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BOUNDARY VALUE PROBLEMS WITH BOUNDED φ-LAPLACIAN AND NONLOCAL CONDITIONS OF INTEGRAL TYPE

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In memory of Professor Jaroslav Kurzweil

Abstract. We study the existence of solutions to nonlinear boundary value problems for second order quasilinear ordinary differential equations involving bounded φ-Laplacian, subject to integral boundary conditions formulated in terms of Riemann-Stieltjes integrals.

Keywords: boundary value problem; φ-Laplacian; functions of bounded variation; Riemann-Stieltjes integral; prescribed curvature

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1. Introduction

Differential equations of one-dimensional prescribed curvature problems of the form

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = f(t,u)$$

with f a continuous or Carathéodory function and Dirichlet boundary conditions on $[0,T]$ have been considered by Kusahara-Usame (see [8]) and Habets-Omari (see [6]) using the method of lower and upper solutions and time-maps, and by Bonheure-Habets-Obersnel-Omari (see [4]) using lower and upper solutions and variational methods. Existence and multiplicity results for more general quasilinear problems of the form

$$(\varphi(u'))' = f(t,u,u')$$

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with $\varphi\colon \mathbb{R} \to (-a, a)$ an increasing homeomorphism such that $\varphi(0) = 0$ and Dirichlet or Neumann boundary conditions have been given in [2], [3], [11] using topological degree techniques and lower and upper solutions. With the exception of [4], [6], who discuss the case of solutions singular at 0 and T, all those papers consider classical solutions such that u is of class C^1 on $[0, T]$ and $\phi \circ u'$ is of class C^1 or absolutely continuous on $[0, T]$.

In papers (see [2], [3], [10]) the boundary value problems are reduced to some nonlinear integral equations in the Banach space of continuous functions verifying the linear two-point boundary conditions, in order to apply some topological fixed point theorem in order to prove the existence of a solution and to show that such a fixed point is a solution of the boundary value problem in the sense mentioned above.

It is therefore, a natural question to consider more general boundary conditions involving linear continuous functionals on the space of continuous functions. In this paper, we consider some perturbations of Dirichlet boundary conditions by a linear functional expressed in terms of a Riemann-Stieltjes integral. Let us add that boundary value problems with nonlocal boundary conditions defined by Riemann-Stieltjes integrals were studied, for example, in [1], [5], [7], [13].

By refining some technical tools used in [3], we are able to extend its results to those more general nonlocal boundary conditions in such a way that the existence results are improved even for classical Dirichlet conditions and are sharp in a sense described in the paper.

2. Preliminaries

In what follows, $C[0, T]$ denotes the Banach space of all real-valued continuous functions on $[0, T]$ with the usual supremum norm $\|u\|_\infty$ and $C^1[0, T]$ denotes the Banach space of all continuously differentiable real-valued functions on $[0, T]$, with the norm $\|u\|_1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$.

By $\mathrm{BV}[0, T]$ we will denote the Banach space of all real-valued functions u defined on $[0, T]$ of bounded variation in the sense of Jordan (briefly: BV-functions), endowed with the norm

$$\|u\|_{\mathrm{BV}} = |u(0)| + \mathrm{Var}(u, [0, T]).$$

The symbol $\mathrm{Var}(u, [0, T])$ denotes the Jordan variation of the function $u\colon [0, T] \to \mathbb{R}$, that is,

$$\mathrm{Var}(u, [0, T]) = \sup_{\pi} \sum_{i=1}^{n} |u(t_i) - u(t_{i-1})|,$$

where the supremum is taken over all finite partitions $\pi\colon\ 0 = t_0 < t_1 < \ldots < t_n = T$ of the interval $[0, T]$. Let us recall that BV-functions are bounded and

$$\|u\|_\infty \leqslant \|u\|_{\mathrm{BV}} \quad \text{for every} \quad u \in \mathrm{BV}[0, T].$$

Recall also that if $v \in \mathrm{BV}[0, T]$, then any function $u \in C[0, T]$ is integrable in the Riemann-Stieltjes sense and the following inequality holds:

$$\left| \int_0^T u(t)\, \mathrm{d}v(t) \right| \leqslant \|u\|_\infty \operatorname{Var}(v, [0, T]).$$

(The interested reader can find more information about functions of bounded variation as well as Riemann-Stieltjes integral, e.g. in [12].)

