\,\mathrm{d}t\right| + \left|\int_{J-[x]-1} f(t)\,\mathrm{d}t\right| \leqslant 2\|f\|_{A,0},$$

where $I = [x, x+h] \cap [[x], [x]+1]$ and $J = [x, x+h] \cap [[x]+1, [x]+2]$.

Let $\bar{p}$, $\bar{p}_k$, $\bar{q}_k$, for $k \in \mathbb{N}$, be 1-periodic functions such that

$$\bar{p}(x) = p(x) \quad \text{for } x \in [0,1), \qquad \bar{p}_k(x) = p_k(x) \quad \text{for } x \in [0,1),$$
$$\bar{q}_k(x) := p_k'(x) \quad \text{for } x \in [0,1).$$

Because

$$\|\bar{q}_k - \bar{q}_l\|_{D^*} \leqslant 2\|p_k' - p_l'\|_{A,0},$$

the sequence $\{\bar{q}_k\}$ is a Cauchy sequence in the space of D^*-almost periodic functions. If there exists D^*-almost periodic function g such that $\|\bar{q}_k - g\|_{D^*} \to 0$, then $p_k(x) = \int_0^x \bar{q}_k(s)\,\mathrm{d}s \to \int_0^x g(s)\,\mathrm{d}s$ uniformly for $x \in [0,1]$, so $p(x) = \int_0^x g(s)\,\mathrm{d}s$ for $x \in [0,1]$.

By the properties of the Henstock-Kurzweil integral, the function p is continuous and almost everywhere differentiable, which gives a contradiction. Hence, the space of D^*-almost periodic functions with the norm $\|\cdot\|_{D^*}$ is not complete.

Now, we are going to prove three lemmas which will help us to prove the main result of this section.

Lemma 3.4. *If f is D^*-almost periodic, then for every $\varepsilon > 0$ the set*

$$E_{D^*}\{\varepsilon, f\} := \{\tau \in \mathbb{R}\colon \ \|f_\tau - f\|_{D^*} \leqslant \varepsilon\} \cap \mathbb{Z}$$

is relatively dense.

Proof. For a function $f\colon \mathbb{R} \to \mathbb{R}$ and $\varepsilon > 0$ let us denote

$$E\{\varepsilon, f\} := \Big\{\tau \in \mathbb{R}\colon \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leqslant \varepsilon\Big\}.$$

Let us consider the 1-periodic function $\sin(2\pi x)$. By the inequality

$$4x \leqslant \sin(2\pi x) \quad \text{for } x \in [0, \tfrac{1}{4}]$$

for $0 < \eta < 1$ we obtain

$$E\{\eta, \sin(2\pi\cdot)\} \subset \Big[-\frac{\eta}{4}, \frac{\eta}{4}\Big] + \mathbb{Z}.$$

Further, by [19], Theorem 2.1 the function

$$\Phi(x) = \int_0^x f(t)\,\mathrm{d}t$$

is uniformly continuous.

Let us fix $\varepsilon > 0$. We choose $0 < \delta_0 < \frac{1}{4}$ such that for $|\delta| \leqslant \delta_0$

$$\sup_{x\in\mathbb{R}} |\Phi(x+\delta) - \Phi(x)| \leqslant \frac{\varepsilon}{8}.$$

This means that for such δ we have

$$\|f_\delta - f\|_{D^*} \leqslant \frac{\varepsilon}{4}.$$

Let $N = [1/\delta_0] + 1$ and $h_i = i/N$ for $i = 1, 2, \ldots, N$. Let $\varepsilon' > 0$ be such that

$$\varepsilon' < \varepsilon, \quad \frac{\varepsilon'}{16} < \delta_0.$$

Let

$$E_i = E\Big\{\frac{\varepsilon'}{4}, \int_x^{x+h_i} f(t)\,\mathrm{d}t\Big\}.$$

Since the functions

$$x \to \int_x^{x+h_i} f(t)\,\mathrm{d}t$$

are uniformly almost periodic, the set

$$E_0 = \bigcap_{i=1}^{N} E_i \cap E\Big\{\frac{\varepsilon'}{4}, \sin(2\pi\cdot)\Big\}$$

is relatively dense.

