A MENON-TYPE IDENTITY USING KLEE'S FUNCTION

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Received August 26, 2020. Published online April 16, 2021.

Abstract. Menon's identity is a classical identity involving gcd sums and the Euler totient function φ. A natural generalization of φ is the Klee's function Φ_s. We derive a Menon-type identity using Klee's function and a generalization of the gcd function. This identity generalizes an identity given by Y. Li and D. Kim (2017).

Keywords: Euler totient function; generalized gcd; Jordan totient function; Klee's function

MSC 2020: 11A07, 11A25, 20D60, 20D99

1. Introduction

The Euler totient function φ appears in many interesting identities in number theory. Probably because of its applications in various branches of number theory, it has been generalized in many ways. The Jordan function $J_s(n)$, the von Sterneck's function $H_s(n)$, the Cohen's function φ_s (see [2]) and the Klee's function Φ_s (see [8]) are some important extensions of φ (see the definitions in the next section). All these functions share several common properties. For example, Euler totient function φ holds a relation with the Möbius function. Similar relations are satisfied by all these generalizations. All these generalizations have a product formulae in terms of the prime factorization of their arguments. Hence, all these are multiplicative and behave similarly to φ on prime powers.

The first author thanks the University Grants Commission of India for providing financial support for carrying out research work through their Junior Research Fellowship (JRF) scheme. The second author thanks the Kerala State Council for Science, Technology and Environment, Thiruvananthapuram, Kerala, India for providing financial support for carrying out research work.

DOI: 10.21136/CMJ.2021.0370-20

Note that Cohen has proved in [2] the equality of the functions J_s, H_s and φ_s, although they are all defined differently. Klee's function Φ_s and Cohen's φ_s are connected by the relation $\varphi_s(n) = \Phi_s(n^s)$. Thus, Φ_s seems to be a natural generalization of φ (as for $s = 1$, the former turns out to be equal to the latter).

The classical Menon's identity which originally appeared in [7] is a gcd sum turning out to be equivalent to a product of the Euler function and the number of divisors function τ. If (m, n) denotes the gcd of m and n, the identity is precisely the following:

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} (m-1, n) = \varphi(n)\tau(n). \tag{1.1}$$

It has been generalized and extended by many authors. Many of the identities were derived using elementary number theory techniques. For example, in a recent paper (see [19]), Zhao and Kao suggested a generalization involving Dirichlet characters $\bmod n$ using elementary number theoretic methods. Their identity is

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} (m-1, n)\chi(m) = \varphi(n)\tau\Big(\frac{n}{d}\Big),$$

where χ is a Dirichlet character $\bmod n$ with conductor d. When one takes χ as the principal character $\bmod n$, this identity turns to be equal to the Menon's identity. After this, a similar type of identity in terms of even functions $\bmod n$ was given by Tóth, see [17]. An arithmetical function f is n-even (or even $\bmod n$) if $f(r) = f((r, n))$. Tóth also used elementary number theory techniques and properties of arithmetical functions to prove his identity. Rao in [11] gave a generalization of the form

$$\sum_{m_i \in U_k(n)} (m_1 - s_1, m_2 - s_2, \ldots, m_k - s_k, n)^k = J_k(n)\tau(n),$$

where $U_k(n)$ is the unit goup modulo n and $s_i \in \mathbb{Z}$. He used Cauchy composition and finite Fourier representations to establish this result.

