ON GENERALIZED SQUARE-FULL NUMBERS IN AN ARITHMETIC PROGRESSION

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Abstract. Let a and $b \in \mathbb{N}$. Denote by $R_{a,b}$ the set of all integers $n > 1$ whose canonical prime representation $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ has all exponents α_i $(1 \leqslant i \leqslant r)$ being a multiple of a or belonging to the arithmetic progression $at+b$, $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All integers in $R_{a,b}$ are called generalized square-full integers. Using the exponent pair method, an upper bound for character sums over generalized square-full integers is derived. An application on the distribution of generalized square-full integers in an arithmetic progression is given.

Keywords: arithmetic progression; character sum; exponent pair method; square-full number

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1. Introduction and results

An integer $n > 1$ is called *square-full*, if in the canonical representation of n each exponent is not less than 2. Let Q_2 denote the set of all square-full integers, $q_2(n)$ the characteristic function of the set Q_2 and $Q_2(x)$ the number of square-full integers $n \leqslant x$. The investigation of square-full integers has a long and rich history. Erdős and Szekeres in [5] proved that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \tag{1.1}$$

In 1958, Bateman and Grosswald in [1] improved (1.1) and showed that

$$Q_2(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/6} \exp(-C(\log x)^{3/5} (\log\log x)^{-1/5})) \tag{1.2}$$

for some absolute constant $C > 0$. For a given integer $q \geqslant 2$ and an integer l such that $\gcd(l, q) = 1$, $Q_2(x; l, q)$ denotes the number of square-full numbers $n \leqslant x$ and

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$n \equiv l \pmod q$. The study on the distribution of square-full numbers in an arithmetic progression attracts the interests of many authors; see, e.g. [2], [3], [6], [7] and [10].

It would be interesting to extend this problem on generalized square-full integers. Let a and b be fixed positive integers and $R_{a,b}$ the set of all integers $n > 1$ with the property that, for the canonical representation of $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, each exponent α_i $(1 \leqslant i \leqslant r)$ is either a multiple of a or is contained in the progression $at + b$, $t \geqslant 0$. Let $r_{a,b}$ be the characteristic function for the element in the set $R_{a,b}$ and $R_{a,b}(x)$ the number of integers $n \leqslant x$ such that $n \in R_{a,b}$. Cohen was the first mathematician to introduce generalized square-full integers and he derived asymptotic formulas for $R_{a,b}(x)$ in various cases, see [4]. It is clear that the set $R_{2,3}$ is the set of square-full integers. Thus, all integers in $R_{a,b}$ may be called *generalized square-full integers*.

In this paper, we use the exponent pair method of [8] and [9] to bound the character sums over generalized square-full numbers and apply it to study the distribution for a generalized square-full numbers in an arithmetic progression.

In [10], Srichan used the exponent pair method to study the appearance of square-full numbers in an arithmetic progression and showed that

$$\begin{aligned}(1.3)\qquad Q_2(x;l,q) &= \frac{L(3/2,\chi_0) + \sum_{\chi_1} \bar{\chi_1}(l) L(3/2,\chi_1)}{qL(3,\chi_0)} x^{1/2} \\ &\quad + \frac{L(2/3,\chi_0) + \sum_{\chi_2} \bar{\chi_2}(l) L(2/3,\chi_2)}{qL(2,\chi_0)} x^{1/3} + O(q^{1/2+\varepsilon} x^{1/6}),\end{aligned}$$

where χ_0, χ_1 and χ_2 denote the principal, quadratic and cubic character modulo q, respectively. However, there is a mistake in the O-term of (1.3). Srichan overlooked the value of $\varphi(q)$. In particular, the O-term in the formula for $Q_2(x;l,q)$ should be corrected to $q^{4/3+\varepsilon} x^{1/6}$. In the present paper, we also give this correction.

Let a and b be fixed positive integers, and $a < b < 2a$. For given integers $q \geqslant 2$ and $0 < l < q$ such that $\gcd(l,q) = 1$, let $R_{a,b}(x;l,q)$ be the number of integers $n \leqslant x$ such that $n \in R_{a,b}$ and $n \equiv l \pmod q$. We obtain the following theorem.

Theorem 1.1. *Let a and b be fixed positive integers and $a < b < 2a$. For given integers $q \geqslant 2$, $0 < l < q$, such that $\gcd(l,q) = 1$ and $x \geqslant q^{a+b}$, we have*

$$\begin{aligned}R_{a,b}(x;l,q) &= \left(\frac{L(b/a,\chi_0)}{qL(2b/a,\chi_0)} + \sum_{\chi \in A_\chi} \bar{\chi}(l) \frac{L(b/a,\chi^b)}{qL(2b/a,\chi^{2b})} \right) x^{1/a} \\ &\quad + \left(\frac{L(a/b,\chi_0)}{qL(2,\chi_0)} + \sum_{\chi \in B_\chi} \bar{\chi}(l) \frac{L(a/b,\chi^a)}{qL(2,\chi^{2b})} \right) x^{1/b} + O(q^{4/3+\varepsilon} x^{1/(2b)}),\end{aligned}$$

where A_χ and B_χ denote the set of all non-principal characters modulo q of order d such that $d \mid a$ and $d \mid b$, respectively.

Remark 1.1. Using Theorem 1.1 we obtain the correct error term in [10] for $a = 2$ and $b = 3$.