Let us consider $0 < h \leqslant 1$. For $h_1 < h < 1$ there exists $1 \leqslant i \leqslant N-1$ such that $h \in (h_i, h_{i+1}]$ and for $\tau \in E_0$ we obtain

$$\begin{aligned}
&\Big|\int_x^{x+h} [f(t+\tau) - f(t)]\,\mathrm{d}t\Big| \\
&\qquad \leqslant \Big|\int_x^{x+h_i} [f(t+\tau) - f(t)]\,\mathrm{d}t| + \Big|\int_{x+h_i}^{x+h} f(t)\,\mathrm{d}t\Big| + \Big|\int_{x+\tau+h_i}^{x+\tau+h} f(t)\,\mathrm{d}t\Big| \\
&\qquad \leqslant \frac{\varepsilon'}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \leqslant \frac{\varepsilon}{2}.
\end{aligned}$$

For $0 < h \leqslant h_1$, since $h_1 \leqslant \delta_0$, we obtain

$$\Big|\int_x^{x+h} [f(t+\tau) - f(t)]\,\mathrm{d}t\Big| \leqslant \Big|\int_x^{x+h} f(t)\,\mathrm{d}t\Big| + \Big|\int_{x+\tau}^{x+\tau+h} f(t)\,\mathrm{d}t\Big| \leqslant \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \leqslant \frac{\varepsilon}{2}.$$

For $h = 1$, we have

$$\Big|\int_x^{x+h} [f(t+\tau) - f(t)]\,\mathrm{d}t\Big| \leqslant \frac{\varepsilon'}{4} \leqslant \frac{\varepsilon}{2}.$$

Thus, in each case we get

$$\Big|\int_x^{x+h} [f(t+\tau) - f(t)]\,\mathrm{d}t\Big| \leqslant \frac{\varepsilon}{2},$$

which means that

$$\|f_\tau - f\|_{D^*} \leqslant \frac{\varepsilon}{2}.$$

Let $z(x)$ denote the nearest integer number to x for $x \in (-\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}$. Since $\varepsilon' < 4$,

$$E_0 = \bigcap_{i=1}^{N} E_i \cap E\Big\{\frac{\varepsilon'}{4}, \sin(2\pi\cdot)\Big\} \subset \Big[-\frac{\varepsilon'}{16}, \frac{\varepsilon'}{16}\Big] + \mathbb{Z}$$

for $\tau \in E_0$ we have

$$\|f_{z(\tau)} - f\|_{D^*} \leqslant \|f_{z(\tau)} - f_\tau\|_{D^*} + \|f_\tau - f\|_{D^*} \leqslant \frac{3}{4}\varepsilon < \varepsilon.$$

Obviously the set $\{z(\tau) \colon \tau \in E_0\}$ is relatively dense, which completes the proof. $\square$

Lemma 3.5. *For each D^*-almost periodic function f and each interval $I \subset [0,1]$, the function*

$$f\chi_{I+\mathbb{Z}},$$

where $\chi_{I+\mathbb{Z}}$ denotes the characteristic function of the set $I+\mathbb{Z}$, is D^*-almost periodic.

P r o o f. Let us fix $I = [a,b] \subset 1$. For each $x\in \mathbb{R}$ and $0 < h \leqslant 1$, let us consider the intersection

$$[x, x+h] \cap [I + \mathbb{Z}] = A \cup B,$$

where

$$A = [x, x+h] \cap ([x] + I), \quad B = [x, x+h] \cap ([x] + 1 + I),$$

the interior of the set $A \cap B$ is empty. For

$$\tau \in E_{D^*}\Big\{f, \frac{\varepsilon}{2}\Big\} \cap \mathbb{Z}$$

we have

$$\begin{aligned}\Big|\int_x^{x+h} & f(t+\tau)\chi_{I+\mathbb{Z}}(t+\tau)\,\mathrm{d}t - \int_x^{x+h} f(t)\chi_{I+\mathbb{Z}}(t)\,\mathrm{d}t\Big| \\ &= \Big|\int_A (f(t+\tau) - f(t))\,\mathrm{d}t + \int_B (f(t+\tau) - f(t))\,\mathrm{d}t\Big| \leqslant 2\|f_\tau - f\|_{D^*} \leqslant \varepsilon.\end{aligned}$$

This means that

$$E_{D^*}\Big\{f, \frac{\varepsilon}{2}\Big\} \cap \mathbb{Z} \subset E_{D^*}\{f\chi_{I+\mathbb{Z}}, \varepsilon\}.$$

By Lemma 3.4, $E_{D^*}\{f\chi_{I+\mathbb{Z}}, \varepsilon\}$ is relatively dense, which completes the proof. $\square$