We finally need the following technical result, which sharpens Lemma 3 in [3]. For each $h \in C[0, T]$, we denote by $H \in C^1[0, T]$ its indefinite integral

$$H(t) := \int_0^t h(s)\, \mathrm{d}s \quad (t \in [0, T])$$

and by $\operatorname{osc}_{[0,T]} H$ we denote the oscillation of H on $[0, T]$, that is, $\operatorname{osc}_{[0,T]} H = \max\limits_{t\in[0,T]} H(t) - \min\limits_{t\in[0,T]} H(t)$. Since now on, $(-a, a)$, $a > 0$ will denote a bounded interval.

Theorem 2.1. *Let $\varphi\colon \mathbb{R} \to (-a, a)$, be an increasing homeomorphism such that $\varphi(0) = 0$. For each $h \in C[0, T]$ such that $\operatorname{osc}_{[0,T]} H < a$, there exists a unique $Q_\varphi[h]$ such that $H(s) - Q_\varphi[h] \in (-a, a)$ for all $s \in [0, T]$ and*

$$\int_0^T \varphi^{-1}(H(s) - Q_\varphi[h])\, \mathrm{d}s = 0.$$

Furthermore, $Q_\varphi\colon \{h \in C[0, T]\colon \operatorname{osc}_{[0,T]} H < a\} \to \mathbb{R}$ is continuous, and if $\varphi\colon \mathbb{R} \to (-a, a)$ is a diffeomorphism, then Q_φ is of class C^1.

P r o o f. Let $\tau_m \in [0, T]$ and $\tau_M \in [0, T]$ be such that

$$H(\tau_m) = \min_{s\in[0,T]} H(s), \quad H(\tau_M) = \max_{s\in[0,T]} H(s).$$

Then for all $s \in [0, T]$,

$$H(s) - H(\tau_M) \leqslant 0 \leqslant H(s) - H(\tau_m),$$

and hence, using the increasing character of φ^{-1} and the fact that $\varphi^{-1}(0) = 0$,

$$\int_0^T \varphi^{-1}(H(s) - H(\tau_M))\, \mathrm{d}s \leqslant 0 \leqslant \int_0^T \varphi^{-1}(H(s) - H(\tau_m))\, \mathrm{d}s.$$

Consequently, by the continuity of the integral depending on a parameter, there exists $\tau^* \in [0,T]$ such that

$$\psi(\tau^*) := \int_0^T \varphi^{-1}(H(s) - H(\tau^*))\,\mathrm{d}s = 0.$$

Now, if $\psi(\tau^*) = \psi(\tau^{**}) = 0$, then

$$\int_0^T [\varphi^{-1}(H(s) - H(\tau^*)) - \varphi^{-1}(H(s) - H(\tau^{**}))]\,\mathrm{d}s = 0$$

and there exists $s_0 \in [0,T]$ such that

$$\varphi^{-1}(H(s_0) - H(\tau^*)) - \varphi^{-1}(H(s_0) - H(\tau^{**})) = 0,$$

which implies that

$$H(\tau^*) = H(\tau^{**}).$$

Consequently, for each $h \in C[0,T]$ such that $\operatorname{osc}_{[0,T]} H < a$, there exists a unique $Q_\varphi[h] := H(\tau^*)$ such that

$$\int_0^T \varphi^{-1}(H(s) - Q_\varphi[h])\,\mathrm{d}s = 0.$$

To show that Q_φ is continuous, let $(h_k)_{k\in\mathbb{N}}$ be a sequence in $C[0,T]$ such that $\operatorname{osc}_{[0,T]} H_k < a$ for all $k \in \mathbb{N}$, $h_k \to h^*$, $h^* \in C[0,T]$, $\operatorname{osc}_{[0,T]} H^* < a$, and

$$\int_0^T \varphi^{-1}(H_k(s) - Q_\varphi[h_k])\,\mathrm{d}s = 0 \quad (k \in \mathbb{N}).$$

Then there is a sequence $(\tau_k)_{k\in\mathbb{N}}$ such that

$$\int_0^T \varphi^{-1}\biggl(\int_{\tau_k}^s h_k(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}s = 0 \quad (k \in \mathbb{N}).$$