2. Preliminary results and notations

As usual, let $\mu(n)$ and $\varphi(n)$ denote the Möbius function and the Euler phi function, respectively, and $\tau_{a,b}(n)$ denotes the number of ordered pairs (d, δ) of positive integers d and δ such that $d^a\delta^b = n$. Let $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$. The following lemmas are well known and the proofs can be found in their references.

Lemma 2.1 ([4], Lemma 2.1)**.** *If $a \nmid b$, then*

$$r_{a,b}(n) = \sum_{d^{2b}\delta = n} \mu(d)\tau_{a,b}(\delta).$$

Lemma 2.2 ([8], Lemma 2)**.** *Let ω and κ be two real numbers, $\omega > 0$, $0 < \kappa \neq 1$. Then*

$$\sum_{n \leqslant \omega} n^{-\kappa} = \zeta(\kappa) - \frac{1}{\kappa - 1}\omega^{1-\kappa} - \psi(\omega)\omega^{-\kappa} + O(\omega^{-\kappa-1}).$$

Lemma 2.3. *For real $x > 1$ and any non-principal character χ modulo q,*

$$\sum_{k \leqslant x} \chi(k) = \sum_{j \leqslant q} \chi(j) \Big\lfloor \frac{x}{q} - \frac{j}{q} + 1 \Big\rfloor \tag{2.1}$$

and

$$\sum_{k \leqslant x} \chi(k)\lfloor k \rfloor = \sum_{j \leqslant q} \chi(j) \sum_{\substack{k \leqslant x \\ k \equiv j \pmod{q}}} \lfloor k \rfloor. \tag{2.2}$$

Proof. Equations (2.1) and (2.2) follow from the periodicity of the character χ modulo q. □

Lemma 2.4 ([9], Lemma 13)**.** *If $f(n)$ is an arithmetic function, then*

$$\sum_{\substack{n \leqslant \omega, \\ \gcd(n,q)=1}} f(n) = \sum_{d \mid q} \mu(d) \sum_{m \leqslant \omega/d} f(md).$$

Lemma 2.5. *For $\alpha > 0$, $\alpha \neq 1$, and $0 < \beta \leqslant 1$, we have*

$$\sum_{\substack{n\leqslant x\\ n\equiv l \pmod q}} n^{-\alpha} = q^{-\alpha}\zeta\Big(\alpha, \frac{l}{q}\Big) + \frac{1}{1-\alpha}\frac{x^{1-\alpha}}{q} - \psi\Big(\frac{x-l}{q}\Big)x^{-\alpha} + O(qx^{-\alpha-1}),$$

where

$$\zeta(s,\beta) = \sum_{n=0}^{\infty}(n+\beta)^{-s}, \quad \Re(s) > 1.$$

Proof. It follows from Lemma 14 in [9]. □

Lemma 2.6 ([9], Lemma 17). *Let x, η, α, ω be real numbers, j and q positive numbers, where $x \geqslant 1$, $\alpha > 0$, $\eta \geqslant 1$, $1 \leqslant j \leqslant q$, and (k,l) an exponent pair with $k > 0$ and*

$$R(x,\eta,\alpha;q,j;\omega) = \sum_{\substack{n\leqslant \eta\\ n\equiv j \pmod q}} \psi\Big(\frac{x}{n^\alpha} + \omega\Big),$$

if ω is independent on n. Then

$$\begin{aligned} R(x,\eta,\alpha;q,j;\omega) = O(1) + O(x^{-1/2}\eta^{1+\alpha/2}q^{-1}) \\ + \begin{cases} O(x^{k/(k+1)}\eta^{(l-\alpha k)/(k+1)}q^{-l/(k+1)}) & \text{for } l > \alpha k, \\ O(x^{k/(k+1)}\log \eta q^{-\alpha k/(k+1)}) & \text{for } l = \alpha k, \\ O((xq^{-\alpha})^{k/(1+(1+\alpha)k-l)}) & \text{for } l < \alpha k, \end{cases}\end{aligned}$$

where the O-constants are dependent on only α.

3. Proof of Theorem 1.1

Proof. Let a and b be fixed positive integers and $a < b < 2a$. Let q and l be integers such that $q \geqslant 2$, $0 < l < q$ and $\gcd(l,q) = 1$. By the orthogonality relation for Dirichlet characters modulo q, we have

$$R_{a,b}(x;l,q) = \sum_{\substack{n\leqslant x\\ n\equiv l \pmod q}} r_{a,b}(n) = \frac{1}{\varphi(q)}\sum_{\chi}\bar\chi(l)C_{a,b}(x,\chi), \tag{3.1}$$

where

$$C_{a,b}(x,\chi) := \sum_{n\leqslant x} r_{a,b}(n)\chi(n).$$

In view of Lemma 2.1, we have

$$
\begin{aligned}
(3.2)\qquad C_{a,b}(x,\chi) &= \sum_{d^{2b}\delta\leqslant x} \mu(d)\chi^{2b}(d)\tau_{a,b}(\delta)\chi(\delta) \\
&= \sum_{d\leqslant (xq^{-(a+b)})^{1/(2b)}} \mu(d)\chi^{2b}(d) \sum_{\delta\leqslant xd^{-2b}} \tau_{a,b}(\delta)\chi(\delta) \\
&\quad + \sum_{(xq^{-(a+b)})^{1/(2b)}<d\leqslant x^{1/(2b)}} \mu(d)\chi^{2b}(d) \sum_{\delta\leqslant xd^{-2b}} \tau_{a,b}(\delta)\chi(\delta) \\
&:= U_1 + U_2.
\end{aligned}
$$

