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A ROLLER COASTER APPROACH TO INTEGRATION AND PEANO'S EXISTENCE THEOREM

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Dedicated to the memory of Professor Jaroslav Kurzweil

Abstract. This is a didactic proposal on how to introduce the Newton integral in just three or four sessions in elementary courses. Our motivation for this paper were Talvila's work on the continuous primitive integral and Koliha's general approach to the Newton integral. We introduce it independently of any other integration theory, so some basic results require somewhat nonstandard proofs. As an instance, showing that continuous functions on compact intervals are Newton integrable (or, equivalently, that they have primitives) cannot lean on indefinite Riemann integrals. Remarkably, there is a very old proof (without integrals) of a more general result, and it is precisely that of Peano's existence theorem for continuous nonlinear ODEs, published in 1886. Some elements in Peano's original proof lack rigor, and that is why his proof has been criticized and revised several times. However, modern proofs are based on integration and do not use Peano's original ideas. In this note we provide an updated correct version of Peano's original proof, which obviously contains the proof that continuous functions have primitives, and it is also worthy of remark because it does not use the Ascoli-Arzelà theorem, uniform continuity, or any integration theory.

Keywords: primitive; Newton integral; Peano's existence theorem

MSC 2020: 26A27, 26A36, 26A39

1. Introduction

We owe many important advances on integration and differential equations to Jaroslav Kurzweil, so we thought that a paper combining ideas from those two fields would be a good way to honor him.

In this paper we present a didactic approach to Koliha's new version of the classical Newton integral (see [13]) and we update Peano's original proof of his celebrated

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existence theorem for ODEs to obtain, in particular, a direct proof that continuous functions are Newton integrable.

Let us start by recalling that if F is differentiable at each point then the Newton integral of its derivative is

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

i.e., half of the fundamental theorem of calculus. Is this old notion of an integral really worthy of attention nowadays? The Newton integral, whose drawbacks (specially in limiting processes) forced a reformulation of the notion of an integral, has kept certain interest through times (see [5], [11]) because, from its very definition, it integrates all derivatives, and this is something impossible with Riemann or Lebesgue integrals.

Finding an integral which can "integrate all derivatives" and does not suffer Newton integral's drawbacks was the problem which motivated the search for the Henstock-Kurzweil integral, and we cannot refrain from quoting reference [15], where Kurzweil described his integral for the first time. Even more recently, variations on that problem have drawn the attention of many experts, see [2], [3], [4], [6]. There has also been an interest in characterizing different notions of integrability in terms of generalized derivatives or adequate sequences of primitives. See [1], where Henstock-Kurzweil integrals are defined by means of primitives using the so-called monotonically controlled derivatives. See also [24], where Lebesgue integrable functions are proven to be limits of sequences of derivatives, and [16], where the Henstock-Kurzweil integral is redefined in terms of special sequences of primitives of Lebesgue integrable functions.

Far from remaining as a mere historical topic, the Newton integral gained relevance thanks to its reformulation in terms of distributions by Mikusiński and Sikorski, see [18], also [17], and, more recently, by Talvila, see [23]. Indeed, one can see in [23] that if f is a distribution on the real line and $F\colon \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies $F' = f$ in the sense of distributions and has finite limits $F(\pm\infty)$, then the simple Newton-type definition

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = F(\infty) - F(-\infty) \tag{1.1}$$

yields an integral which contains Henstock-Kurzweil's. Therefore, the distributional Newton integral (or, as Talvila puts it, the "continuous primitive integral") deserves a prominent treatment in advanced mathematical analysis courses. In turn, and this is the point in this paper, elementary courses should pay some more attention to the classical Newton integral, and not only for historical reasons but also for its importance in the more advanced setting of distributions and for its influence in the development of other integration theories, as commented in the previous paragraphs.

There are many bibliographical references where the essential contents on the Newton integral can be looked up, such as [5], [10]. However, Koliha's approach (see [13]) is, in our opinion, the best, as it leans on primitives which need not be so on countable sets, thus getting a more general, though still elementary, Newton integral. What is new in this paper? Just a slight revision of Koliha's approach to the Newton integral, aiming to be self-contained, independent of any other integration theory, and, hopefully, useful as a couple of teaching sessions for undergraduates not acquainted with Lebesgue integration. In particular, we provide two new proofs of an important basic result (see namely, Theorem 2.1) and a direct proof that continuous functions are Newton integrable which does not lean on any other integration theory or on uniform continuity. To do so, we update Peano's original proof of his classical existence theorem for ODEs, thus getting a more general result and connecting this paper with a remarkable academic controversy which took place in the seventies of the last century.

Peano's original proof dates back to 1886 (see [19]) and it has been revised many times since, starting by Peano himself who also discovered a completely different proof for the system case, see [20]. We can quote old important related papers such as that by Perron (see [21]), who wiped out some inaccuracies in Peano's original proof by using Dini derivatives and integration. Peano's and Perron's proofs, though essentially correct, lacked some rigor at certain steps, so their validity remained at issue, and they were replaced by the usual more general proofs for the vector case established in terms of sequences of Euler polygons. In the early seventies of the last century, as a reaction to a challenging question posed by Kennedy (see [12]), there was an intense production of elementary proofs of the scalar Peano theorem. We quote Dow and Výborný (see [7]), Gardner (see [8]), J. Walter (see [25]), and W. Walter (see [26]). However, all these elementary proofs are essentially different from the original, in the sense that all of them, excepting (see [8]), use Riemann integration and Gardner employs a sequence of Euler-type polygons introduced by W. Walter instead of upper and lower functions as in Peano's original proof.

The proof in this paper is old and new at the same time: we simply provide an updated and technically correct version of Peano's original proof. In our opinion, this is the simplest proof of the scalar case of Peano's theorem, as it only needs the notions of least upper bound, continuity and right derivative of real valued real functions, and it does not depend on uniform continuity, on the Ascoli-Arzelà theorem and, last but not least, on integrals of any type. Therefore, our proof includes as a particular case a proof that continuous functions have primitives which is independent of integration theories.

This paper is organized as follows: in Section 2 we describe Koliha's revision of the Newton integral (see [13]) with some simplifications and alternative proofs indicated at relevant places; in Section 3 we present an updated version of Peano's original proof of his existence theorem for scalar ODEs and we use it to prove that continuous functions are integrable in the sense described in Section 2; in Section 4, we include a uniform convergence theorem for the Koliha-Newton integral which is useful, in particular, for another proof of the integrability of continuous functions and it also helps to see that integrals of positive functions are areas, which is not at all evident from the definition. Finally, an elementary result about continuous right derivatives needed in Section 3 is proven in the appendix.