Lemma 3.6. *Let us fix $0 \leqslant a < b \leqslant 1$. If f is D^*-almost periodic, then the function*

$$y \to f\chi_{[a,y]+\mathbb{Z}} \quad \text{for } a < y \leqslant b$$

is continuous. Moreover,

$$\lim_{y\to a^+} \|f\chi_{[a,y]+\mathbb{Z}}\|_{D^*} = 0.$$

Remark 3.7. Since for $0 \leqslant a < b \leqslant 1$ and $x\in \mathbb{R}, h > 0$, we have

$$\begin{aligned}\int_x^{x+h} f(t)\chi_{[a,y]+\mathbb{Z}}(t)\,\mathrm{d}t &= \int_x^{x+h} f(t)\chi_{[a,y)+\mathbb{Z}}(t)\,\mathrm{d}t = \int_x^{x+h} f(t)\chi_{(a,y]+\mathbb{Z}}(t)\,\mathrm{d}t \\ &= \int_x^{x+h} f(t)\chi_{(a,y)+\mathbb{Z}}(t)\,\mathrm{d}t\end{aligned}$$

for the intervals $(a,y]$, $[a,y)$, (a,y) we obtain the same conclusions.

P r o o f. By Lemma 3.5, for every $y \in (a,b]$, the function $f\chi_{[a,y]+\mathbb{Z}}$ is D^*-almost periodic. Denote

$$\Phi(x) = \int_0^x f(t)\,\mathrm{d}t \quad \text{for } x \in \mathbb{R}.$$

Let us fix $\varepsilon > 0$. There exist $\delta_0 > 0$ such that for $0 < \delta \leqslant \delta_0$

$$\sup_{x\in\mathbb{R}} |\Phi(x+\delta) - \Phi(x)| \leqslant \frac{\varepsilon}{2}.$$

Let us consider $y \in (a,b]$ and $\delta > 0$ such that $a \leqslant y - \delta \leqslant y \leqslant b$. For $x\in \mathbb{R}$ and $0 < h \leqslant 1$ let us denote

$$A = [x, x+h] \cap ([y-\delta, y] + [x]), \quad B = [x, x+h] \cap ([y-\delta, y] + [x] + 1).$$

We have

$$[x, x+h] \cap ([y-\delta, y] + \mathbb{Z}) = A \cup B,$$

and the interior of the set $A \cap B$ is empty. Therefore,

$$\left|\int_x^{x+h} f(t)\chi_{[a,y]+\mathbb{Z}}(t)\,\mathrm{d}t - \int_x^{x+h} f(t)\chi_{[a,y-\delta]+\mathbb{Z}}(t)\,\mathrm{d}t\right| = \left|\int_A f(t)\,\mathrm{d}t + \int_B f(t)\,\mathrm{d}t\right| \leqslant \varepsilon.$$

This means that

$$\|f\chi_{[a,y]+\mathbb{Z}} - f\chi_{[a,y-\delta]+\mathbb{Z}}\|_{D^*} \leqslant \varepsilon.$$

This means that the function

$$y \to f\chi_{[a,y]+\mathbb{Z}} \quad \text{for } a < y \leqslant b$$

is continuous at every point $y \in (a,b]$. Moreover, if we consider $a < y$ such that $y - a < \delta_0$, then the similar arguments show that

$$\left|\int_x^{x+h} f(t)\chi_{[a,y](t)}\,\mathrm{d}t\right| \leqslant \varepsilon.$$

Hence,

$$\|f\chi_{[a,y]+\mathbb{Z}}\|_{D^*} \leqslant \varepsilon,$$

and consequently

$$\lim_{y\to a^+} \|f\chi_{[a,y]+\mathbb{Z}}\|_{D^*} = 0.$$

□

Now, we are going to prove the main result of this section which, according to Proposition 2.10, means that the space D^* is barrelled.

Theorem 3.8. *Let B be a weak* bounded subset of the dual space of the D^*-almost periodic functions. Then B is norm bounded.*

The main idea of the proof below is similar to that of the proof of barrelledness of the space $\mathrm{HK}([a,b])$, see [26].