Going if necessary to a subsequence, we can assume that $\tau_k \to \tau^* \in [0,T]$, and hence, by the dominated convergence theorem

$$0 = \int_0^T \varphi^{-1}\biggl(\int_{\tau^*}^s h^*(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}s = \int_0^T \varphi^{-1}(H^*(s) - H^*(\tau^*))\,\mathrm{d}s,$$

so that $H^*(\tau^*) = Q_\varphi[h^*]$, a result independent of the subsequence of $(\tau_k)_{k\in\mathbb{N}}$. Hence, $Q_\varphi[h^*] = \lim_{k\to\infty} Q_\varphi[h_k]$. Finally, if φ is a diffeomorphism, the implicit function theorem applied to the implicit equation

$$\Psi(h,c) := \int_0^T \varphi^{-1}(H(s) - c)\,\mathrm{d}s = 0$$

implies that Q_φ is of class C^1. □

Remark 2.2. It follows from the proof of Theorem 2.1 that $Q_\varphi[h]$ can always be written in the form $\int_0^\tau h(\xi)\,\mathrm{d}\xi$ for some $\tau \in [0,T]$.

3. BVP with nonlocal boundary conditions involving Riemann-Stieltjes integral

Let $\varphi\colon \mathbb{R} \to (-a,a)$ be an increasing homeomorphism, $f\colon [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous and $A \in \mathrm{BV}[0,T]$. Consider the second-order differential equation

$$(\varphi(u'))' = f(t,u,u') \tag{3.1}$$

with the nonlocal boundary conditions

$$u(0) = u(T) = \int_0^T u(t)\,\mathrm{d}A(t). \tag{3.2}$$

By a solution of (3.1)–(3.2) we mean a function $u \in C^1[0,T]$ such that $\varphi \circ u' \in C^1[0,T]$ and equations (3.1) and (3.2) are satisfied. Clearly, if $\varphi\colon \mathbb{R} \to (-a,a)$ is a diffeomorphism, then such a solution is of class C^2 in $[0,T]$.

Lemma 3.1. *Assume that f satisfies the condition*

$$\left|\int_\tau^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\right| < a \quad \text{for all } u \in C^1[0,T],\ t\in[0,T],\ \tau\in[0,T]. \tag{3.3}$$

Then the BVP (3.1)–(3.2) *is equivalent to*

$$u(t) = \int_0^T u(t)\,\mathrm{d}A(t) + \int_0^t \varphi^{-1}\biggl(\int_0^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - Q_\varphi[f(\cdot,u(\cdot),u'(\cdot))]\biggr)\,\mathrm{d}s, \tag{3.4}$$

that is, every solution $u\colon [0,T]\to\mathbb{R}$ of (3.1)–(3.2) *is a solution in $C^1[0,T]$ to* (3.4) *and vice-versa.*

P r o o f. First, let us assume that $u\colon [0,T] \to \mathbb{R}$ is a solution to (3.1)–(3.2). By (3.2) and Rolle's Theorem, there exists $\tau \in (0,T)$ such that $u'(\tau) = 0$. Integrating both sides of (3.1) on the interval $[\tau,t]$ we get

$$\varphi(u'(t)) = \int_\tau^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi.$$

Since condition (3.3) holds, then

$$u'(t) = \varphi^{-1}\biggl(\int_\tau^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\biggr).$$

Integrating both sides of the above equation on $[0,t]$ and using the first boundary condition, we obtain for all $t \in [0,T]$,

(3.5)

$$\begin{aligned} u(t) &= u(0) + \int_0^t \varphi^{-1}\Big(\int_\tau^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Big)\,\mathrm{d}s \\ &= \int_0^T u(t)\,\mathrm{d}A(t) + \int_0^t \varphi^{-1}\Big(\int_0^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - \int_0^\tau f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Big)\,\mathrm{d}s. \end{aligned}$$