A different approach was used by Sury in [14]. He used the method of group actions to derive the following identity:

$$\sum_{\substack{1 \leqslant m_1, m_2, \ldots, m_k \leqslant n \\ (m_1,n)=1}} (m_1 - 1, m_2, \ldots, m_k, n) = \varphi(n)\sigma_{k-1}(n), \tag{1.2}$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Miguel in [10] extended this identity of Sury from $\mathbb{Z}$ to any residually finite Dedekind domains. A further extension of Miguel's result was given by Li and Kim in [9], Theorem 1.1. For the case of $\mathbb{Z}$, their result reads as follows [9], Corollary 1.3:

$$\begin{aligned}(1.3)\qquad &\sum_{\substack{a_1,a_2,\ldots,a_s\in U(\mathbb{Z}_n)\\ b_1,b_2,\ldots,b_r\in\mathbb{Z}_n}} (a_1-1,\ldots,a_s-1,b_1,\ldots,b_r,n)\\ &\qquad = \varphi(n)\prod_{i=1}^{m}(\varphi(p_i^{k_i})^{s-1}p_i^{k_i r} - p_i^{k_i(s+r-1)} + \sigma_{s+r-1}(p_i^{k_i})),\end{aligned}$$

where $n = p_1^{k_1}p_2^{k_2}\ldots p_m^{k_m}$ is the prime factorization of n. Note that this identity generalizes the classical Menon's identity and Menon-Sury identity.

Various other generalizations of Menon's identity were provided by many authors, see for example [3], [5], [12], [16] and the more recent papers [4] and [18].

A natural question, arising if one considers a generalization of the usual gcd function (which we define in the next section) in the place of the gcd function appearing in Menon's identity (1.1), is what could be the possible change that can happen to this identity as well as the other generalizations of it. We propose a very natural generalization of the Li-Kim identity (1.3) involving generalized gcd function and Klee's function in this paper (which in turn generalizes Menon's identity as well). We prove it using elementary number theory techniques.

2. Notations and basic results

Most of the notations, functions, and identities we use in this paper are standard and their definitions can be found in [1]. For a finite set A, by $\#A$ we mean the number of elements in A.

The Jordan totient function $J_s(n)$ defined for positive integers s and n gives the number of ordered sets of s elements from a complete residue system $(\bmod\, n)$ such that the greatest common divisor of each set is prime to n, see [6], pages 95–97. Von Sterneck's function H_s is defined as

$$H_s(n) = \sum_{n=[d_1,d_2,\ldots,d_s]} \varphi(d_1)\varphi(d_2)\ldots\varphi(d_s),$$

where the summation ranges over all ordered sets of s positive integers $d_1, d_2, \ldots, d_s$ with their least common multiple equal to n. Note that $[a, b]$ denotes the lcm of integers a, b.

For $m, n \in \mathbb{N}$, (m, n) will denote the gcd of m and n. Generalizing this notion, for positive integer s, integers a, b, not both zero, the largest l^s (where $l \in \mathbb{N}$) dividing both a and b will be denoted by $(a, b)_s$. Following Cohen in [2] we call this function on $\mathbb{N} \times \mathbb{N}$ as the generalized gcd function. When $s = 1$, this will be equal to the usual gcd function. Like the gcd function, $(a, b)_s = (b, a)_s$, $a \in \mathbb{N}$ is said to be s-power free or s-free if no l^s, where $l \in \mathbb{N}$, divides a.

Cohen's function φ_s is defined as follows. If $(a, b)_s = 1$, a, b are said to be relatively s-prime. The subset N of a complete residue system $M \pmod{n^s}$ consisting of all elements of M that are relatively s-prime to n^s is called an *s-reduced residue system* $\pmod n$. The number of elements of an s-reduced residue system is denoted by $\varphi_s(n)$.

The functions $J_s(n)$ and $\varphi_s(n)$ are the same [2], Theorem 5, although their definitions look different.

Using the above generalization of the gcd function, for positive integers s and n Klee's function $\Phi_s(n)$ is defined to give the cardinality of the set $\{m \in \mathbb{N} \colon 1 \leqslant m \leqslant n, (m, n)_s = 1\}$.

Note that $\Phi_1 = \varphi$, the usual Euler totient function on $\mathbb{N}$. Some interesting properties of Φ_s are the following.

(1) For $n, s \in \mathbb{N}$, $\Phi_s(n) = \sum\limits_{d^s \mid n} \mu(d) n / d^s$.