P r o o f. Let A be a bounded subset of the D^*-almost periodic functions with $\alpha = \sup\{\|f\|_{D^*}\colon f \in A\}$. It suffices to show that $\beta = \sup\{|v(f)|\colon v \in B,\ f \in A\} < \infty$. Suppose that $\beta = \infty$. Then there exist $v_1 \in B$, $f_1 \in A$ such that $|v_1(f_1)| > 2$. Let us observe that for each interval $I \subset [0,1]$ the value of a functional v on the functions $f\chi_{I+\mathbb{Z}}$ and $f\chi_{\overline{I}+\mathbb{Z}}$ is the same since the functions $f\chi_{I+\mathbb{Z}}$ and $f\chi_{\overline{I}+\mathbb{Z}}$ differ on a set, the Lebesgue measure of which is equal to zero. Let us observe that if for $[a,b] \subset [0,1]$ we have

$$\sup\{|v(f\chi_{[a,b]+\mathbb{Z}})|\colon v \in B,\ f \in A\} = \infty,$$

then for all closed intervals I, K such that $I \cup K = [a,b]$, $\mathrm{card}(I \cap K) = 1$, we have

$$\sup\{|v(f\chi_{I+\mathbb{Z}})|\colon v \in B,\ f \in A\} = \infty \quad \text{or} \quad \sup\{|v(f\chi_{K+\mathbb{Z}})|\colon v \in B,\ f \in A\} = \infty.$$

We can therefore, for convenience, assume that the above property is satisfied on the subinterval denoted by the letter I (if both intervals have such property, then in particular the interval I has such property). After this remark, we are ready to construct sequences of closed intervals (I_k) and (K_k).

By Lemma 3.6, the function $y \to v_1(f_1\chi_{[0,y]+\mathbb{Z}})$ is continuous, so there exists a partition of $[0,1]$, that is, the closed intervals I_1, K_1 such that $I_1 \cup K_1 = [0,1]$, $\mathrm{card}(I_1 \cap K_1) = 1$ and such that $|v_1(f_1\chi_{I_1+\mathbb{Z}})| > 1$, $|v_1(f_1\chi_{K_1+\mathbb{Z}})| > 1$, $\sup\{|v(f\chi_{I_1+\mathbb{Z}})|\colon v \in B,\ f \in A\} = \infty$. Now there exist $v_2, \in B$, $f_2 \in A$ such that $|v_2(f_2\chi_{I_1+\mathbb{Z}})| > 2^4$. By the argument as above, there exist a partition of I_1 on 2 closed subintervals I_2, K_2 such that $|v_2(f_2\chi_{I_2+\mathbb{Z}})| > 2^3$, $|v_2(f_2\chi_{K_2+\mathbb{Z}})| > 2^3$ and $\sup\{|v(f\chi_{I_2+\mathbb{Z}})|\colon v \in B,\ f \in A\} = \infty$. Continuing this construction we obtain sequences of closed intervals (I_k), (K_k) with the following property:

- $[0,1] = I_1 \cup K_1$;
- $I_k = I_{k+1} \cup K_{k+1}$, $k \in \mathbb{N}$;
- $\mathrm{card}(I_k \cap K_k) = 1$, so $\mathrm{Int}\, I_k \cap \mathrm{Int}\, K_k = \emptyset$, $k \in \mathbb{N}$;
- $|v_k(f_k\chi_{K_k+\mathbb{Z}})| > k^3$, $k \in \mathbb{N}$.

The existence of such intervals cannot be proved using just the linearity of v_k and the triangle inequality. This is because in each step we need two estimations simultaneously. Let us denote $J_k = \mathrm{Int}\, K_k$ for $k \in \mathbb{N}$. Since $I_{k+1} \subset I_k$ and $K_{k+1} \subset I_k$, we have for $k, l \in \mathbb{N}$

$$K_{k+l} \subset I_{k+l-1} \subset I_k.$$

Therefore,

$$K_{k+l} \cap K_k \subset I_k \cap K_k,$$

and consequently,

$$J_{k+l} \cap J_k = \emptyset.$$

In that way we obtain a pairwise disjoint sequence of open intervals (J_k), $J_k \subset [0,1]$, (f_k), $f_k \in A$, (v_k), $v_k \in B$ such that

$$|v_k(f_k \chi_{J_k+\mathbb{Z}})| > k^3, \quad k \in \mathbb{N}. \tag{3.3}$$

Since for $k \neq l$ the sets J_k, J_l are open disjoint intervals, we also have

$$K_k \cap J_l = \emptyset \quad \text{for } k \neq l. \tag{3.4}$$

Because we will use Proposition 2.12, now we are going to prove that the boundary ∂X of the set

$$X := [0,1] \setminus \bigcup_{k=1}^{\infty} J_k$$

is countable. Let us observe that the above set is closed, so $\partial X \subset X$. Let us denote $K_k = [a_k, b_k]$, $k \in \mathbb{N}$. Observe that

$$[0,1] \setminus \bigcup_{k=1}^{\infty} J_k = \bigcap_{k=1}^{\infty} I_k \cup \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\}. \tag{3.5}$$