2. A roller coaster approach to integration

This is a didactic proposal on how to introduce the Newton integral in an elementary course, essentially along the lines of [13]. With this section's appealing title we intend to highlight the fact that we have to start by proving only one "hard" result in order to climb high enough so that we can perform a super fast easy descent towards the remaining basic details of the theory. Most of the results in this section can be looked up in [13], either as theorems or as exercises, and they are included for the sake of clearer presentation.

Let $A \subset \mathbb{R}$. Following [13], [14], we say that a condition holds for "nearly all $x \in A$", or that it holds "nearly everywhere on A", if it holds for all $x \in A \setminus C$, where $C \subset A$ is, at most, countable.

Our steep first ascent is the following result, which readers can find in a more general form in [14] and which we prove in two different ways in this section. The proof given in [14] for the general version uses full covers and Cousin's lemma, but things can be simplified a little and can be made self-contained in our simplified setting. Due to the nature of this special issue, it is more than appropriate to highlight that our first proof of the following result, which is the base of the integration theory to be described next, leans on Kurzweil's δ-fine tagged partitions.

Theorem 2.1. *Let $a, b \in \mathbb{R}$, $a < b$, and let $F\colon [a,b] \to \mathbb{R}$ be a continuous function.*

(i) *If $F'(x) \geqslant 0$ for nearly all $x \in [a,b]$, then F is monotone nondecreasing on $[a,b]$.*

(ii) *If $F'(x) \leqslant 0$ for nearly all $x \in [a,b]$, then F is monotone nonincreasing on $[a,b]$.*

(iii) *If $F'(x) = 0$ for nearly all $x \in [a,b]$, then F is constant on $[a,b]$.*

Proof. Observe that (ii) follows from (i) applied to $-F$, and (iii) is immediate from (i) and (ii).

To prove (i), we fix arbitrary points $x, y \in [a,b]$, $x < y$, and we have to show that $F(x) \leqslant F(y)$.

Let $\{s_n\colon n \in \mathbb{N}\}$ be the countable subset of $[x, y]$, where F' does not exist or $F' < 0$.

Let $\varepsilon > 0$ be fixed. Since F is continuous, for each $z = s_n$, $n \in \mathbb{N}$, there exists $\delta(z) > 0$ such that

$$F(u) - F(v) \geqslant -\frac{\varepsilon}{2^n} \quad \text{provided that } u, v \in [z - \delta(z), z + \delta(z)] \cap [x, y]. \tag{2.1}$$

If, on the other hand, $z \in [x, y] \setminus \{s_n\colon n \in \mathbb{N}\}$, then $F'(z) \geqslant 0$, so we can find $\delta(z) > 0$ such that

$$\frac{F(u) - F(z)}{u - z} = \frac{F(z) - F(u)}{z - u} > -\varepsilon \text{ for all } u \in [z - \delta(z), z + \delta(z)] \cap [x, y],\ u \neq z. \tag{2.2}$$

We deduce from (2.2) that

$$F(u) - F(z) \geqslant -\varepsilon(u - z) \quad \text{for all } u \in [z, z + \delta(z)] \cap [x, y] \tag{2.3}$$

and

$$F(z) - F(u) \geqslant -\varepsilon(z - u) \quad \text{for all } u \in [z - \delta(z), z] \cap [x, y]. \tag{2.4}$$

The collection of intervals $\{(z - \delta(z), z + \delta(z))\}_{z \in [x,y]}$ is an open cover of the compact set $[x, y]$, hence there exist finitely many points in $[x, y]$, say $z_1 < z_2 < \ldots < z_m$ ($m \in \mathbb{N}$), such that

$$[x, y] \subset \bigcup_{j=1}^{m} (z_j - \delta(z_j), z_j + \delta(z_j)).$$

We may (and we do) assume that no interval $(z_j - \delta(z_j), z_j + \delta(z_j))$ is contained in another interval of the previous union. This guarantees that if for some $j \in \{1, 2, \ldots, m - 1\}$ we have $z_j + \delta(z_j) < z_{j+1}$, then $z_{j+1} - \delta(z_{j+1}) < z_j + \delta(z_j)$, and therefore, F satisfies either (2.1) or (2.4) with $z = z_{j+1}$ in the interval $[z_j + \delta(z_j), z_{j+1}]$.

Let us consider the partition of $[x, y]$ which contains $z_1, z_2, \ldots, z_m$ and all those points $z_j + \delta(z_j)$ such that $z_j + \delta(z_j) < z_{j+1}$. We denote the points in this partition by $x_0 = x < x_1 < x_2 < \ldots < x_p = y$ and let $I_k = [x_k, x_{k+1}]$, $k = 0, 1, \ldots, p - 1$.

Observe that for each I_k there is some j such that either $x_k = z_j$ or $x_{k+1} = z_j$ and, in both cases, $I_k \subset [z_j - \delta(z_j), z_j + \delta(z_j)] \cap [x, y]$. Therefore, either F satisfies condition (2.1) in I_k for $z = z_j = s_n$ for some n, or F satisfies one of the conditions (2.3) or (2.4) in I_k for $z = z_j$.

Let $\mathcal{I}_1$ be the family of the I_k's, where the condition (2.1) holds and let $\mathcal{I}_2$ be the complementary family. Now we have

$$\begin{aligned} F(y)-F(x) &= \sum_{k=0}^{p-1}(F(x_{k+1})-F(x_k)) \\ &= \sum_{I_k\in\mathcal{I}_1}(F(x_{k+1})-F(x_k)) + \sum_{I_k\in\mathcal{I}_2}(F(x_{k+1})-F(x_k)) \\ &\geqslant -\varepsilon\sum_{n=1}^{\infty}2^{-n} - \varepsilon\sum_{k=0}^{p-1}(x_{k+1}-x_k) = -\varepsilon(1+y-x). \end{aligned}$$

Since $\varepsilon > 0$ can be arbitrarily small, we deduce that $F(y)-F(x)\geqslant 0$. □

The following new proof of Theorem 2.1 uses a Cantor ternary set.