(2) For $n, s \in \mathbb{N}$, $\Phi_s(n) = n \prod\limits_{\substack{p^s \mid n \\ p \,\text{prime}}} (1 - 1/p^s)$, where by convention, empty product is taken to be equal to 1.

(3) $\Phi_s(p^a) = \begin{cases} p^a - p^{a-s} & \text{if } a \geqslant s, \\ p^a & \text{otherwise,} \end{cases}$ where p is prime and $a \in \mathbb{N}$.

(4) $\Phi_s(n)$ is multiplicative in n.

(5) $\Phi_s(n)$ is not completely multiplicative in n.

(6) If a divides b and $(a, b/a) = 1$, then $\Phi_s(a)$ divides $\Phi_s(b)$.

(7) For a prime p, $\Phi_s(p) = p$. So $\Phi_s(n)$ need not be even, whereas $\varphi(n)$ is even for $n > 2$.

(8) If 2^{s+1} divides n or $2^{s-1} \mid n$ and $2^s \nmid n$, then $\Phi_s(n)$ is even.

(9) If p is an odd prime such that p^s divides n, then $\Phi_s(n)$ is even.

(10) If $n = 2^s a$, where a is odd and a is s-free, then Φ_s is odd.

Many of the above properties are listed in [8]. The rest can be verified easily via elementary techniques.

By $\tau_s(n)$, where $s, n \in \mathbb{N}$, we mean the number of l^s with $l \in \mathbb{N}$ dividing n. The function $\tau_s(n)$ is multiplicative in n, because for $(m, n) = 1$, $\tau_s(mn) = \sum\limits_{d^s \mid mn} 1 = \sum\limits_{d_1^s \mid m} 1 \sum\limits_{d_2^s \mid n} 1 = \tau_s(m)\tau_s(n)$. But $\tau_s(n)$ is not completely multiplicative as for example $m = p_1^s p_2$ and $n = p_2^{s-1} p_3^s$ gives $\tau_s(mn) \neq \tau_s(m)\tau_s(n)$. The usual sum of divisors function can be generalized as follows: for $k, s, n \in \mathbb{N}$ define $\sigma_{k,s}(n)$ to be the kth power sum of the sth power divisors of n. That is, $\sigma_{k,s}(n) = \sum\limits_{d^s \mid n} (d^s)^k$. Note that $\sigma_{k,s}(n) \neq \sigma_{ks}(n)$.

The principle of cross-classification lies in counting the number of elements in certain sets. Since we use it in our proofs, we state it below.

Theorem 2.1 ([1], Theorem 5.31). *If $A_1, A_2, \ldots, A_n$ are given subsets of a finite set A, then*

$$\#\Big(A - \bigcup_{i=1}^{n} A_i\Big) = \#A - \sum_{1 \leqslant i \leqslant n} \#A_i + \sum_{1 \leqslant i < j \leqslant n} \#(A_i \cap A_j) - \sum_{1 \leqslant i < j < k \leqslant n} \#(A_i \cap A_j \cap A_k) + \ldots + (-1)^n \#(A_1 \cap A_2 \cap \ldots \cap A_n).$$

3. Main results and proofs

We state below the main results we prove in this paper and provide the proof after that.

As a consequence of the principle of cross-classification, we prove the following.

Theorem 3.1. *Let $n, d, s, r \in \mathbb{N}$, $d^s \mid n$. Let $(r, d^s)_s = 1$. Number of elements in $A = \{r + td^s \colon t = 1, 2, \ldots, n/d^s\}$ such that $(r + td^s, n)_s = 1$ is $\Phi_s(n)/\Phi_s(d^s)$.*