First, we bound U_2. For any character modulo q, we have

$$
\begin{aligned}
(3.3)\qquad U_2 &= \sum_{(xq^{-(a+b)})^{1/(2b)}<d\leqslant x^{1/(2b)}} \mu(d)\chi^{2b}(d) \sum_{m^a n^b\leqslant xd^{-2b}} \chi(m^a n^b) \\
&\ll \sum_{(xq^{-(a+b)})^{1/(2b)}<d\leqslant x^{1/(2b)}} \frac{1}{q}(xd^{-2b})^{1/a} \\
&\ll x^{1/(2b)} q^{-(a^2+ab-2b^2)/(2ab)}.
\end{aligned}
$$

Now we calculate U_1. Put

$$
T_{a,b}(x,\chi) := \sum_{n\leqslant x} \tau_{a,b}(n)\chi(n).
$$

Thus

$$
(3.4)\qquad U_1 = \sum_{d\leqslant (xq^{-(a+b)})^{1/(2b)}} \mu(d)\chi^{2b}(d) T_{a,b}(xd^{-2b},\chi).
$$

Then we need the asymptotic formula for $T_{a,b}(x,\chi)$. We consider the corresponding Dirichlet series of $r_{a,b}(n)\chi(n)$, namely,

$$
\sum_{n=1}^{\infty} \frac{r_{a,b}(n)\chi(n)}{n^s} = \frac{L(as,\chi^a)L(bs,\chi^b)}{L(2bs,\chi^{2b})},
$$

which converges absolutely for $\Re(s) > 1/a$. Since

$$
\frac{1}{L(2bs,\chi^{2b})} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi^{2b}(n)}{n^{2bs}} \quad \text{for } \Re(s) \geqslant \frac{1}{2b}
$$

has a comparatively small abscissa of absolute convergence with $1 \leqslant r < 2b$, we put

$$
D_b(x,\chi,r) := \sum_{d\leqslant x^{1/(2b)}} \frac{\mu(d)\chi^{2b}(d)}{d^{2b/r}}.
$$

Thus, for $1 \leqslant r < 2b$, we have

$$\text{(3.5)}\quad D_b(x,\chi,r) = \frac{1}{L(2bs,\chi^{2b})} - \sum_{d > x^{1/(2b)}} \frac{\mu(d)\chi^{2b}(d)}{d^{2b/r}} = \frac{1}{L(2bs,\chi^{2b})} + O(x^{1/(2b)-1/r}).$$

Since

$$\sum_{n=1}^{\infty} \frac{\tau_{a,b}(n)\chi(n)}{n^s} = L(as,\chi^a)L(bs,\chi^b)$$

and from the Euler product for $L(s,\chi)$, we can deduce

$$\begin{aligned}\text{(3.6)}\qquad T_{a,b}(x,\chi) = \sum_{m^a n^b \leqslant x} \chi^a(m)\chi^b(n) = &\sum_{m \leqslant x^{1/(a+b)}} \chi^a(m) \sum_{n \leqslant (x/m^a)^{1/b}} \chi^b(n) \\ &+ \sum_{n \leqslant x^{1/(a+b)}} \chi^b(n) \sum_{m \leqslant (x/n^b)^{1/a}} \chi^a(m) \\ &- \sum_{m \leqslant x^{1/(a+b)}} \chi^a(m) \sum_{n \leqslant x^{1/(a+b)}} \chi^b(n).\end{aligned}$$

Let A_χ and B_χ be the set of all non-principal characters $\bmod q$ of order d such that $d \mid a$ and $d \mid b$, respectively. Now we consider the sum (3.6) in the following five cases:

Case 1: $\chi = \chi_0$, the principal character modulo q. In view of (3.6), we have

$$\begin{aligned}T_{a,b}(x,\chi_0) = &\sum_{\substack{m \leqslant x^{1/(a+b)} \\ \gcd(m,q)=1}} \sum_{\substack{n \leqslant (x/m^a)^{1/b} \\ \gcd(n,q)=1}} 1 + \sum_{\substack{n \leqslant x^{1/(a+b)} \\ \gcd(n,q)=1}} \sum_{\substack{m \leqslant (x/n^b)^{1/a} \\ \gcd(m,q)=1}} 1 \\ &- \sum_{\substack{m \leqslant x^{1/(a+b)} \\ \gcd(m,q)=1}} 1 \sum_{\substack{n \leqslant x^{1/(a+b)} \\ \gcd(n,q)=1}} 1.\end{aligned}$$

In view of Lemma 2.4, we have

$$\begin{aligned}T_{a,b}(x,\chi_0) = &\sum_{\substack{m \leqslant x^{1/(a+b)} \\ \gcd(m,q)=1}} \sum_{d|q} \mu(d) \sum_{n \leqslant x^{1/b}/(dm^{a/b})} 1 + \sum_{\substack{n \leqslant x^{1/(a+b)} \\ \gcd(n,q)=1}} \sum_{d|q} \mu(d) \sum_{m \leqslant x^{1/a}/(dn^{b/a})} 1 \\ &- \sum_{d|q} \mu(d) \sum_{m \leqslant x^{1/(a+b)}/d} 1 \sum_{t|q} \mu(t) \sum_{m \leqslant x^{1/(a+b)}/t} 1 \\ = &\sum_{d|q} \mu(d) \sum_{\substack{m \leqslant x^{1/(a+b)} \\ \gcd(m,q)=1}} \left\lfloor \frac{x^{1/b}}{dm^{a/b}} \right\rfloor + \sum_{d|q} \mu(d) \sum_{\substack{n \leqslant x^{1/(a+b)} \\ \gcd(n,q)=1}} \left\lfloor \frac{x^{1/a}}{dn^{b/a}} \right\rfloor \\ &- \sum_{d|q} \mu(d) \left\lfloor \frac{x^{1/(a+b)}}{d} \right\rfloor \sum_{t|q} \mu(t) \left\lfloor \frac{x^{1/(a+b)}}{t} \right\rfloor.\end{aligned}$$