First, we will show the inclusion

$$[0,1] \setminus \bigcup_{k=1}^{\infty} J_k \subset \bigcap_{k=1}^{\infty} I_k \cup \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\}. \tag{3.6}$$

Let $x \in [0,1] \setminus \bigcup_{k=1}^{\infty} J_k$. If simultaneously $x \in \bigcap_{k=1}^{\infty} I_k$, then (3.6) is satisfied. If $x \notin \bigcap_{k=1}^{\infty} I_k$, then there exists k_0 such that $x \notin I_{k_0}$. Because for each $k \in \mathbb{N}$ we have $I_k = I_{k+1} \cup K_{k+1}$ and $K_k = J_k \cup \{a_k, b_k\}$, so

$$[0,1] = I_{k_0} \cup K_1 \cup K_2 \cup \ldots \cup K_{k_0} = I_{k_0} \cup J_1 \cup J_2 \cup \ldots \cup J_{k_0} \cup \{a_k \colon k \leqslant k_0\} \cup \{b_k \colon k \leqslant k_0\}.$$

Then because $x \notin J_k$ for all $k \in \mathbb{N}$, so $x \in \{a_k \colon k \leqslant k_0\} \cup \{b_k \colon k \leqslant k_0\}$, which means (3.6).

Now we are going to prove that

$$\bigcap_{k=1}^{\infty} I_k \cup \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\} \subset [0,1] \setminus \bigcup_{k=1}^{\infty} J_k. \tag{3.7}$$

If $x \in \bigcap_{k=1}^{\infty} I_k$, then because $\operatorname{card}(I_k \cap K_k) = 1$, we know that $I_k \cap J_k = \emptyset$, which means $x \notin J_k$ for $k \in \mathbb{N}$. Now, assume that $x \in \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\}$. Let

$x = a_{k_0}$ or $x = b_{k_0}$ for some $k_0 \in \mathbb{N}$. This means that $x \notin \bigcup\limits_{k=1}^{\infty} J_k$. Indeed, if for some k_1 we have $x \in J_{k_1}$, then because $x \notin J_{k_0}$, we infer that $k_0 \neq k_1$. We would get $x \in K_{k_0} \cap J_{k_1}$, but this is impossible according to (3.4). Therefore, $x \notin \bigcup\limits_{k=1}^{\infty} J_k$ and (3.5) is satisfied.

It is easy to see that the boundary of the set $\bigcap\limits_{k=1}^{\infty} I_k \cup \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\}$ is countable. If $\bigcap\limits_{k=1}^{\infty} I_k$ is a singleton, then the set X is countable and closed. If $\bigcap\limits_{k=1}^{\infty} I_k = [a,b]$ is a nondegenerate interval, then $(a,b) \subset \operatorname{Int} X$, so

$$\partial X \subset \{a,b\} \cup \{a_k \colon k \in \mathbb{N}\} \cup \{b_k \colon k \in \mathbb{N}\},$$

which means that ∂X is countable.

Set $h_k = k^{-2} f_k$, $k \in \mathbb{N}$, $h = \sum\limits_{k=1}^{\infty} h_k \chi_{J_k+\mathbb{Z}}$ (pointwise) and let $s_n = \sum\limits_{k=1}^{n} h_k \chi_{J_k+\mathbb{Z}}$, $n \in \mathbb{N}$. Then $\|h_k\|_{D^*} = k^{-2}\|f_k\|_{D^*} \leqslant \alpha/k^2$. Hence, $\sum\limits_{k=1}^{\infty} \|h_k\|_{D^*} < \infty$.

Consider the matrix $M = [a_{jk}] = [j^{-1} v_j(h_k \chi_{J_k+\mathbb{Z}})] = [j^{-1} v_j(k^{-2} f_k \chi_{J_k+\mathbb{Z}})]$, $j,k \in \mathbb{N}$. We claim that M satisfies (i) and (ii) of Theorem 2.13. First, (i) holds by the weak* boundedness of B.