Theorem 3.2 (Generalization of Li-Kim identity (1.3)). *Let $m_1, m_2, \ldots, m_k$, $b_1, b_2, \ldots, b_r, n, s \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}$ such that $(a_i, n^s)_s = 1$, $i = 1, 2, \ldots, k$. Then*

$$\sum_{\substack{1 \leqslant m_1, m_2, \ldots, m_k \leqslant n^s \\ (m_1, n^s)_s = 1 \\ (m_2, n^s)_s = 1 \\ \ldots \\ (m_k, n^s)_s = 1 \\ 1 \leqslant b_1, b_2, \ldots, b_r \leqslant n^s}} (m_1 - a_1, m_2 - a_2, \ldots, m_k - a_k, b_1, b_2, \ldots, b_r, n^s)_s = \Phi_s(n^s)^k \sum_{d^s \mid n^s} \frac{(d^s)^r}{\Phi_s(n^s/d^s)^{k-1}}. \tag{3.1}$$

Note that the Menon-Sury identity (1.2) is a special case of the above generalization with $k = 1$, $a_1 = 1$ and $s = 1$.

We proceed to prove our results. First we prove Theorem 3.1, which is essential in the proof of our generalization.

P r o o f of Theorem 3.1. This result is a generalization of Theorem 5.32 appearing in [1]. We use the same techniques used there to justify our claim.

We have to find the number of elements $r+td^s$ such that $(n, r+td^s)_s = 1$. Hence, we need to remove elements from A that have $(r+td^s, n)_s > 1$. If for an element $r+td^s$ of A, $p^s \mid n$ and $p^s \mid r+td^s$, then since $(r,n)_s = 1$, $p^s \nmid d^s$. Therefore, the number we require is the number of elements in A with $p^s \mid n$ and $p^s \nmid d^s$ for some prime p. Let these primes be $p_1, p_2, \ldots, p_m$. Write $l = p_1^s p_2^s \ldots p_m^s$. Let $A_i = \{x\colon\ x \in A$ and $p_i^s \mid x\}$, $i = 1, 2, \ldots, m$. If $x \in A_i$ and $x = r + td^s$, then $r + td^s \equiv 0 \pmod{p_i^s}$. This means that $td^s \equiv -r \pmod{p_i^s}$. Since $p_i^s \nmid d^s$ (which is if and only if $p_i \nmid d$), there is a unique $t \bmod p_i^s$ satisfying this congruence equation. Therefore, there exists exactly one t in each of the intervals $[1, p_i^s]$, $[p_i^s+1, 2p_i^s], \ldots, [(q-1)p_i^s+1, qp_i^s]$, where $qp_i^s = n/d^s$. Therefore

$$\#(A_i) = q = \frac{n/d^s}{p_i^s}.$$

Similarly,

$$\#(A_i \cap A_j) = \frac{n/d^s}{p_i^s p_j^s}, \ldots, \#(A_1 \cap A_2 \cap \ldots \cap A_m) = \frac{n/d^s}{p_1^s p_2^s \ldots p_m^s}.$$

Hence, by cross classification principle, the number of elements we seek is equal to

$$\begin{aligned}
\#\Big(A - \bigcup_{i=1}^{m} A_i\Big) &= \#(A) - \sum_{i=1}^{m} \#(A_i) + \sum_{1\leqslant i<j\leqslant m} \#(A_i \cap A_j) \\
&\quad - \ldots + (-1)^m \#(A_1 \cap A_2 \cap \ldots \cap A_m) \\
&= \frac{n}{d^s} - \sum \frac{n/d^s}{p_i^s} + \sum \frac{n/d^s}{p_i^s p_j^s} + \ldots + (-1)^m \frac{n/d^s}{p_1^s p_2^s \ldots p_m^s} \\
&= \frac{n}{d^s}\Big(1 - \sum \frac{1}{p_i^s} + \sum \frac{1}{p_i^s p_j^s} + \ldots + \frac{(-1)^m}{p_1^s p_2^s \ldots p_m^s}\Big) \\
&= \frac{n}{d^s}\Big(1 - \frac{1}{p_1^s}\Big)\Big(1 - \frac{1}{p_2^s}\Big) \ldots \Big(1 - \frac{1}{p_m^s}\Big) \\
&= \frac{n}{d^s} \prod_{p^s \mid l}\Big(1 - \frac{1}{p^s}\Big) = \frac{n}{d^s} \frac{\prod_{p^s\mid n}(1 - 1/p^s)}{\prod_{p^s \mid d^s}(1 - 1/p^s)} = \frac{\Phi_s(n)}{\Phi_s(d^s)}.
\end{aligned}$$

$\square$

Next we prove the generalization we proposed. Here we use elementary number theoretic techniques in the proof. Li and Kim used direct computations involving Dedekind domains to derive their identity.