We apply Lemma 2.4 again to the first two inner sums and note that the last term is the product of the same sums. Thus, we have

$$\begin{aligned} T_{a,b}(x,\chi_0) = {} & \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\left\lfloor\frac{x^{1/b}}{d(tm)^{a/b}}\right\rfloor \\ & + \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\left\lfloor\frac{x^{1/a}}{d(tm)^{b/a}}\right\rfloor \\ & - \Big(\sum_{d|q}\mu(d)\left\lfloor\frac{x^{1/(a+b)}}{d}\right\rfloor\Big)^2. \end{aligned}$$

From $\lfloor x\rfloor = x - \psi(x) - \frac{1}{2}$, we have

$$\begin{aligned} T_{a,b}(x,\chi_0) = {} & \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\Big(\frac{x^{1/b}}{d(tm)^{a/b}} - \psi\Big(\frac{x^{1/b}}{d(tm)^{a/b}}\Big) - \frac{1}{2}\Big) \\ & + \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\Big(\frac{x^{1/a}}{d(tm)^{b/a}} - \psi\Big(\frac{x^{1/a}}{d(tm)^{b/a}}\Big) - \frac{1}{2}\Big) \\ & - \Big(\sum_{d|q}\mu(d)\Big(\frac{x^{1/(a+b)}}{d} - \psi\Big(\frac{x^{1/(a+b)}}{d}\Big) - \frac{1}{2}\Big)\Big)^2. \end{aligned}$$

In view of the well known identities $\sum_{d|q}\mu(d) = 0$ for $q > 1$ and $\sum_{d|q}\mu(d)/d = \varphi(q)/q$, we have

$$\begin{aligned} T_{a,b}(x,\chi_0) = {} & x^{1/b}\frac{\varphi(q)}{q}\sum_{t|q}\frac{\mu(t)}{t^{a/b}}\sum_{m\leqslant t^{-1}x^{1/(a+b)}} m^{-a/b} \\ & - \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/b}}{d(tm)^{a/b}}\Big) \\ & + x^{1/a}\frac{\varphi(q)}{q}\sum_{t|q}\frac{\mu(t)}{t^{b/a}}\sum_{m\leqslant t^{-1}x^{1/(a+b)}} m^{-b/a} \\ & - \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/a}}{d(tm)^{b/a}}\Big) \\ & + 2\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\psi\Big(\frac{x^{1/(a+b)}}{d}\Big)x^{1/(a+b)} \\ & - \frac{\varphi^2(q)}{q^2}x^{2/(a+b)} + O(q^{\varepsilon}). \end{aligned}$$

Now we use Lemma 2.2 to get

$$
\begin{aligned}
T_{a,b}(x,\chi_0) = x^{1/b}\frac{\varphi(q)}{q}\sum_{t|q}\frac{\mu(t)}{t^{a/b}}\Big(\zeta\Big(\frac{a}{b}\Big) + b\frac{x^{1/b(a+b)}}{t^{1/b}} \\
&- \psi\Big(\frac{x^{1/(a+b)}}{t}\Big)x^{-a/(a+b)b}t^{a/b} + O\Big(x^{-1/b}t^{(a+b)/b}\Big)\Big) \\
&- \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/b}}{d(tm)^{a/b}}\Big) \\
&+ x^{1/a}\frac{\varphi(q)}{q}\sum_{t|q}\frac{\mu(t)}{t^{b/a}}\Big(\zeta\Big(\frac{b}{a}\Big) - a\frac{x^{-1/a(a+b)}}{t^{-1/a}} \\
&- \psi\Big(\frac{x^{1/(a+b)}}{t}\Big)x^{-b/a(a+b)}t^{b/a} + O\Big(x^{-1/a}t^{(a+b)/a}\Big)\Big) \\
&- \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/a}}{d(tm)^{b/a}}\Big) \\
&- \frac{\varphi^2(q)}{q^2}x^{2/(a+b)} + 2\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\psi\Big(\frac{x^{1/(a+b)}}{d}\Big)x^{1/(a+b)} + O(q^{\varepsilon}).
\end{aligned}
$$