P r o o f of Theorem 2.1. (i) First, we prove the following result, which is interesting in its own right: *if F is continuous on $[a,b]$ and $F'(x) > 0$ for nearly all $x\in[a,b]$, then F is increasing on $[a,b]$ (i.e., $a\leqslant x<y\leqslant b \Rightarrow F(x)<F(y)$).* Reasoning by contradiction, suppose that there exist $x,y\in[a,b]$, $x<y$, such that $F(x)\geqslant F(y)$. Since F cannot be constant on $[x,y]$, we may assume that $F(x)>F(y)$. Now, put $\bar{x}=\sup\{s\in[x,y]\colon F(s)\geqslant F(x)\}$ and $\bar{y}=\inf\{s\in[\bar{x},y]\colon F(s)\leqslant F(y)\}$. Since F is continuous, we have $\bar{x}<\bar{y}$, $F(\bar{x})=F(x)>F(y)=F(\bar{y})$, and

$$F(\bar{x})>F(s)>F(\bar{y}) \quad \text{for all } s\in(\bar{x},\bar{y}). \tag{2.5}$$

From the interval $[\bar{x},\bar{y}]$ we extract recursively infinitely many subintervals with vanishing lengths and each one satisfying the corresponding version of (2.5). To do so, we proceed as follows: put $I_0=[\bar{x},z_*]$ and $I_1=[z^*,\bar{y}]$, where $z_*=\inf\{s\in[\bar{x},\bar{x}+\frac{1}{3}(\bar{y}-\bar{x})]\colon F(s)=F(\bar{x}+\frac{1}{3}(\bar{y}-\bar{x}))\}$ and

$$z^*=\sup\{s\in[\bar{y}-\tfrac{1}{3}(\bar{y}-\bar{x}),\bar{y}]\colon F(s)=F(\bar{y}-\tfrac{1}{3}(\bar{y}-\bar{x}))\}.$$

Note that $I_0\cap I_1=\emptyset$ and $l(I_i)\leqslant\frac{1}{3}(\bar{y}-\bar{x})$, where $l(I_i)$ stands for the length of the interval I_i $(i=0,1)$. Moreover, we deduce from (2.5), the definition of z_*, and the continuity of F, that

$$F(\bar{x})>F(s)>F(z_*)=F(\bar{x}+\tfrac{1}{3}(\bar{y}-\bar{x})) \quad \text{for all } s\in(\bar{x},z_*).$$

Analogously, $F(z^*)=F\frac{1}{3}(\bar{y}-(\bar{y}-\bar{x}))>F(s)>F(\bar{y})$ for all $s\in(z^*,\bar{y})$.

Repeating the previous process with I_0 and I_1, we get four pairwise disjoint subintervals I_{00} and I_{01} (both inside I_0) and I_{10} and I_{11} (both inside I_1) which fulfill the following properties: if $I_{c_1c_2}=[\alpha,\beta]$ is one of these four subintervals $(c_i\in\{0,1\}$, $i=1,2)$, then $l(I_{c_1c_2})\leqslant\frac{1}{9}(\bar{y}-\bar{x})$ and $F(\alpha)>F(s)>F(\beta)$ for all $s\in(\alpha,\beta)$.

Continuing this process *ad infinitum* by extracting two subintervals from each interval produced in the previous step, we get as many different families of nested intervals as sequences of 0's and 1's. More specifically, for each sequence $\theta = \{\theta_n\}_{n=1}^{\infty}$, where each θ_n is either 0 or 1, we have the associated sequence of nested intervals $\{J_n\}_{n=1}^{\infty}$ defined by $J_n = I_{\theta_1\theta_2\ldots\theta_n}$ for each $n \in \mathbb{N}$.

Now, if we denote $J_n = [a_n, b_n]$, $n \in \mathbb{N}$, then we have $F(a_n) > F(s) > F(b_n)$ for all $s \in (a_n, b_n)$ and $l(J_n) \leqslant \frac{1}{3}(\overline{y} - \bar{x})^n$. Therefore, there is a unique $x_\theta \in [\bar{x}, \overline{y}]$ such that

$$\bigcap_{n=1}^{\infty} J_n = \{x_\theta\}$$

and so, for any $n \in \mathbb{N}$, we have at least one of the inequalities

$$\frac{F(a_n) - F(x_\theta)}{a_n - x_\theta} < 0 \quad \text{or} \quad \frac{F(b_n) - F(x_\theta)}{b_n - x_\theta} < 0.$$

Since both the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ tend to x_θ, we deduce that either $F'(x_\theta)$ does not exist or $F'(x_\theta) \leqslant 0$. This is a contradiction with the assumption, because there are as many different points $x_\theta \in [\bar{x}, \overline{y}]$ as different sequences $\theta = \{\theta_n\}$ of 0's and 1's, which are uncountable.

Finally, we prove (i). For any $\varepsilon > 0$ the function $F_\varepsilon(x) = F(x) + \varepsilon x$ is continuous and $F_\varepsilon' = F' + \varepsilon > 0$ nearly everywhere on $[a, b]$. Therefore, for $x, y \in [a, b]$, $x \leqslant y$, we have $F_\varepsilon(x) \leqslant F_\varepsilon(y)$, hence $F(x) - F(y) \leqslant \varepsilon(y - x) \leqslant \varepsilon(b - a)$. Since $\varepsilon > 0$ was arbitrary, we deduce that F is nondecreasing on $[a, b]$. □

Remark 2.2. Theorem 2.1 is false if we replace "nearly everywhere" by "almost everywhere" (in the Lebesgue sense of "everywhere with the possible exception of a null measure set"). A well-known counterexample is Lebesgue's singular function (see [22], pages 129 and 130), which is continuous, it has zero derivative almost everywhere, and yet it is not constant. See also [14] for more information.

Unless stated otherwise, integrability, primitives and integrals in this paper are considered in the following sense only.

Definition 2.3. A function $f\colon D(f) \subset [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if $[a, b] \setminus D(f)$ is countable and there exists a continuous function $F\colon [a, b] \to \mathbb{R}$ such that

$$\exists\, F'(x) = f(x) \quad \text{for nearly all } x \in [a, b].$$

In this case, we say that F is a primitive of f on $[a, b]$, and we denote and define the integral of f on $[a, b]$ as

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) \quad (\text{notation: } F|_a^b = F(b) - F(a)).$$

As usual, we employ the notations

$$\int_c^c f(x)\,\mathrm{d}x = 0 \quad\text{and}\quad \int_d^c f(x)\,\mathrm{d}x = -\int_c^d f(x)\,\mathrm{d}x \quad (a \leqslant c \leqslant d \leqslant b).$$

Koliha's definition in [13] also includes integration on unbounded intervals. We have decided to remain in this more elementary setting for simplicity and also because, in our opinion, integration on unbounded intervals should be presented in elementary courses as an afterthought.