Proof of Theorem 3.2. We know that [13], Section V.3, $n^s = \sum_{d|n} J_s(d)$ and $J_s(n) = \Phi_s(n^s)$. Hence,

$$n^s = \sum_{d|n} \Phi_s(d^s) = \sum_{d^s|n^s} \Phi_s(d^s).$$

Now $(m_1 - a_1, m_2 - a_2, \ldots, m_k - a_k, b_1, b_2, \ldots, b_r, n^s)_s$ is the sth power of some natural number. So

$$\begin{aligned}
&\sum_{\substack{1\leqslant m_1,m_2,\ldots,m_k\leqslant n^s\\ (m_1,n^s)_s=1\\ (m_2,n^s)_s=1\\ \cdots\\ (m_k,n^s)_s=1\\ 1\leqslant b_1,b_2,\ldots,b_r\leqslant n^s}} (m_1 - a_1, m_2 - a_2, \ldots, m_k - a_k, b_1, b_2, \ldots, b_r, n^s)_s\\
&= \sum_{\substack{m_1=1\\ (m_1,n^s)_s=1}}^{n^s} \cdots \sum_{\substack{m_k=1\\ (m_k,n^s)_s=1}}^{n^s} \sum_{b_1=1}^{n^s} \cdots \sum_{b_r=1}^{n^s} (m_1 - a_1, m_2 - a_2, \ldots, m_k - a_k, b_1, b_2, \ldots, b_r, n^s)_s\\
&= \sum_{\substack{m_1=1\\ (m_1,n^s)_s=1}}^{n^s} \cdots \sum_{\substack{m_k=1\\ (m_k,n)_s=1}}^{n^s} \sum_{b_1=1}^{n^s} \cdots \sum_{b_r=1}^{n^s} \sum_{d^s|(m_1-a_1,m_2-a_2,\ldots,m_k-a_k,b_1,b_2,\ldots,b_r,n^s)_s} \Phi_s(d^s)\\
&= \sum_{d^S|n^s} \Phi_s(d^s) \sum_{\substack{b_1=1\\ d^s|b_1}}^{n^s} \cdots \sum_{\substack{b_r=1\\ d^s|b_r}}^{n^s} \sum_{\substack{m_1=1\\ (m_1,n)_s=1\\ m_1\equiv 1 (\mathrm{mod}\ d^s)}}^{n^s} \cdots \sum_{\substack{m_k=1\\ (m_k,n^s)=1\\ m_k\equiv 1 (\mathrm{mod}\ d^s)}}^{n^s} 1\\
&= \sum_{d^s|n^s} \Phi_s(d^s) \sum_{\substack{b_1=1\\ d^s|b_1}}^{n^s} \cdots \sum_{\substack{b_r=1\\ d^s|b_r}}^{n^s} \left(\frac{\Phi_s(n^s)}{\Phi_s(d^s)}\right)^k \quad \text{(using Theorem 3.1)}\\
&= \Phi_s(n^s)^k \sum_{d^s|n^s} \frac{1}{\Phi_s(d^s)^{k-1}} \sum_{\substack{b_1=1\\ d^s|b_1}}^{n^s} \cdots \sum_{\substack{b_r=1\\ d^s|b_r}}^{n^s} 1 = \Phi_s(n^s)^k \sum_{d^s|n^s} \frac{1}{\Phi_s(d^s)^{k-1}} \left(\frac{n^s}{d^s}\right)^r\\
&= \Phi_s(n^s)^k \sum_{d^s|n^s} \frac{(d^s)^r}{\Phi_s(\frac{n^s}{d^s})^{k-1}},
\end{aligned}$$

which completes the proof. □

We will now show that the above identity is indeed the same as Li-Kim identity (1.3) when $s = a_i = 1$. For that we need to show that the RHS of identities (3.1) and (1.3) are equal to the LHS of our identity, which can be quickly seen to be equal to the LHS of (1.3) when $s = a_i = 1$. To prove that the RHS are also equal, we require the following identity.