In view of $\sum_{t|q}\mu(t)/t^{\alpha} = \prod_{p|q}(1-p^{-\alpha})$, $\alpha\in\mathbb{R}$, and $\zeta(s)\prod_{p|q}(1-p^{-s}) = L(s,\chi_0)$, $s\neq 1$, we have

$$
\begin{aligned}
(3.7)\quad T_{a,b}(x,\chi_0) &= \frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi_0\Big)x^{1/b} + b\frac{\varphi^2(q)}{q^2}x^{2/(a+b)} \\
&- \frac{\varphi(q)}{q}\sum_{t|q}\mu(t)\psi\Big(\frac{x^{1/(a+b)}}{t}\Big)x^{1/(a+b)} + O(q^{1+\varepsilon}) \\
&- \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/b}}{d(tm)^{a/b}}\Big) \\
&+ \frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi_0\Big)x^{1/a} - a\frac{\varphi^2(q)}{q^2}x^{2/(a+b)} \\
&- \frac{\varphi(q)}{q}\sum_{t|q}\mu(t)\psi\Big(\frac{x^{1/(a+b)}}{t}\Big)x^{1/(a+b)} + O(q^{1+\varepsilon}) \\
&- \sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant t^{-1}x^{1/(a+b)}}\psi\Big(\frac{x^{1/a}}{d(tm)^{b/a}}\Big) - \frac{\varphi^2(q)}{q^2}x^{2/(a+b)} \\
&+ 2\frac{\varphi(q)}{q}\sum_{d|q}\mu(d)\psi\Big(\frac{x^{1/(a+b)}}{d}\Big)x^{1/(a+b)} + O(q^{\varepsilon}) \\
&:= \frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi_0\Big)x^{1/a} + \frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi_0\Big)x^{1/b} + O(q^{1+\varepsilon}) - S_1 - S_2,
\end{aligned}
$$

where

$$S_1=\sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\psi\Big(\frac{x^{1/b}}{d(tm)^{a/b}}\Big)$$

and

$$S_2=\sum_{d|q}\sum_{t|q}\mu(d)\mu(t)\sum_{m\leqslant x^{1/(a+b)}/t}\psi\Big(\frac{x^{1/a}}{d(tm)^{b/a}}\Big).$$

If we use Lemma 2.6 with the exponent pair $(\frac{2}{7},\frac{4}{7})$, then for $q\leqslant x^{1/(a+b)}$ we have

$$\begin{aligned}S_1&=\sum_{d|q}\sum_{t|q}\mu(d)\mu(t)R\Big(\frac{x^{1/b}}{dt^{a/b}},\frac{x^{1/(a+b)}}{t},\frac{a}{b},1,0,0\Big)\\&\ll\sum_{d|q}\sum_{t|q}(x^{1/(2a+2b)}d^{1/2}t^{-1}+x^{2/(3a+3b)}d^{-2/9}t^{-4/9})\\&=O(x^{2/(3a+3b)}q^{1/3+\varepsilon})\end{aligned}$$

and

$$\begin{aligned}S_2&=\sum_{d|q}\sum_{t|q}\mu(d)\mu(t)R\Big(\frac{x^{1/a}}{dt^{b/a}},\frac{x^{1/(a+b)}}{t},\frac{b}{a},1,0,0\Big)\\&\ll\sum_{d|q}\sum_{t|q}(x^{1/(2a+2b)}d^{1/2}t^{-1}+x^{2/(3a+3b)}d^{-2/9}t^{-4/9})\\&=O(x^{2/(a+b)}q^{1/3+\varepsilon}).\end{aligned}$$

Substituting for S_1 and S_2 in (3.7), we have, for $q\leqslant x^{1/(a+b)}$,

$$(3.8)\quad T_{a,b}(x,\chi_0)=\frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi_0\Big)x^{1/a}+\frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi_0\Big)x^{1/b}+O(x^{2/(3a+3b)}q^{1/3+\varepsilon}).$$

Case 2: the non-principal characters χ modulo q such that $\chi\in A_\chi-B_\chi$. In view of (3.6), we have

$$\begin{aligned}T_{a,b}(x,\chi)=&\sum_{\substack{m\leqslant x^{1/(a+b)}\\ \gcd(m,q)=1}}\sum_{n\leqslant(x/m^a)^{1/b}}\chi^b(n)\\&+\sum_{n\leqslant x^{1/(a+b)}}\chi^b(n)\sum_{\substack{m\leqslant(x/n^b)^{1/a}\\ \gcd(m,q)=1}}1\\&-\sum_{\substack{m\leqslant x^{1/(a+b)}\\ \gcd(m,q)=1}}1\sum_{n\leqslant x^{1/(a+b)}}\chi^b(n).\end{aligned}$$

We use (2.1) of Lemma 2.3 for the inner sum of the first term and then apply Lemma 2.1. For the second term we first use Lemma 2.1 for its inner sum and then use (2.2) of Lemma 2.3. We apply Lemmas 2.1 and 2.3 separately for the last term. Thus, we have

$$T_{a,b}(x,\chi) = \sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{n\leqslant x^{1/(a+b)}/d}\Big\lfloor\frac{x^{1/b}}{q(nd)^{a/b}}-\frac{j}{q}+1\Big\rfloor$$
$$+\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{\substack{n\leqslant x^{1/(a+b)}\\ n\equiv j \pmod q}}\Big\lfloor\frac{x^{1/a}}{dn^{b/a}}\Big\rfloor$$
$$-\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\Big\lfloor\frac{x^{1/(a+b)}}{d}\Big\rfloor\Big\lfloor\frac{x^{1/(a+b)}}{q}-\frac{j}{q}+1\Big\rfloor.$$