On the other hand, our Definition 2.3 allows for functions defined only nearly everywhere, which makes it more flexible in practice and emphasizes the fact that countable sets are negligible in this theory.

Next we point out some simple examples of integrable functions, primitives and integrals (observe that some of the functions under the integral sign are not defined at every point of the interval):

$$\int_0^1 (\alpha x + \beta)\,\mathrm{d}x = \Big(\frac{\alpha}{2}x^2 + \beta x\Big)\Big|_0^1 = \frac{\alpha}{2} + \beta \quad (\alpha, \beta \in \mathbb{R}),$$

$$\int_{-1}^1 \frac{x}{|x|}\,\mathrm{d}x = |x|\big|_{-1}^1 = 0, \qquad \int_0^1 \frac{\mathrm{d}x}{2\sqrt{x}} = \sqrt{x}\big|_0^1 = 1.$$

The "potential energy" delivered by Theorem 2.1 trivializes the proofs of the usual basic integration results. It is an illuminating exercise to prove the following properties using Definition 2.3 and Theorem 2.1 (note that some properties can be deduced from the previous ones) and to compare the proofs with the corresponding ones in the setting of the Riemann integral.

(1) The integral is well defined, i.e., given two primitives we get the same result.

(2) The integral is nonnegative, i.e., if $f(x)$ is integrable on $[a,b]$ and $f \geqslant 0$ nearly everywhere on $[a,b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \geqslant 0.$$

(3) If f and g are integrable on $[a,b]$, then for every $\alpha, \beta \in \mathbb{R}$ the function $\alpha f + \beta g$ is integrable on $[a,b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

(4) The integral is monotone nondecreasing: if f and g are integrable on $[a,b]$ and $f \leqslant g$ nearly everywhere on $[a,b]$, then

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x. \tag{2.6}$$

(5) If f and $|f|$ are integrable on $[a,b]$ (see Subsection 2.1 for the importance of these two assumptions), then

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x.$$

(6) If f is integrable on $[a,b]$ and $[c,d] \subset [a,b]$, then f is integrable on $[c,d]$.

(7) For every $c \in (a,b)$ we have

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x,$$

where the existence of each side of the identity implies the existence of the other.

(8) Integration by parts. If f and g are integrable on $[a,b]$ with primitives F and G, respectively, and the product Fg (or fG) is integrable on $[a,b]$, then

$$\int_a^b F(x)g(x)\,\mathrm{d}x = FG|_a^b - \int_a^b f(x)G(x)\,\mathrm{d}x.$$

(9) Change of variable. If $g\colon D(g) \subset [a,b] \to \mathbb{R}$ has a primitive $G\colon [a,b] \to [\alpha,\beta]$, $\alpha < \beta$, and $f\colon [\alpha,\beta] \to \mathbb{R}$ has a primitive $F\colon [\alpha,\beta] \to \mathbb{R}$ such that $F' = f$ everywhere on $[\alpha,\beta]$, then

$$\int_{G(a)}^{G(b)} f(x)\,\mathrm{d}x = \int_a^b f(G(t))g(t)\,\mathrm{d}t.$$

Next we show that Definition 2.3 already includes improper integrals on bounded intervals. In the folklore we say that this integral satisfies Hake's property.

Theorem 2.4. *For any $f\colon D(f) \subset [a,b] \to \mathbb{R}$ the following conditions are equivalent:*

(i) *f is integrable on $[a,b]$ and*

$$\int_a^b f(x)\,\mathrm{d}x = I \in \mathbb{R}.$$

(ii) *f is integrable on $[a,c]$ for every $c \in (a,b)$ and there exists*

$$\lim_{c\to b^-} \int_a^c f(x)\,\mathrm{d}x = I \in \mathbb{R}.$$

P r o o f. Obviously, (i) implies (ii). To prove the converse, let $\{b_n\}_{n=1}^{\infty}$ be an increasing sequence in (a,b) tending to b as n tends to infinity. We define a primitive of f on $[a,b]$ recursively as follows. Let $F_1\colon\ [a,b_1]\to\mathbb{R}$ be a primitive of f on $[a,b_1]$ and put $F=F_1$ on $[a,b_1]$. Assume that F has been defined as a primitive of f on $[a,b_n]$ for some $n\in\mathbb{N}$, take F_{n+1} as a primitive of f on $[b_n,b_{n+1}]$ such that $F_{n+1}(b_n)=F(b_n)$, and put $F=F_{n+1}$ on $[b_n,b_{n+1}]$. Finally, the condition (ii) guarantees that we can define

$$F(b)=\lim_{n\to\infty}F(b_n)=F(a)+\lim_{n\to\infty}\int_a^{b_n}f(x)\,\mathrm{d}x=F(a)+\lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x.$$

We have thus, constructed a primitive of F on $[a,b]$ and therefore, f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,\mathrm{d}x=F(b)-F(a)=\lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x.$$

□

Remark 2.5. Theorem 2.4 remains valid, with obvious changes, if we consider limits to a^-.

2.1. Comparison with other integrals and some warnings. In the first part of this section we describe briefly the relation of Definition 2.3 with Riemann and Lebesgue integrals, so it might be outside elementary courses, as intended in this paper. In the second part of this section we point out some important limitations of Definition 2.3.

Definition 2.3 allows us to integrate many functions which are not even Lebesgue integrable. As an instance, take the derivative of $F(x)=x\sin(1/x)$, $x\neq 0$, which is integrable on $[0,1]$ in the sense of Definition 2.3 but it is not Lebesgue integrable on $[0,1]$. Notice that F' is improperly Riemann integrable on $(0,1]$, but improper Riemann integrals do not suffice to cover Definition 2.3 either. Indeed, on each interval $(\pi^{-1}\,(n+1)^{-1},\pi^{-1}\,n^{-1})$, $n\in\mathbb{N}$, define $\widehat{F}(x)=F(x-\pi^{-1}(n+1)^{-1})$ and put $\widehat{F}(x)=0$ elsewhere in $[0,\pi^{-1}]$. Now, $f=\widehat{F}'$ is integrable on $[0,\pi^{-1}]$ in the sense of Definition 2.3 but it is not improperly Riemann integrable on $[0,\pi^{-1}]$, because the integral cannot be expressed as a finite sum of improper Riemann integrals.