To prove (ii), first we want to establish that $v_k(h)$ is well defined for $k \in \mathbb{N}$, which actually means that h is D^*-almost periodic. Let us fix $z \in \mathbb{Z}$. We have $h_k|_{[z,z+1]} \in \mathrm{HK}([z,z+1])$, $k \in \mathbb{N}$. Moreover,

$$\sum_{k=1}^{\infty} ||h_k|_{[z,z+1]}||_{A,z} = \sum_{k=1}^{\infty} ||h_k||_{A,z} \leqslant \sum_{k=1}^{\infty} \|h_k\|_{D^*} < \infty.$$

Consider the sequence $(J_k + z)$ of open subintervals of the interval $[z, z+1]$. By Proposition 2.12, for the function $h|_{[z,z+1]} = \sum\limits_{k=1}^{\infty} h_k|_{[z,z+1]} \chi_{J_k+\mathbb{Z}}|_{[z,z+1]}$ the value

$$H(J) = \sum_{k=1}^{\infty} \int_J h_k|_{[z,z+1]} \chi_{J_k+\mathbb{Z}}|_{[z,z+1]} = \sum_{k=1}^{\infty} \int_J h_k \chi_{J_k+\mathbb{Z}}$$

exists for any interval $J \subset [z, z+1]$ and $\lim\limits_{|J|\to 0} H(J) = 0$ (of course the quantity $H(J)$ depends on the $z \in \mathbb{Z}$). Next, since the set $[z, z+1] \setminus \bigcup\limits_{k=1}^{\infty} (J_k + z)$ is just a translation of the set X, so the boundary of the set $[z, z+1] \setminus \bigcup\limits_{k=1}^{\infty} (J_k + z)$ is countable, then from Theorem 2.11 we know that $h|_{[z,z+1]} \in \mathrm{HK}([z, z+1])$ and for $J \subset [z, z+1]$ we know $\int_J h = \int_J h|_{[z,z+1]} = H(J)$, which can be written as

$$\int_J h = \int_J \sum_{k=1}^{\infty} h_k \chi_{J_k+\mathbb{Z}} = \sum_{k=1}^{\infty} \int_J h_k \chi_{J_k+\mathbb{Z}}. \tag{3.8}$$

(The above conclusion can be also obtained by the (iii) of Proposition 2.12, because by Theorem 2.11 we know that $h|_{[z,z+1]} \in \mathrm{HK}([z,z+1])$.) Because for any subinterval $J \subset [z, z+1]$ we have

$$\int_J s_n = \int_J \sum_{k=1}^n h_k \chi_{J_k+\mathbb{Z}} = \sum_{k=1}^n \int_J h_k \chi_{J_k+\mathbb{Z}},$$

so

$$\begin{aligned}\int_J \sum_{k=n+1}^\infty h_k \chi_{J_k+\mathbb{Z}} = \int_J (h - s_n) &= \sum_{k=1}^\infty \int_J h_k \chi_{J_k+\mathbb{Z}} - \sum_{k=1}^n \int_J h_k \chi_{J_k+\mathbb{Z}} \\ &= \sum_{k=n+1}^\infty \int_J h_k \chi_{J_k+\mathbb{Z}}.\end{aligned}$$

By (2.3), we have

$$\sum_{k=n+1}^\infty \sup_{x\in\mathbb{R}} \|h_k\|'_{A,x} = \sum_{k=n+1}^\infty \sup_{x\in\mathbb{R}} \|h_k\|_{A,x},$$

therefore for each $J \subset [z, z+1]$ we obtain

$$\begin{aligned}\left|\int_J (h - s_n)\right| &= \left|\int_J \sum_{k=n+1}^\infty h_k \chi_{J_k+\mathbb{Z}}\right| = \left|\sum_{k=n+1}^\infty \int_J h_k \chi_{J_k+\mathbb{Z}}\right| \leqslant \sum_{k=n+1}^\infty \left|\int_J h_k \chi_{J_k+\mathbb{Z}}\right| \\ &= \sum_{k=n+1}^\infty \left|\int_{J\cap J_k+z} h_k\right| \leqslant \sum_{k=n+1}^\infty \|h_k\|'_{A,z} \leqslant \sum_{k=n+1}^\infty \sup_{x\in\mathbb{R}} \|h_k\|'_{A,x} \\ &= \sum_{k=n+1}^\infty \sup_{x\in\mathbb{R}} \|h_k\|_{A,x} = \sum_{k=n+1}^\infty \|h_k\|_{D^*} \leqslant \sum_{k=n+1}^\infty \frac{\alpha}{k^2}.\end{aligned}$$