We write $\lfloor x\rfloor$ as $x-\psi(x)-\frac{1}{2}$ and $\psi(x+1)=\psi(x)$. We obtain

$$T_{a,b}(x,\chi) = \sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{n\leqslant x^{1/(a+b)}/d}\Big(\frac{x^{1/b}}{q(nd)^{a/b}}-\frac{j}{q}+\frac{1}{2}-\psi\Big(\frac{x^{1/b}}{q(nd)^{a/b}}-\frac{j}{q}\Big)\Big)$$
$$+\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{\substack{n\leqslant x^{1/(a+b)}\\ n\equiv j \pmod q}}\Big(\frac{x^{1/a}}{dn^{b/a}}-\psi\Big(\frac{x^{1/a}}{dn^{b/a}}\Big)-\frac{1}{2}\Big)$$
$$-\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\Big\lfloor\frac{x^{1/(a+b)}}{d}\Big\rfloor\Big(\frac{x^{1/(a+b)}}{q}-\frac{j}{q}+\frac{1}{2}-\psi\Big(\frac{x^{1/(a+b)}}{q}-\frac{j}{q}\Big)\Big).$$

Using the following identities, $\sum\limits_{j\leqslant q}\chi(j) = 0$ for a non-principal character χ, and $\sum\limits_{d|q}\mu(d)=0$, $q>1$, we have

$$T_{a,b}(x,\chi) = \sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{n\leqslant x^{1/(a+b)}/d}\Big(-\frac{j}{q}-\psi\Big(\frac{x^{1/b}}{q(nd)^{a/b}}-\frac{j}{q}\Big)\Big) \tag{3.9}$$
$$+\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{\substack{n\leqslant x^{1/(a+b)}\\ n\equiv j \pmod q}}\Big(\frac{x^{1/a}}{dn^{b/a}}-\psi\Big(\frac{x^{1/a}}{dn^{b/a}}\Big)\Big)$$
$$-\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\Big\lfloor\frac{x^{1/(a+b)}}{d}\Big\rfloor\Big(-\frac{j}{q}-\psi\Big(\frac{x^{1/(a+b)}}{q}-\frac{j}{q}\Big)\Big)$$
$$= -\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{n\leqslant x^{1/(a+b)}/d}\psi\Big(\frac{x^{1/b}}{q(nd)^{a/b}}-\frac{j}{q}\Big)$$
$$+\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\sum_{\substack{n\leqslant x^{1/(a+b)}\\ n\equiv j \pmod q}}\Big(\frac{x^{1/a}}{dn^{b/a}}-\psi\Big(\frac{x^{1/a}}{dn^{b/a}}\Big)\Big)$$
$$+\sum_{j\leqslant q}\chi^b(j)\sum_{d|q}\mu(d)\Big\lfloor\frac{x^{1/(a+b)}}{d}\Big\rfloor\Big(\psi\Big(\frac{x^{1/(a+b)}}{q}-\frac{j}{q}\Big)\Big)$$

$$= \frac{\varphi(q)}{q} x^{1/a} \sum_{j \leqslant q} \chi^b(j) \sum_{\substack{n \leqslant x^{1/(a+b)} \\ n \equiv j \pmod q}} n^{-b/a}$$

$$+ \frac{\varphi(q)}{q} x^{1/(a+b)} \sum_{j \leqslant q} \chi^b(j) \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q}\Big) + O(q^2) - S_3 - S_4,$$

where

$$S_3 = \sum_{j \leqslant q} \chi^b(j) \sum_{d|q} \mu(d) \sum_{n \leqslant x^{1/(a+b)}/d} \psi\Big(\frac{x^{1/b}}{q(nd)^{a/b}} - \frac{j}{q}\Big),$$

$$S_4 = \sum_{j \leqslant q} \chi^b(j) \sum_{d|q} \mu(d) \sum_{\substack{n \leqslant x^{1/(a+b)} \\ n \equiv j \pmod q}} \psi\Big(\frac{x^{1/a}}{dn^{b/a}}\Big).$$

Using Lemma 2.5 and writting the Hurwitz zeta-function as the Dirichlet L-function $q^{-s} \sum_{j \leqslant q} \chi(j)\zeta(s, j/q) = L(s, \chi)$ for $s \neq 1$, we have

$$x^{1/a} \frac{\varphi(q)}{q} \sum_{j \leqslant q} \chi(j)^b \sum_{\substack{n \leqslant x^{1/(a+b)} \\ n \equiv j \pmod q}} n^{-b/a} \tag{3.10}$$

$$= x^{1/a} \frac{\varphi(q)}{q} \sum_{j \leqslant q} \chi^b(j) \Big(q^{-b/a} \zeta\Big(\frac{b}{a}, \frac{j}{q}\Big) + \frac{a}{a-b} x^{(a-b)/a}$$

$$- \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q}\Big) x^{-b/(a(a+b))} + O(qx^{-1/a})\Big)$$

$$= x^{1/a} \frac{\varphi(q)}{q} \sum_{j \leqslant q} \chi^b(j) \Big(q^{-b/a} \zeta\Big(\frac{b}{a}, \frac{j}{q}\Big)$$

$$- \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q}\Big) x^{-b/(a(a+b))} + O(qx^{-1/a})\Big)$$

$$= - \frac{\varphi(q)}{q} \sum_{j \leqslant q} \chi^b(j) \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q}\Big) x^{1/(a+b)}$$

$$+ \frac{\varphi(q)}{q} L\Big(\frac{b}{a}, \chi^b\Big) x^{1/a} + O(q^2).$$

Substituting (3.10) into (3.9), we get

$$T_{a,b}(x, \chi) = \frac{\varphi(q)}{q} L\Big(\frac{b}{a}, \chi^b\Big) x^{1/a} + O(q^2) - S_3 - S_4. \tag{3.11}$$