The information in the previous paragraph may induce the wrong idea that Definition 2.3 yields an integral which is better than Lebesgue's, but this is far from being true. In fact, Definition 2.3 does not even include the Riemann integral. As we are going to show next, to justify our last statement shortly we need a more specialized result from [13] which connects the Newton and the Lebesgue integrals.

Let us denote by χ_C the characteristic function of Cantor's ternary set $C \subset [0,1]$, which is Riemann integrable on $[0,1]$ because it is bounded, continuous on $[0,1]\setminus C$, and $m(C)=0$, where m stands for the Lebesgue measure. To show that χ_C is not integrable in the sense of Definition 2.3 we use a contradiction argument: if χ_C has a primitive F on $[0,1]$ (in the sense of Definition 2.3), then we have $F' \geqslant 0$ nearly everywhere on $[0,1]$ and

$$\int_0^1 \chi_C(x)\,\mathrm{d}x = F(1) - F(0) \geqslant 0.$$

By Theorem 14.12 of [13], the Newton and the Lebesgue integral must agree, so $F(1)-F(0)=m(C)=0$, but then F must be a constant and so it cannot be a primitive of χ_C (because C is not countable).

We close this section with some warnings, all of them being consequences of the first one:

(1) The absolute value of an integrable function need not be integrable. Indeed, we show that $f(x)=x^{-1}\cos(x^{-1})$ is integrable on $[0,1]$ but $|f|$ is not. To do so, we use that continuous functions on compact intervals have primitives (which can be proven without any integration theory, see Section 3) and therefore, they are integrable in the sense of Definition 2.3. Let $F(x)=x\sin(x^{-1})$ and observe that for each $c\in(0,1)$ we have

$$F'(x)=\sin(x^{-1})-f(x) \quad \text{for all } x\in[c,1],$$

hence,

$$F(1)-F(c)=\int_c^1 (\sin(x^{-1})-f(x))\,\mathrm{d}x = \int_c^1 \sin(x^{-1})\,\mathrm{d}x - \int_c^1 f(x)\,\mathrm{d}x.$$

Notice that for $0<c<c'<1$ we have

$$\left|\int_c^{c'} \sin(x^{-1})\,\mathrm{d}x\right| \leqslant c'-c,$$

which implies that the Cauchy condition for the existence of limit as $c\to 0^+$ is satisfied. Therefore,

$$\exists \lim_{c\to 0^+} \int_c^1 f(x)\,\mathrm{d}x = \lim_{c\to 0^+} \int_c^1 \sin(x^{-1})\,\mathrm{d}x - F(1),$$

thus, proving that f is integrable on $[0,1]$, by virtue of Theorem 2.4 and Remark 2.5. In order to show that $|f|$ is not integrable on $[0,1]$, let us take into account that

$$|f(x)| \geqslant \frac{\sqrt{2}}{2}\Big(n\pi + \frac{3\pi}{4}\Big)$$

for

$$x \in I_n = [a_n, b_n] = \Big[\Big(n\pi + \frac{5\pi}{4}\Big)^{-1}, \Big(n\pi + \frac{3\pi}{4}\Big)^{-1}\Big], \quad n \in \mathbb{N}.$$

Put $g(x) = \frac{1}{2}\sqrt{2}(n\pi + \frac{3}{4}\pi)$ for $x \in I_n$, $n \in \mathbb{N}$, and $g = 0$ elsewhere in $[0, 1]$. For any $n \in \mathbb{N}$ we have

$$\int_{a_n}^{1} |f(x)|\,\mathrm{d}x \geqslant \int_{a_n}^{1} g(x)\,\mathrm{d}x = \frac{\sqrt{2}}{2}\sum_{j=1}^{n}\Big(j\pi + \frac{3\pi}{4}\Big)(b_j - a_j),$$

which tends to ∞ as n tends to infinity, so $|f|$ is not integrable on $[0, 1]$ by virtue of Theorem 2.4 and Remark 2.5.

(2) As a consequence of the previous observation, we deduce that compositions $g \circ f$ with continuous g and integrable f need not be integrable.

(3) Pointwise maxima (or minima) of integrable functions need not be integrable (remember that $|f| = \max\{f, -f\}$).

(4) Products of integrable functions need not be integrable. Just consider f as in the first observation and

$$g(x) = \frac{f(x)}{|f(x)|} \quad \text{for nearly all } x \in [0, 1].$$

We have $f(x)g(x) = |f(x)|$ nearly everywhere on $[0, 1]$, which is not integrable. However, g is integrable on $[0, 1]$ because, first, it is integrable on $[c, 1]$ for every $c \in (0, 1)$, and, second, we have

$$\Big|\int_{c}^{c'} g(x)\,\mathrm{d}x\Big| \leqslant c' - c \quad \text{whenever } 0 < c \leqslant c' < 1.$$

3. Integrability of continuous functions and Peano's existence theorem at one stroke

Our goal in this section is a direct proof that continuous functions on compact intervals are Newton integrable. By "direct" proof we mean a proof which does not depend on any other integration theory.

Remarkably, proving it *without integrals* was already done by Peano in a more general form as early as in 1886 (see [19]), when he proved that for any bounded and continuous function $f\colon\ [a, b] \times \mathbb{R} \to \mathbb{R}$ there is at least one differentiable function $y\colon\ [a, b] \to \mathbb{R}$ such that

$$y'(x) = f(x, y(x)) \quad \text{for all } x \in [a, b].$$

In the particular case of $f(x, y) = f(x)$ we get the result we want, but the reduction of Peano's original proof to this particular case is not so much easier as to make it worthy to forget about the more important general Peano's theorem.

Peano's original proof was considered faulty in some technical aspects (see [12]) and many other elementary proofs were given, as we already mentioned in Introduction. In the author's opinion, Peano's original proof is essentially correct and it just needs a couple of minor technical adjustments at relevant places.

The proof given here uses the notation and main ideas from [19], along with an improvement by Goodman (see [9]) and an adequate definition of the involved set of functions. We use no integration theory and our arguments do not depend on uniform continuity or the Ascoli-Arzelà theorem.

Theorem 3.1. *If $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded, then for each $y_a\in\mathbb{R}$ there exists at least one function $y\in\mathcal{C}^1([a,b])$ such that*

$$y'(x) = f(x, y(x)) \quad \text{for all } x\in[a,b], \text{ and } y(a) = y_a. \tag{3.1}$$

In particular, every continuous function $f\colon [a,b]\to\mathbb{R}$ has primitives (hence, f is integrable on $[a,b]$).