This means that $\|h - s_n\|'_{A,z} \leqslant \sum\limits_{k=n+1}^\infty \alpha/k^2$. Then because for $x \in \mathbb{R}$ we have

$$[x, x+1] \subset [[x], [x]+1] \cup [[x]+1, [x]+2], \tag{3.9}$$

where $[x]$ denotes the floor of the number x, so

$$\|h - s_n\|_{A,x} \leqslant \|h - s_n\|'_{A,[x]} + \|h - s_n\|'_{A,[x]+1}$$

and

$$\|h - s_n\|_{D^*} = \sup_{x\in\mathbb{R}} \|h - s_n\|_{A,x} \leqslant 2 \sup_{z\in\mathbb{Z}} \|h - s_n\|'_{A,z} \leqslant 2 \sum_{k=n+1}^\infty \frac{\alpha}{k^2}.$$

By (3.8) for any interval $J \subset [z, z+1]$ we have

$$\left|\int_J h\right| = \left|\sum_{k=1}^\infty \int_J h_k \chi_{J_k+\mathbb{Z}}\right| \leqslant \sum_{k=1}^\infty \left|\int_J h_k \chi_{J_k+\mathbb{Z}}\right| \leqslant \sum_{k=1}^\infty \|h_k\|' < \infty,$$

so using again (3.9), we obtain

$$\|h\|_{D^*} = \sup_{x\in\mathbb{R}} \|h\|_{A,x} \leqslant 2 \sup_{z\in\mathbb{Z}} \|h\|'_{A,z} \leqslant 2\sum_{k=1}^{\infty} \|h_k\|' < \infty.$$

Let us observe that if $\|s_n - h\|_{D^*} \to 0$, where s_n are D^*-a.p. functions for $n \in \mathbb{N}$ and $h \in D^*_{\mathrm{loc}}$ is such that $\|h\|_{D*} < \infty$, then h is a D^*-a.p. function. Indeed, we have

$$\|h_\tau - h\|_{D^*} \leqslant \|h_\tau - (s_{n_0})_\tau\|_{D^*} + \|(s_{n_0})_\tau - s_{n_0}\|_{D^*} + \|s_{n_0} - h\|_{D^*}.$$

By Lemma 3.5 and the fact that D^* is a linear space we know that the finite sums s_n are D^*-a.p. functions. Because the set

$$\{\tau \in \mathbb{R}\colon\ \|(s_{n_0})_\tau - s_{n_0}\|_{D^*} < \varepsilon\}$$

is relatively dense for each $\varepsilon > 0$ if $\|s_{n_0} - h\|_{D^*} < \frac{1}{3}\varepsilon$, then

$$\Big\{\tau \in \mathbb{R}\colon\ \|(s_{n_0})_\tau - s_{n_0}\|_{D^*} < \frac{\varepsilon}{3}\Big\} \subset \{\tau \in \mathbb{R}\colon\ \|h_\tau - h\|_{D^*} < \varepsilon\}.$$

Therefore, h is a D^*-a.p. function. This means that the value of $v_j(h)$ is well defined for $j \in \mathbb{N}$. By the continuity of each v_j on the set of D^*-almost periodic functions since $s_n \to h$,

$$\lim_{n\to\infty} \frac{1}{j} v_j(s_n) = \frac{1}{j} v_j(h),$$

but

$$\frac{1}{j} v_j(s_n) = \frac{1}{j}\sum_{k=1}^{n} v_j(h_k \chi_{J_k+\mathbb{Z}}) = \sum_{k=1}^{n} a_{jk}.$$

This means $\sum\limits_{k=1}^{\infty} a_{jk} = j^{-1}v_j(h)$ and because $j^{-1}v_j(h) \to 0$, by the weak* boundedness of B, we obtain $\lim\limits_{j} \sum\limits_{k=1}^{\infty} a_{jk} = 0$.

Now let us observe that the same arguments can be applied to any subsequence (h_{n_k}) of the sequence (h_k) and (J_{n_k}). One can define the function $h' = \sum\limits_{k=1}^{\infty} h_{n_k}\chi_{J_{n_k}+\mathbb{Z}}$ and repeat the above arguments. Obviously, $\sum\limits_{k=1}^{\infty} \|h_{n_k}\|_{D^*} < \infty$. Let us observe that for any subsequence (J_{n_k}) of the sequence of intervals (J_k) if we denote

$$X' = [0,1] \setminus \bigcup_{k=1}^{\infty} J_{n_k},$$

then X' is closed and therefore $\partial X' \subset X'$. Moreover, if we denote $L = \mathbb{N}\setminus\{n_k\colon\ k\in\mathbb{N}\}$, then we have

$$[0,1] \setminus \bigcup_{k=1}^{\infty} J_{n_k} = \bigcap_{k=1}^{\infty} I_k \cup \{a_k\colon\ k \in \mathbb{N}\} \cup \{b_k\colon\ k \in \mathbb{N}\} \cup \bigcup_{k\in L} J_k$$