If we use Lemma 2.6 with the exponent pair $(\frac{2}{7}, \frac{4}{7})$, then for $q \leqslant x^{1/(a+b)}$ we have

$$\begin{aligned} S_3 &= \sum_{j\leqslant q} \chi^b(j) \sum_{d|q} \mu(d) R\Big(\frac{x^{1/b}}{qd^{a/b}}, \frac{x^{1/(a+b)}}{d}, \frac{a}{b}, 1, 1, \frac{-j}{q}\Big) \\ &\ll \sum_{j\leqslant q} \sum_{d|q} \Big(x^{1/(2a+2b)} q^{1/2} d^{-1} + x^{2/(3a+3b)} q^{-2/9} d^{-4/9}\Big) \\ &= O(x^{2/(3a+3b)} q^{4/3+\varepsilon}) \end{aligned}$$

and similarly

$$\begin{aligned} S_4 &= \sum_{j\leqslant q} \chi^b(j) \sum_{d|q} \mu(d) R\Big(\frac{x^{1/a}}{d}, x^{1/(a+b)}, \frac{b}{a}, q, j, 0\Big) \\ &\ll \sum_{j\leqslant q} \sum_{d|q} (x^{1/(2a+2b)} q^{1/2} d^{-1} + x^{2/(3a+3b)} q^{-2/9} d^{-4/9}) \\ &= O(x^{2/(3a+3b)} q^{4/3+\varepsilon}). \end{aligned}$$

Replacing S_3 and S_4 in (3.11), we have, for $q \leqslant x^{1/(a+b)}$,

$$T_{a,b}(x,\chi) = \frac{\varphi(q)}{q} L\Big(\frac{b}{a}, \chi^b\Big) x^{1/a} + O(x^{2/(3a+3b)} q^{4/3+\varepsilon}). \tag{3.12}$$

Case 3: non-principal characters χ modulo q such that $\chi \in B_\chi - A_\chi$. We prove this case by the same steps as Case 2. We obtain, for $q \leqslant x^{1/(a+b)}$,

$$T_{a,b}(x,\chi) = \frac{\varphi(q)}{q} L\Big(\frac{a}{b}, \chi^a\Big) x^{1/b} + O(x^{2/(3a+3b)} q^{4/3+\varepsilon}). \tag{3.13}$$

Case 4: non-principal characters χ modulo q such that $\chi \in A_\chi \cap B_\chi$. In this case we obtain the same result as in Case 1. Thus we have, for $q \leqslant x^{1/(a+b)}$,

$$T_{a,b}(x,\chi) = \frac{\varphi(q)}{q} L\Big(\frac{b}{a}, \chi_0\Big) x^{1/a} + \frac{\varphi(q)}{q} L\Big(\frac{a}{b}, \chi_0\Big) x^{1/b} + O(x^{2/(3a+3b)} q^{1/3+\varepsilon}). \tag{3.14}$$

Case 5: non-principal characters χ modulo q such that $\chi \notin A_\chi \cup B_\chi$. In view of (3.6), we use (2.1) of Lemma 2.3 and get that

$$\begin{aligned} T_{a,b}(x,\chi) &= \sum_{j\leqslant q} \chi^b(j) \sum_{n\leqslant x^{1/(a+b)}} \chi^a(n) \Big\lfloor \frac{x^{1/b}}{qn^{a/b}} - \frac{j}{q} + 1 \Big\rfloor \\ &\quad + \sum_{j\leqslant q} \chi^a(j) \sum_{m\leqslant x^{1/(a+b)}} \chi^b(m) \Big\lfloor \frac{x^{1/a}}{qm^{b/a}} - \frac{j}{q} + 1 \Big\rfloor \\ &\quad - \sum_{j\leqslant q} \sum_{h\leqslant q} \chi^a(j) \chi^b(h) \Big\lfloor \frac{x^{1/(a+b)}}{q} - \frac{j}{q} + 1 \Big\rfloor \Big\lfloor \frac{x^{1/(a+b)}}{q} - \frac{h}{q} + 1 \Big\rfloor. \end{aligned}$$

Since $\lfloor x \rfloor = x - \psi(x) - \frac{1}{2}$, $\sum_{j \leqslant q} \chi(j) = 0$ for non-principal characters and $\psi(x) = \psi(x+1)$, we have

$$
\begin{aligned}
T_{a,b}(x,\chi) = & \sum_{j \leqslant q} \chi^b(j) \sum_{n \leqslant x^{1/(a+b)}} \chi^a(n) \Big(\frac{x^{1/b}}{qn^{a/b}} - \frac{j}{q} + \frac{1}{2} - \psi\Big(\frac{x^{1/b}}{qn^{a/b}} - \frac{j}{q} \Big) \Big) \\
& + \sum_{j \leqslant q} \chi^a(j) \sum_{m \leqslant x^{1/(a+b)}} \chi^b(m) \Big(\frac{x^{1/a}}{qm^{b/a}} - \frac{j}{q} + \frac{1}{2} - \psi\Big(\frac{x^{1/a}}{qm^{b/a}} - \frac{j}{q} \Big) \Big) \\
& - \sum_{j \leqslant q} \sum_{h \leqslant q} \chi^a(j) \chi^b(h) \Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q} + \frac{1}{2} - \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q} \Big) \Big) \\
& \times \Big(\frac{x^{1/(a+b)}}{q} - \frac{h}{q} + \frac{1}{2} - \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{h}{q} \Big) \Big) \\
= & - \sum_{j \leqslant q} \chi^b(j) \sum_{n \leqslant x^{1/(a+b)}} \chi^a(n) \Big(\frac{j}{q} + \psi\Big(\frac{x^{1/b}}{qn^{a/b}} - \frac{j}{q} \Big) \Big) \\
& - \sum_{j \leqslant q} \chi^a(j) \sum_{m \leqslant x^{1/(a+b)}} \chi^b(m) \Big(\frac{j}{q} + \psi\Big(\frac{x^{1/a}}{qm^{b/a}} - \frac{j}{q} \Big) \Big) \\
& - \sum_{j \leqslant q} \sum_{h \leqslant q} \chi^a(j) \chi^b(h) \Big(\frac{j}{q} + \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{j}{q} \Big) \Big) \Big(\frac{h}{q} + \psi\Big(\frac{x^{1/(a+b)}}{q} - \frac{h}{q} \Big) \Big).
\end{aligned}
$$