Proof. (Respectfully and slightly updated from Peano's original paper of 1886). Let $M > 0$ be such that $|f(x,y)| \leqslant M$ on $[a,b]\times\mathbb{R}$. The function $u(x) = y_a + M(x-a)$ ($x\in[a,b]$) satisfies $u(a) = y_a$ and $u'(x) = M \geqslant f(x, u(x))$ for all $x\in[a,b]$. Define the set[1]

$$\Lambda = \{u\in\mathcal{C}([a,b])\colon\ u(a) = y_a,\ M\geqslant u'\geqslant f(x,u) \text{ on } [a,b]\setminus N_u,\ N_u \text{ empty or finite}\}.$$

Observe that $u\in\Lambda$ implies $u(x)\geqslant y_a - M(x-a)\geqslant y_a - M(b-a)$ for all $x\in[a,b]$, and therefore we can put

$$y(x) = \inf\{u(x)\colon\ u\in\Lambda\} \quad \text{for any } x\in[a,b]. \tag{3.2}$$

In particular, $y(a) = y_a$. Let us prove that $y\in\mathcal{C}([a,b])$. This part of the proof is based on the fact that elements of Λ are Lipschitz continuous on $[a,b]$ with Lipschitz constant M. We fix $\varepsilon > 0$ and we prove that if $x_1, x_2\in[a,b]$ are such that

[1] Peano considers the set of functions u satisfying $u' > f(x,u)$ everywhere. However, at certain step he uses two of them to produce a new one piece by piece, but the resulting function need not be differentiable everywhere and may go outside the set of relevant functions. On the other hand, here we employ Goodman's idea (see [9]) and we use non strict inequalities, which has the advantage that every solution belongs to Λ, making it evident that the solution $y(x)$ to be defined in the proof is the least one, a relevant information which we chose not to include in the statement for the sake of a less technical and clearer presentation.

$|x_1 - x_2| < \varepsilon/(3M)$, then $|y(x_1) - y(x_2)| < \varepsilon$. Let x_1, x_2 be as before, assume that $x_1 < x_2$, and take $u_i \in \Lambda$ $(i = 1, 2)$ such that $y(x_i) \leqslant u_i(x_i) < y(x_i) + \frac{1}{3}\varepsilon$. Now we define $u_3 \in \Lambda$ as follows: if $u_1(x_2) \leqslant u_2(x_2)$, then we take $u_3 = u_1$; if $u_1(x_2) > u_2(x_2)$ and $u_1(x_1) \geqslant u_2(x_1)$, then we take $u_3 = u_2$; finally, if $u_1(x_2) > u_2(x_2)$ and $u_1(x_1) < u_2(x_1)$, then we can find $x_3 \in (x_1, x_2)$ such that $u_1(x_3) = u_2(x_3)$, and we put $u_3 = u_1$ on $[a, x_3]$ and $u_3 = u_2$ on $[x_3, b]$. In any case, $u_3 \in \Lambda$ and we have

$$|y(x_1) - y(x_2)| \leqslant |y(x_1) - u_3(x_1)| + |u_3(x_1) - u_3(x_2)| + |u_3(x_2) - y(x_2)| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Next, we prove that $\exists\, y'_+(x) = f(x, y(x))$ for all $x \in [a, b)$, where y'_+ is the right derivative of y.

Let $x_0 \in [a, b)$ be fixed and denote $m = f(x_0, y(x_0))$. Let $\varepsilon > 0$ be fixed and consider the function

$$\varphi(x) = y(x_0) + (x - x_0)(m - \varepsilon).$$

We have $\varphi'(x) = m - \varepsilon < m$, hence there exists $x_1 > x_0$ such that $\varphi' < f(x, \varphi)$ on $[x_0, x_1)$.

Let us prove that $\varphi \leqslant u$ on $[x_0, x_1]$ for every $u \in \Lambda$. For any $u \in \Lambda$ we have $u(x_0) \geqslant y(x_0) = \varphi(x_0)$ and, moreover, $u' \geqslant f(x, u)$ on $(x_0, x_1) \setminus N_u$, N_u being empty or finite. Assume, reasoning by contradiction, that there is an interval $(x_2, x_3) \subset (x_0, x_1)$ such that $u(x_2) = \varphi(x_2)$ and

$$u(x) < \varphi(x), \quad x \in (x_2, x_3). \tag{3.3}$$

Since $\varphi'(x_2) < f(x_2, \varphi(x_2)) = f(x_2, u(x_2))$, and φ' and $f(\cdot, u(\cdot))$ are continuous at $x = x_2$, there exists $x_4 \in (x_2, x_3)$ such that $\varphi'(x) < f(x, u(x)) \leqslant u'(x)$ for all $x \in (x_2, x_4)$ ($N_u \cap (x_2, x_4) = \emptyset$ provided that x_4 is sufficiently close to x_2). Hence, $u - \varphi$ is increasing on $[x_2, x_3]$ and therefore $u - \varphi \geqslant (u - \varphi)(x_2) = 0$ on $[x_2, x_4]$, a contradiction with (3.3).

Since $u \in \Lambda$ was arbitrary, we deduce that $\varphi \leqslant y$ on $[x_0, x_1]$, i.e.,

$$y(x_0) + (x - x_0)(m - \varepsilon) \leqslant y(x), \quad x \in [x_0, x_1],$$

hence

$$\frac{y(x) - y(x_0)}{x - x_0} \geqslant m - \varepsilon, \quad x \in (x_0, x_1]. \tag{3.4}$$

On the other hand, for a fixed $\varepsilon > 0$, the function

$$H(\alpha, x) = m + \varepsilon - f(x, y(x_0) + \alpha + (m + \varepsilon)(x - x_0))$$

is continuous and $H(0, x_0) = \varepsilon$. Hence, we can find $\varrho > 0$ and $x_1 > x_0$ such that $H(\alpha, x) > 0$ for every $\alpha \in [0, \varrho)$ and every $x \in [x_0, x_1)$. Now, for any $\alpha \in (0, \varrho)$ we can find $u \in \Lambda$ such that $u(x_0) = y(x_0) + \widetilde{\alpha}$ for some $\widetilde{\alpha} \in [0, \alpha)$, and we consider a function $\psi(x)$ such that $\psi = u$ on $[a, x_0]$,

$$\psi(x) = y(x_0) + \widetilde{\alpha} + (m + \varepsilon)(x - x_0), \quad x \in (x_0, x_1),$$

and $\psi(x) = y(x_0) + \widetilde{\alpha} + (m + \varepsilon)(x_1 - x_0) + M(x - x_1)$ for $x \in [x_1, b]$. Obviously, $\psi \in \Lambda$, and therefore,

$$y(x) \leqslant \psi(x) < y(x_0) + \alpha + (m + \varepsilon)(x - x_0), \quad x \in (x_0, x_1).$$