Now, introducing the second boundary condition in (3.5), we obtain

$$0 = \int_0^T \varphi^{-1}\Big(\int_0^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - \int_0^\tau f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Big)\,\mathrm{d}s,$$

so that, by Theorem 2.1,

$$\int_0^\tau f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi = Q_\varphi[f(\cdot,u(\cdot),u'(\cdot))]$$

and u is a solution of (3.4). Now, let us assume that $u \in C^1[0,T]$ and solves (3.4). Then

$$u'(t) = \varphi^{-1}\Big(\int_0^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - Q_\varphi[f(\cdot,u(\cdot),u'(\cdot))]\Big), \quad t \in [0,T],$$

so

$$\varphi(u'(t)) = \int_0^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - Q_\varphi[f(\cdot,u(\cdot),u'(\cdot))],$$

and hence,

$$(\varphi(u'))'(t) = f(t,u(t),u'(t)) \quad (t \in [0,T]).$$

Further, by (3.4), we have

$$u(0) = \int_0^T u(t)\,\mathrm{d}A(t)$$

and by the definition of Q_φ,

$$\begin{aligned} u(T) &= \int_0^T u(t)\,\mathrm{d}A(t) + \int_0^T \varphi^{-1}\Big(\int_0^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi - Q_\varphi[f(\cdot,u(\cdot),u'(\cdot))]\Big)\,\mathrm{d}s \\ &= \int_0^T u(t)\,\mathrm{d}A(t). \end{aligned}$$

Thus, u satisfies (3.1)–(3.2). □

Theorem 3.2. *Let $\varphi\colon \mathbb{R}\to(-a,a)$ be an increasing homeomorphism, $f\colon [0,T]\times\mathbb{R}^2\to\mathbb{R}$ be continuous, $A\in \mathrm{BV}[0,T]$ and assume that the following conditions hold:*

(i) *there exists $b\in(0,a)$ such that*

$$\Bigl|\int_\tau^t f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Bigr|\leqslant b \quad \text{for all } t\in[0,T],\ \tau\in[0,T],\ u\in C^1[0,T], \tag{3.6}$$

(ii) $\int_0^T \mathrm{d}A(s)\neq 1$.

Then problem (3.1)–(3.2) *has at least one solution.*

P r o o f. For $\lambda\in[0,1]$, consider the family of BVP

$$(\varphi(u'))'=\lambda f(t,u,u'),\quad u(0)=u(T)=\int_0^T u(s)\,\mathrm{d}A(s), \tag{3.7}$$

which reduces to (3.1)–(3.2) for $\lambda=1$. The nonlinear operator $\mathcal{M}$ given by

$$\begin{aligned}\mathcal{M}[\lambda,u](t)=&\int_0^T u(s)\,\mathrm{d}A(s)\\ &+\int_0^t\varphi^{-1}\Bigl(\int_0^s\lambda f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi-Q_\varphi[\lambda f(\cdot,u(\cdot),u'(\cdot))]\Bigr)\,\mathrm{d}s\end{aligned} \tag{3.8}$$

is well defined on $[0,1]\times C^1[0,T]$, and from Lemma 3.1, its fixed points are the solutions of BVP (3.7). The fact that $\mathcal{M}$ is completely continuous on $C^1[0,T]$ directly follows from Ascoli-Arzela's theorem. Furthermore, for all $u\in C^1[0,T]$ and $\lambda\in[0,1]$ we have

$$\begin{aligned}&\Bigl|\varphi^{-1}\Bigl(\int_0^t\lambda f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi-Q_\varphi(\lambda f(\cdot,u(\cdot),u'(\cdot)))\Bigr)\Bigr|\\ &\quad=\Bigl|\varphi^{-1}\Bigl(\int_0^t\lambda f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi-\int_0^{\tau_\lambda}\lambda f(\xi,u(\xi),u'(\xi)\,\mathrm{d}\xi\Bigr)\Bigr|\\ &\quad=\Bigl|\varphi^{-1}\Bigl(\int_{\tau_\lambda}^t\lambda f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Bigr)\Bigr|<\max\{-\phi^{-1}(-b),\phi^{-1}(b)\}:=M.\end{aligned} \tag{3.9}$$

Let $\lambda\in[0,1]$ and u be a possible fixed point of $\mathcal{M}[\lambda,\cdot]$. Then we deduce from (3.9) that for all $t\in[0,T]$,

$$|u'(t)|=\Bigl|\varphi^{-1}\Bigl(\int_0^t\lambda f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi-Q_\varphi[\lambda f(\cdot,u(\cdot),u'(\cdot))]\Bigr)\Bigr|<M.$$