Lemma 3.1.

$$\sum_{d|p^t} d^r\Big(\frac{\varphi(p^t)}{\varphi(p^t/d)}\Big)^k = \sum_{j=0}^{t-1} p^{j(k+r)} + p^{t(k+r)}\Big(1-\frac{1}{p}\Big)^k.$$

Proof.

$$\begin{aligned}\sum_{d|p^t} d^r\Big(\frac{\varphi(p^t)}{\varphi(p^t/d)}\Big)^k &= \sum_{j=0}^{t}(p^j)^r\Big(\frac{\varphi(p^t)}{\varphi(p^{t-j})}\Big)^k \\ &= \sum_{j=0}^{t-1} p^{jr}\Big(\frac{p^t(1-1/p)}{p^{t-j}(1-1/p)}\Big)^k + p^{tr}p^{tk}\Big(1-\frac{1}{p}\Big)^k \\ &= \sum_{j=0}^{t-1} p^{j(k+r)} + p^{t(k+r)}\Big(1-\frac{1}{p}\Big)^k.\end{aligned}$$

□

Now we show what we claimed, that is

$$\sum_{d|n} d^r\Big(\frac{\varphi(n)}{\varphi(n/d)}\Big)^{k-1} = \prod_{i=1}^{q}(\varphi(p_i^{t_i})^{k-1}p_i^{t_i r} - p_i^{t_i(k+r-1)} + \sigma_{k+r-1}(p_i^{t_i})),$$

where $n = p_1^{t_1}p_2^{t_2}\dots p_q^{t_q}$.

Starting from the RHS, we have

$$\begin{aligned}&\prod_{i=1}^{q}(\varphi(p_i^{t_i})^{k-1}p_i^{t_i r} - p_i^{t_i(k+r-1)} + \sigma_{k+r-1}(p_i^{t_i})) \\ &\quad= \prod_{i=1}^{q}\Big(p_i^{t_i(k-1)}\Big(1-\frac{1}{p_i}\Big)^{k-1}p_i^{t_i r} - p_i^{t_i(k+r-1)} + 1 + p_i^{k+r-1} \\ &\qquad\qquad + p_i^{2(k+r-1)} + \ldots + p_i^{(t_i-1)(k+r-1)} + p_i^{t_i(k+r-1)}\Big) \\ &\quad= \prod_{i=1}^{q}\Big(1 + p_i^{k+r-1} + p_i^{2(k+r-1)} + \ldots + p_i^{(t_i-1)(k+r-1)} \\ &\qquad\qquad + p_i^{t_i(k+r-1)}\Big(1-\frac{1}{p_i}\Big)^{k-1}\Big) \\ &\quad= \prod_{i=1}^{q}\Big(\sum_{j=0}^{t_i-1} p_i^{j(k+r-1)} + p_i^{t_i(k+r-1)}\Big(1-\frac{1}{p_i}\Big)^{k-1}\Big) \\ &\quad= \prod_{i=1}^{q}\sum_{d_i|p_i^{t_i}} d_i^r\Big(\frac{\varphi(p_i^{t_i})}{\varphi(p_i^{t_i}/d_i)}\Big)^{k-1} \quad \text{(by Lemma 3.1)}\end{aligned}$$