Since the previous inequality is satisfied for any $\alpha \in (0, \varrho)$, we deduce that

$$\frac{y(x) - y(x_0)}{x - x_0} \leqslant m + \varepsilon, \quad x \in (x_0, x_1). \tag{3.5}$$

We have thus proven that y has a derivative from the right at every point $x \in [a, b)$ and $y'_+(x) = f(x, y(x))$. In particular, the right derivative y'_+ is continuous on $[a, b)$ and it has a finite limit as x tends to b^-, hence (see Appendix) y is differentiable everywhere[2] on $[a, b]$ and $y'(x) = f(x, y(x))$ for all $x \in [a, b]$. □

Continuous functions on (a, b) need not be integrable on $[a, b]$, but they are provided that we can find an integrable bound.

Corollary 3.2. *Let $a, b \in \mathbb{R}$, $a < b$. If $f\colon (a, b) \to \mathbb{R}$ is continuous on (a, b) and there exists $g\colon D(g) \subset [a, b] \to \mathbb{R}$, g integrable on $[a, b]$, and*

$$|f(x)| \leqslant g(x) \quad \text{for nearly all } x \in (a, b),$$

then f is integrable on $[a, b]$.

Proof. We have to show that f has a primitive $F\colon [a, b] \to \mathbb{R}$ in the sense of Definition 2.3. Let $I_n = [a_n, b_n] \subset (a, b)$, $n \in \mathbb{N}$, and assume that $\{a_n\}_n$ is decreasing and tends to a, and that $\{b_n\}_n$ is increasing and tends to b. Define F on $[a_1, b_1]$ as a primitive of f on $[a_1, b_1]$, which exists by Peano's theorem. Assume that we have already defined F on $[a_n, b_n]$ for some $n \in \mathbb{N}$. Now, define F on $[a_{n+1}, a_n]$ as a primitive of f on that interval which coincides with F at $x = a_n$, and define F on $[b_n, b_{n+1}]$ as a primitive of f which assumes at b_n the value $F(b_n)$. So far, we have inductively defined a primitive $F(x)$ for all $x \in (a, b)$. We need to show that F can be extended to $[a, b]$ as a continuous function and, to do so, it suffices to prove that F is uniformly continuous on (a, b). Let $G\colon [a, b] \to \mathbb{R}$ be a primitive of g.

[2] Our argumentation here is again different from the original.

For any $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in (a,b)$, $|x-y| < \delta$, we have $|G(x) - G(y)| < \varepsilon$, hence

$$|F(x) - F(y)| = \left| \int_x^y f(s)\,\mathrm{d}s \right| \leqslant \left| \int_x^y g(s)\,\mathrm{d}s \right| = |G(x) - G(y)| < \varepsilon.$$

□

4. Convergence, integrability of continuous functions revisited and areas

We have an analogue of the uniform convergence theorem for the Riemann integral. In fact, the following result is almost a translation into integrals of a well-known theorem on uniform convergence of sequences of derivatives, see [22], Theorem 4.56, page 214. However, some minor differences arise due to exceptional countable sets, which makes it advisable to include the proof here for convenience of the reader.

Theorem 4.1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $[a,b]$.*
If there is a countable set $C \subset [a,b]$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b] \setminus C$ to some function $f\colon [a,b] \setminus C \to \mathbb{R}$, then f is integrable on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

Proof. For any $n \in \mathbb{N}$, let F_n be the primitive of f_n such that $F_n(a) = 0$. We assume, without loss of generality, that $F_n'(x) = f_n(x)$ for all $x \in [a,b] \setminus C$ and all $n \in \mathbb{N}$.

It suffices to prove that $\{F_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to some continuous function F which, moreover, satisfies $F'(x) = f(x)$ for all $x \in [a,b] \setminus C$.

The assumptions ensure that

$$\|f_m - f_n\| := \sup\{|f_m(x) - f_n(x)|\colon\ x \in [a,b] \setminus C\} \to 0 \quad \text{as } m,n \to \infty.$$

For any x and t in $[a,b]$ and any $m,n \in \mathbb{N}$, we use that our integral is monotone in the sense of (2.6), to get

$$|(F_m - F_n)(x) - (F_m - F_n)(t)| = \left| \int_t^x (f_m - f_n)(s)\,\mathrm{d}s \right| \leqslant \|f_m - f_n\| |x - t|. \tag{4.1}$$

If we take $t = a$ in (4.1), we have

$$|(F_m - F_n)(x)| \leqslant \|f_m - f_n\|(b - a) \quad \text{for all } x \in [a,b] \text{ and all } m,n \in \mathbb{N}.$$

It follows that $\{F_n\}_{n=1}^{\infty}$ converges uniformly to some continuous function $F\colon [a,b] \to \mathbb{R}$.

Next we prove that $F'(x) = f(x)$ for all $x \in [a,b] \setminus C$. Fix $x \in [a,b] \setminus C$ and $\varepsilon > 0$. First, choose $N \in \mathbb{N}$ such that

$$\|f_N - f\| < \frac{\varepsilon}{3} \tag{4.2}$$

and then choose $\delta > 0$ so that

$$\Big|\frac{F_N(t) - F_N(x)}{t - x} - f_N(x)\Big| < \frac{\varepsilon}{3} \quad \text{when } 0 < |t - x| < \delta,\ t \in [a,b]. \tag{4.3}$$

Fix any such t. Using (4.1), we have

$$\Big|\frac{F_m(t) - F_m(x)}{t - x} - \frac{F_N(t) - F_N(x)}{t - x}\Big| \leqslant \|f_m - f_N\| \quad \text{for any } m \in \mathbb{N}. \tag{4.4}$$

Letting $m \to \infty$ in (4.4) we get

$$\Big|\frac{F(t) - F(x)}{t - x} - \frac{F_N(t) - F_N(x)}{t - x}\Big| \leqslant \|f - f_N\| < \frac{\varepsilon}{3}. \tag{4.5}$$

Using the triangle inequality along with (4.5), (4.3) and (4.2), we deduce that, if $0 < |t - x| < \delta$, $t \in [a,b]$, then

$$\Big|\frac{F(t) - F(x)}{t - x} - f(x)\Big| < \varepsilon.$$

□

As a consequence of Theorem 4.1 we easily obtain the following information: first, another proof that continuous functions on compact intervals are integrable (though this time we need uniform continuity) and, second, if a continuous function on a compact interval is nonnegative, then its integral can be interpreted as the area under its graph and above the x axis. Unsurprisingly, we simply use approximating functions whose Newton integrals are lower or upper Darboux sums.