Now the Fredholm linear equation in $C[0,T]$

$$u(t)=\int_0^T u(s)\,\mathrm{d}A(s) \tag{3.10}$$

has of course only constant solutions $u(t) = c$ with c such that $c = c\int_0^T \mathrm{d}A(s)$, and hence, only the trivial solution $c = 0$ by assumption (ii). Consequently, the linear operator

$$\mathcal{L}\colon\ C[0,T] \to C[0,T], \quad u \mapsto u - \int_0^T u(s)\,\mathrm{d}A(s)$$

has a continuous inverse and equation $u = \mathcal{M}[\lambda, u]$ can be written equivalently as

$$u = \mathcal{L}^{-1}\Big[\int_0^{\cdot} \varphi^{-1}\Big(\int_0^s \lambda f(\xi, u(\xi), u'(\xi))\,\mathrm{d}\xi - Q_\varphi[\lambda f(\cdot, u(\cdot), u'(\cdot))]\Big)\,\mathrm{d}s\Big],$$

so that by using estimate (3.9),

$$\|u\|_\infty < \|\mathcal{L}^{-1}\|M.$$

If we define the open bounded set $\Omega \subset C^1[0,T]$ by

$$\Omega = \{u \in C^1[0,T]\colon\ \|u\|_\infty < \|\mathcal{L}^{-1}\|M, \quad \|u'\|_\infty < M\},$$

we deduce from the homotopy invariance and the reduction formula of the Leray-Schauder degree (see [9], [10]) that

$$\begin{aligned} d_{\mathrm{LS}}[I - \mathcal{M}(1,\cdot), \Omega, 0] &= d_{\mathrm{LS}}[I - \mathcal{M}(0,\cdot), \Omega, 0] = d_{\mathrm{B}}[(I - \mathcal{M}(0,\cdot))|_{\mathbb{R}}, \Omega \cap \mathbb{R}, 0] \\ &= \operatorname{sgn}\Big(1 - \int_0^T \mathrm{d}A(s)\Big) = \pm 1, \end{aligned}$$

where d_{LS} and d_{B}, respectively, denote the Leray-Schauder and Brouwer degree, see [10]. Then the existence property of the Leray-Schauder degree implies that $\mathcal{M}(1,\cdot)$ has a fixed point in Ω, which is a solution of (3.1)–(3.2), by Lemma 3.1. □

Remark 3.3. If f depends only upon t, namely for the problem

$$(\varphi(u'))' = f(t), \quad u(0) = u(T) = \int_0^T u(s)\,\mathrm{d}A(s), \tag{3.11}$$

assumption (i) of Theorem 3.2 becomes

$$\Big|\int_\tau^t f(\xi)\,\mathrm{d}\xi\Big| \leqslant b \quad \text{for some } b \in (0,a) \text{ and all } t \in [0,T],\ \tau \in [0,T]$$

and is clearly equivalent to condition

$$\Big|\int_\tau^t f(\xi)\,\mathrm{d}\xi\Big| < a \quad \text{for all } t \in [0,T],\ \tau \in [0,T].$$

Now, if the BVP (3.11) has a solution u, then there exists $\tau \in [0,T]$ such that

$$\left|\int_\tau^t f(\xi)\,\mathrm{d}\xi\right| = |\varphi(u'(t))| < a,$$

so that a necessary condition for the solvability of BVP (3.11) is our sufficient condition (i) holds for all $t \in [0,T]$ and *some* $\tau \in [0,T]$. This shows the sharpness of assumption (i) in Theorem 3.2.