(if $n_k = k$ for $k \in \mathbb{N}$, then $L = \emptyset$ and $X' = X$). Moreover, $\operatorname{Int}\Bigl(\bigcap_{k=1}^{\infty} I_k\Bigr) \subset \operatorname{Int} X'$, $\bigcup_{k\in L} J_k \subset \operatorname{Int} X'$, so

$$\partial X' \subset \{a, b\} \cup \{a_k \colon\ k \in \mathbb{N}\} \cup \{b_k \colon\ k \in \mathbb{N}\},$$

which means that the boundary of X' is countable. Therefore, we can use Theorem 2.11 and Proposition 2.12 for the subsequence under consideration. We showed that

$$\lim_{j\to\infty} \sum_{k=1}^{\infty} a_{jn_k} = 0$$

for each increasing subsequence (n_k) of positive integers. Theorem 2.13 implies that $a_{jj} \to 0$ contradicting (3.3). $\square$

4. Linear differential equations

An essential tool which will be needed in this section is the following proposition.

Proposition 4.1 ([21]). *Suppose that f is D^*-almost periodic and g is a function of bounded variation in the Jordan sense on $\mathbb{R}$ such that*

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{x\in[k,k+1]} |g(x)| < \infty.$$

Then the convolution $f * g$ is uniformly almost periodic and

$$\|f * g\|_{D^*} \leqslant \|f\|_{D^*} \|g\|_{L^1(\mathbb{R})}, \quad \|f * g\|_{\infty} \leqslant 2\|f\|_{D^*} \|g\|_V,$$

where

$$\|g\|_V = \sum_{k=\infty}^{\infty} \operatorname*{ess\,sup}_{x\in[k,k+1]} |g(x)| + \operatorname{Var}(g, \mathbb{R}).$$

and $\operatorname{Var}(g, \mathbb{R})$ denotes the Jordan variation of g on $\mathbb{R}$.

Remark 4.2. Let us recall that if one considers the Lebesgue integral and Stepanov almost periodic functions, then in the above proposition it is enough to assume that g is integrable in the Lebesgue sense to get a similar conclusion concerning the convolution, see [5], Lemma 2. However, let us notice that, in general, the convolution of a Stepanov almost periodic function with a function integrable in the Lebesgue sense, does not have to be uniformly almost periodic, see [9], Example 2.

The main result of this section is the following theorem.

Theorem 4.3. *If f is a D^*-almost periodic function, then the linear differential equation*

$$y'(x) = \lambda y(x) + f(x), \quad \text{where } \lambda \neq 0 \tag{4.1}$$

possesses a uniformly almost periodic solution.

Remark 4.4. We may assume that $\lambda < 0$, because if y_1 is a solution to (4.1), then $y_2(x) := -y_1(-x)$ for $x \in \mathbb{R}$, is a solution to the equation

$$y'(x) = -\lambda y(x) + \tilde{f}(x),$$

where $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}$.

Proof. For $\lambda < 0$ let us consider the function

$$g_\lambda(x) = \begin{cases} e^{\lambda x} & \text{for } x \geqslant 0; \\ 0 & \text{for } x < 0. \end{cases}$$

Obviously, for this function the following inequality holds:

$$\sum_{k=-\infty}^{\infty} \operatorname*{ess\,sup}_{x \in [k,k+1]} |g_\lambda(x)| < \infty.$$

Let us observe that

$$(f * g_\lambda)(x) = e^{\lambda x} \int_{-\infty}^{x} f(t) e^{-\lambda t} \, dt.$$

By the properties of the Henstock-Kurzweil integral, the above function is continuous and almost everywhere differentiable. The proof of the existence of convolution $f * g_\lambda$ for all $x \in \mathbb{R}$ can be found in [20]. Moreover, this convolution is a solution to equation (4.1). Applying Proposition 4.1 we can infer that for a D^*-almost periodic function f, equation (4.1) possesses a uniformly almost periodic solution. □

Remark 4.5. Let us notice that the above theorem is connected, in a sense, with [3], Theorem 4.1, in which it is assumed that the non-homogeneous term is Henstock-Kurzweil integrable on the extended set of real numbers.

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