To prove our previous claims, let $f\colon\ [a,b] \to \mathbb{R}$ be an arbitrary continuous function and for any $n \in \mathbb{N}$ introduce points

$$x_k = a + k\frac{b-a}{n}, \quad k = 0, 1, 2, \dots, n-1,$$

and for each k define the (Darboux) numbers

$$m_k = \min_{[x_k, x_{k+1}]} f \quad \text{and} \quad M_k = \max_{[x_k, x_{k+1}]} f.$$

The functions

$$f_n(x) = \sum_{k=0}^{n-1} m_k \chi_{[x_k, x_{k+1})}(x), \quad x \in [a,b],$$

and

$$g_n(x) = \sum_{k=0}^{n-1} M_k \chi_{[x_k, x_{k+1})}(x), \quad x \in [a,b],$$

are integrable on $[a, b]$ (because they are piecewise constant). Moreover, by uniform continuity of f on $[a, b]$ it is easy to prove that $f_n \to f$ and $g_n \to f$ uniformly on $[a, b)$, so Theorem 4.1 ensures that f is integrable and

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b g_n(x)\,\mathrm{d}x.$$

Now for the areas. If, in particular, $f(x) \geqslant 0$ for all $x \in [a, b]$, then for every $n \in \mathbb{N}$,

$$\int_a^b f_n(x)\,\mathrm{d}x = \sum_{k=0}^{n-1} m_k \frac{b-a}{n}$$

is exactly the area below the graph of f_n and over the x axis, which should be less than or equal to the "area" (yet to be defined) below the graph of f because $f_n \leqslant f$ on $[a, b]$.

Analogously, the integral of each g_n is larger than the "area" below the graph of f.

Finally, the integral of f on $[a, b]$ is the unique real number between the integrals of the f_n's and the integrals of the g_n's, so it must be equal to the area that we want to define.

5. Appendix

For completeness, this appendix contains the proof of an elementary result used in final part of the proof of Theorem 3.1.

Let $a, b \in \mathbb{R}$, $a < b$. The right derivative of a function $f\colon [a, b] \to \mathbb{R}$ at a point $x \in [a, b)$ is defined as

$$f'_+(x) = \lim_{y\to x^+} \frac{f(y) - f(x)}{y - x},$$

provided that the limit exists.

The definition of $f'_-(x)$, the left derivative at $x \in (a, b]$, is analogous with limits from the left.

Proposition 5.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$. If $f'_+(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*

Proof. Assume, reasoning by contradiction, that for some $a \leqslant x < y \leqslant b$ we have $f(x) \geqslant f(y)$.

If f is constant on $[x, y]$, then $f'_+ = 0$ on $[x, y)$, a contradiction, so we assume that $f(x) \geqslant f(y)$ and f is not constant on $[x, y]$. In this case, we can find some $c \in [x, y]$ such that either $f(x) > f(c)$ or $f(c) > f(y)$. Summing up, we can assume without loss of generality that $f(x) > f(y)$. Since we may have $x = a$ and we have no information on $f'_+(a)$, it is necessary to replace x by a sufficiently close $z \in (x, y)$ for which $f(z) > f(y)$. Such a point z exists by continuity of f at x. Also thanks to continuity (at y), there exists some $\delta \in (0, y - z)$ such that

$$f(z) > f(t) \quad \text{for all } t \in (y - \delta, y]. \tag{5.1}$$

Put $w = \sup\{t \in [z, y]\colon\ f(t) = f(z)\}$. We have $f(w) = f(z)$ and, by (5.1), $w < y$. Moreover, by definition of w, we also have $f(w) > f(t)$ for all $t \in (w, y]$, hence

$$f'_+(w) = \lim_{t \to w^+} \frac{f(t) - f(w)}{t - w} \leqslant 0,$$

a contradiction. □

As a corollary, we get the following mean value theorem for continuous right derivatives.

Corollary 5.2. *If $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, $f'_+(x)$ exists for all $x \in (a, b)$, and f'_+ is continuous on (a, b), then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'_+(c)(b - a).$$

P r o o f. Define $F(x) = f(x) - f(a) - (f(b) - f(a))(x - a)/(b - a)$, $x \in [a, b]$. Observe that F is continuous on $[a, b]$ and there exists

$$F'_+(x) = f'_+(x) - \frac{f(b) - f(a)}{b - a} \quad \text{for all } x \in (a, b).$$

Since $F(a) = F(b) = 0$, F is not increasing on $[a, b]$, so F'_+ cannot be positive everywhere on (a, b). The assumptions imply that F'_+ is continuous on (a, b), hence there must be at least one $c \in (a, b)$ such that $F'_+(c) = 0$. □

Corollary 5.2 is false if we remove the continuity assumption on f'_+. Indeed, consider $f(x) = |x|$, $x \in [-1, 1]$.

Corollary 5.3. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$.*

If f'_+ exists and is continuous on (a, b), then f is differentiable on (a, b). Moreover, if $f'_+(x)$ tends to a limit $f'_+(b^-)$ as x tends to b^-, then there exists $f'_-(b) = f'_+(b^-)$.

P r o o f. We prove all the information in the statement at one stroke. Let $x_0 \in (a,b]$ be fixed; we have to show that $f'_-(x_0)$ exists and is equal to $f'_+(x_0^-)$ (the limit from the left of f'_+ at x_0). To do so, we fix $\varepsilon > 0$ and we take $\delta > 0$ such that

$$c \in (x_0 - \delta, x_0) \Rightarrow |f'_+(c) - f'_+(x_0^-)| < \varepsilon.$$

For any $x \in (x_0 - \delta, x_0)$ we use Corollary 5.2 to ensure that there exists $c_x \in (x, x_0)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} = f'_+(c_x),$$

hence

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'_+(x_0^-)\right| < \varepsilon \quad \text{for all } x \in (x_0 - \delta, x_0).$$

□

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