$$= \sum_{d_1|p_1^{t_1}} d_1^r \Big(\frac{\varphi(p_1^{t_1})}{\varphi(p_1^{t_1}/d_1)}\Big)^{k-1} \sum_{d_2|p_2^{t_2}} d_2^r \Big(\frac{\varphi(p_2^{t_2})}{\varphi(p_2^{t_2}/d_2)}\Big)^{k-1} \dots \sum_{d_q|p_q^{t_q}} d_q^r \Big(\frac{\varphi(p_q^{t_q})}{\varphi(p_q^{t_q}/d_q)}\Big)^{k-1}$$

$$= \sum_{d_1 d_2 \dots d_q | p_1^{t_1} p_2^{t_2} \dots p_q^{t_q}} (d_1 d_2 \dots d_q)^r \Big(\frac{\varphi(p_1^{t_1} p_2^{t_2} \dots p_q^{t_q})}{\varphi(p_1^{t_1} p_2^{t_2} \dots p_q^{t_q}/d_1 d_2 \dots d_q)}\Big)^{k-1}$$

$$= \sum_{d|n} d^r \Big(\frac{\varphi(n)}{\varphi(n/d)}\Big)^{k-1}.$$

Therefore we get the Li-Kim identity (1.3) as a special case of our identity (3.1) when $s = 1$ and $a_i = 1$. Hence, our identity also gives a generalization of the Menon-Sury identity which in turn is a generalization of Menon's identity.

The following identities can be easily deduced from our result by giving special values to a_i, b_i and s_i and may be of independent interest. Note that the first one gives another generalization of the Li-Kim identity and it involves the usual gcd function.

Corollary 3.1.

(1) $$\sum_{\substack{1\leqslant m_1,m_2,\dots,m_k\leqslant n\\(m_1,n)=1\\(m_2,n)=1\\ \dots\\(m_k,n)=1\\1\leqslant b_1,b_2,\dots,b_r\leqslant n}} (m_1-a_1, m_2-a_2, \dots, m_k-a_k, b_1, b_2, \dots, b_r, n)$$
$$= \varphi(n)^k \sum_{d|n} \frac{(d)^r}{\varphi(n/d)^{k-1}},$$

(2) $$\sum_{\substack{1\leqslant m_1,m_2,\dots,m_k\leqslant n^s\\(m_1,n^s)_s=1\\(m_2,n^s)_s=1\\ \dots\\(m_k,n^s)_s=1}} (m_1-a_1, m_2-a_2, \dots, m_k-a_k, n^s)_s = \Phi_s(n^s)^k \sum_{d^s|n^s} \frac{1}{\Phi_s(d^s)^{k-1}},$$

(3) $$\sum_{\substack{m=1\\(m,n^s)_s=1}}^{n^s} (m-1, n^s)_s = \Phi_s(n^s)\tau_s(n^s).$$

4. An alternating way of defining Φ_s and extending it further

In [15] Tărnăuceanu suggested a new generalization of φ using elementary concepts in group theory. His generalization was based on the following idea. An element $m \in \mathbb{Z}_n$ is a generator of the group $(\mathbb{Z}_n, +)$ if and only if $(m, n) = 1$, which is if and only if $o(m) = n = \exp(\mathbb{Z}_n)$, where $o(m)$ is the order of the element m and $\exp(\mathbb{Z}_n)$ is the exponent of the group $(\mathbb{Z}_n, +)$. Thus, $\varphi(n)$ is the number of elements of order n in $\mathbb{Z}_n$. That is, $\varphi(n) = \#\{m \in \mathbb{Z}_n \colon o(m) = \exp(\mathbb{Z}_n)\}$. Tărnăuceanu extended φ to an arbitrary finite group G by defining $\varphi(G) = \#\{m \in G \colon o(m) = \exp(G)\}$.