Remark 3.4. Let us consider the BVP

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \varepsilon, \quad u(0) = u\Big(\frac{1}{2}\Big) = u(1), \tag{3.12}$$

a special case of (3.1)–(3.2) with $\varphi(v) = v/\sqrt{1+v^2}$, $f(t,u,v) = \varepsilon > 0$, $T = 1$, $A = \chi_{(1/2,1]}$, where χ_E denotes the characteristic function of a set E, so that

$$\int_0^1 u(s)\,\mathrm{d}A(s) = u\Big(\frac{1}{2}\Big) \quad \text{and} \quad \int_0^1 \mathrm{d}A(s) = \mathrm{Var}(A,[0,1]) = 1,$$

so that assumption (ii) of Theorem 3.2 is not satisfied. If (3.12) has a solution u, then $u'/\sqrt{1+u'^2}$, and hence u' is increasing on $[0,1]$ and u is strictly convex on $[0,1]$. Consequently,

$$u\Big(\frac{1}{2}\Big) = u\Big(\frac{0}{2}+\frac{1}{2}\Big) < \frac{u(0)+u(1)}{2},$$

a contradiction with the boundary conditions. Hence, the conclusion of Theorem 3.2 may be violated for some f when assumption (ii) does not hold.

Corollary 3.5. *If $\varphi\colon\ \mathbb{R} \to (-a,a)$ is an increasing homeomorphism, $f\colon\ [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and such that*

$$|f(t,u,v)| \leqslant c < \frac{a}{T} \quad \text{for all } (t,u,v) \in [0,T]\times\mathbb{R}^2 \tag{3.13}$$

holds and if $A \in BV[0,T]$ satisfies $\int_0^T \mathrm{d}A(s) \neq 1$, then problem (3.1)–(3.2) has at least one solution.

Remark 3.6. In the special case of the Dirichlet problem (A constant), assumption (ii) is trivially satisfied and assumption (i) of Corollary 3.5 improves the assumption of Theorem 5 in [3] by a factor two.

Example 3.7. If follows from Corollary 3.5 that the Dirichlet problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha(\sin u + \cos t), \quad u(0) = 0 = u\Big(\frac{\pi}{2}\Big)$$

has at least one solution if $|\alpha| < \pi^{-1}$.

4. Other boundary conditions

The preceding arguments can be used to study other BVP like nonlocal perturbations of mixed boundary conditions

$$(\varphi(u'))' = f(t,u,u'), \quad u(0) = \int_0^T u(s)\,\mathrm{d}A(s), \quad u'(T) = 0, \tag{4.1}$$

where $\varphi\colon \mathbb{R} \to (-a,a)$ is an increasing homeomorphism, $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $A \in \mathrm{BV}[0,T]$.

By mimicking the proof of Lemma 3.1, we obtain the corresponding version for BVP (4.1).

Lemma 4.1. *Assume that f satisfies the condition*

$$\Bigl|\int_t^T f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Bigr| < a \quad \text{for all } u \in C^1[0,T] \text{ and } t \in [0,T]. \tag{4.2}$$

Then BVP (4.1) is equivalent to

$$u(t) = \int_0^T u(t)\,\mathrm{d}A(t) + \int_0^t \varphi^{-1}\Bigl(\int_T^s f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Bigr)\,\mathrm{d}s, \tag{4.3}$$

that is, every solution $u\colon [0,T] \to \mathbb{R}$ of (4.1) is a solution in $C^1[0,T]$ to (4.3) and vice-versa.

Notice that because of the second boundary condition there is no need to use Theorem 2.1 and condition (4.2) is slightly simpler than condition (3.3).

An easy adaptation of the proof of Theorem 3.2 provides the following existence conditions for BVP (4.1).

Theorem 4.2. *Let $\varphi\colon \mathbb{R} \to (-a,a)$ be an increasing homeomorphism, $f\colon [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, $A \in \mathrm{BV}[0,T]$ and assume that the following conditions hold:*

(i) *there exists $b \in (0,a)$ such that*

$$\Bigl|\int_t^T f(\xi,u(\xi),u'(\xi))\,\mathrm{d}\xi\Bigr| \leqslant b \quad \text{for all } t \in [0,T],\ u \in C^1[0,T],$$

(ii) $\int_0^T \mathrm{d}A(s) \neq 1$.

Then problem (4.1) has at least one solution.

Corollary 4.3. *If* $\varphi\colon \mathbb{R} \to (-a, a)$ *is an increasing homeomorphism,* $f\colon [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and such that condition* (3.13) *holds and if* $A \in \mathrm{BV}[0, T]$ *satisfies* $\int_0^T \mathrm{d}A(s) \neq 1$*, then the problem* (4.1) *has at least one solution.*

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