We may adapt this technique for defining the generalization Φ_s as follows. An $m \in \mathbb{N}$ can be counted in $\Phi_s(n)$ if and only if $1 \leqslant m \leqslant n$ and $(m, n)_s = 1$. Now $o(m) = n/(m, n)$, and $(m, n)_s = 1$ if and only if m and n do not share any prime factor with power greater than or equal to s. That is, $(m, n) = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$ with $0 \leqslant a_i < s$. Here p_i are prime divisors of n. Therefore

$$o(m) = \frac{n}{(m, n)} = \frac{n}{p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}}, \quad 0 \leqslant a_i < s.$$

By using this fact, we may observe that

$$\Phi_s(n) = \#\Big\{m \in \mathbb{Z}_n \colon o(m) = \frac{n}{p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}},\ p_i^{a_i} \mid n \text{ and } 0 \leqslant a_i < s,\ i = 1, 2, \dots, r\Big\}.$$

Now the extension of Φ_s can be defined as follows. For any arbitrary finite group G, define

$$\Phi_s(G) = \#\Big\{a \in G \colon o(a) = \frac{\exp(G)}{p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}},\ p_i^{a_i} \mid \exp(G) \text{ and } 0 \leqslant a_i < s,\ i = 1, 2, \dots r\Big\}.$$

With this definition we have the following quick observations. For any finite cyclic group G, $\Phi_s(G) = \Phi_s(\#G)$. For relatively prime integers m and n, we have $\Phi_s(\mathbb{Z}_m \times \mathbb{Z}_n) = \Phi_s(\mathbb{Z}_{mn}) = \Phi_s(mn)$. For s-free integers m and n, $\Phi_s(\mathbb{Z}_m \times \mathbb{Z}_n) = \Phi_s(\mathbb{Z}_m)\Phi_s(\mathbb{Z}_n)$. The last statement follows because

$$\begin{aligned} \Phi_s(\mathbb{Z}_m \times \mathbb{Z}_n) &= \#\Big\{a \in \mathbb{Z}_m \times \mathbb{Z}_n \colon o(a) = \frac{\exp(\mathbb{Z}_m \times \mathbb{Z}_n)}{p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}}, \\ &\qquad p_i^{a_i} \mid \exp(\mathbb{Z}_m \times \mathbb{Z}_n) \text{ and } 0 \leqslant a_i < s,\ i = 1, 2, \dots, k\Big\} \\ &= \#\Big\{a \in \mathbb{Z}_m \times \mathbb{Z}_n \colon o(a) = \frac{\operatorname{lcm}(m, n)}{p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}}, \\ &\qquad p_i^{a_i} \mid \operatorname{lcm}(m, n) \text{ and } 0 \leqslant a_i < s,\ i = 1, 2, \dots, k\Big\}. \end{aligned}$$

Note that if m and n are s-free, then $\operatorname{lcm}(m, n)$ is also an s-free integer. We have $\Phi_s(\mathbb{Z}_m \times \mathbb{Z}_n) = \#\{a \in \mathbb{Z}_m \times \mathbb{Z}_n \colon o(a) = d, \text{ where } d \mid \operatorname{lcm}(m, n)\} =$

$mn = \Phi_s(\mathbb{Z}_m) \times \Phi_s(\mathbb{Z}_n)$. It is not very difficult to deduce the following general statement. For s-free integers $m_1, m_2, \ldots, m_k$, $\Phi_s(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_k}) = \Phi_s(\mathbb{Z}_{m_1})\Phi_s(\mathbb{Z}_{m_2}) \ldots \Phi_s(\mathbb{Z}_{m_k})$.

5. Further directions

Since we feel that this is the first time Menon's identity is revisited through the generalized gcd concept, it would be interesting to see what possible results can be obtained if one tries to apply our techniques to other generalizations of the identity. We note that we have investigated how does the identity of Zhao and Cao in [19] change if one uses the generalized gcd, Φ_s and τ_s in a recent (unpublished) work. We expect our generalization to have interesting consequences in group theory considering the definition of Φ_s we gave in the previous section.

Acknowledgements. We would like to thank the reviewer for suggesting the possible generalization of Li-Kim identity before which our attention was restricted to giving a generalization of the Menon-Sury identity alone.

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