In view of (2.2) of Lemma 2.3, we have

$$
\begin{aligned}
T_{a,b}(x,\chi) = & - \sum_{j \leqslant q} \sum_{h \leqslant q} \chi^b(j) \chi^a(h) \sum_{\substack{n \leqslant x^{1/(a+b)} \\ n \equiv h \pmod q}} \psi\Big(\frac{x^{1/(a+b)}}{qn^{a/b}} - \frac{j}{q} \Big) \\
& - \sum_{j \leqslant q} \sum_{h \leqslant q} \chi^a(j) \chi^b(h) \sum_{\substack{m \leqslant x^{1/(a+b)} \\ m \equiv j \pmod q}} \psi\Big(\frac{x^{1/(a+b)}}{qm^{b/a}} - \frac{h}{q} \Big) + O(q^2) \\
=: & -S_5 - S_6 + O(q^2).
\end{aligned}
$$

If we use Lemma 2.6 with the exponent pair $(\frac{2}{7}, \frac{4}{7})$, then, for $q \leqslant x^{1/(a+b)}$, we have

$$
\begin{aligned}
S_5 = & \sum_{j \leqslant q} \sum_{h \leqslant q} \chi^b(j) \chi^a(h) R\Big(\frac{x^{1/b}}{q}, x^{1/(a+b)}, \frac{a}{b}, q, h, \frac{-j}{q} \Big) \\
\ll & \sum_{j \leqslant q} \sum_{h \leqslant q} (x^{1/(2a+2b)} q^{-1/2} + x^{2/(3a+3b)} q^{-2/3}) \\
= & O(x^{2/(3a+3b)} q^{4/3})
\end{aligned}
$$

and similarly

$$
S_6 = O(x^{2/(3a+3b)} q^{4/3}).
$$

Thus,

$$T_{a,b}(x,\chi) = O(x^{2/(3a+3b)}q^{4/3}). \tag{3.15}$$

Considering (3.8), (3.12), (3.13), (3.14) and (3.15), we deduce that

$$T_{a,b}(x,\chi) = \begin{cases} \frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi_0\Big)x^{1/a} + \frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi_0\Big)x^{1/b} \\ \qquad\qquad + O(x^{1/(2b)}q^{1/3+\varepsilon}) & \text{for } \chi = \chi_0, \\ \frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi^b\Big)x^{1/a} + O(x^{1/(2b)}q^{4/3+\varepsilon}) & \text{for } \chi \in A_\chi - B_\chi, \\ \frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi^a\Big)x^{1/b} + O(x^{1/(2b)}q^{4/3+\varepsilon}) & \text{for } \chi \in B_\chi - A_\chi, \\ \frac{\varphi(q)}{q}L\Big(\frac{b}{a},\chi_0\Big)x^{1/a} + \frac{\varphi(q)}{q}L\Big(\frac{a}{b},\chi_0\Big)x^{1/b} \\ \qquad\qquad + O(x^{1/(2b)}q^{1/3+\varepsilon}) & \text{for } \chi \in A_\chi \cap B_\chi, \\ O(x^{1/(2b)}q^{4/3}) & \text{for } \chi \notin A_\chi \cup B_\chi. \end{cases}$$

We return to the asymptotics for (3.4). In view of (3.8), we obtain the asymptotic formula for the first case $\chi = \chi_0$,

$$\begin{aligned} C_{a,b}(x,\chi_0) &= \sum_{d \leqslant x^{1/(2b)}} \mu(d)\chi_0^{2b}(d)T_{a,b}(xd^{-2b},\chi_0) \\ &= \frac{\varphi(q)}{q}\frac{L(b/a,\chi_0)}{L(2b/a,\chi_0)}x^{1/a} + \frac{\varphi(q)}{q}\frac{L(a/b,\chi_0)}{L(2,\chi_0)}x^{1/b} + O(x^{1/(2b)}q^{1/3+\varepsilon}). \end{aligned} \tag{3.16}$$

For other non-principal characters, we proceed in a similar way. For $q \leqslant x^{1/(a+b)}$, we have

$$C_{a,b}(x,\chi) = \begin{cases} \frac{\varphi(q)}{q}\frac{L(b/a,\chi^b)}{L(2b/a,\chi^{2b})}x^{1/a} + O(x^{1/(2b)}q^{4/3+\varepsilon}) & \text{for } \chi \in A_\chi - B_\chi, \\ \frac{\varphi(q)}{q}\frac{L(a/b,\chi^a)}{L(2,\chi^{2b})}x^{1/b} + O(x^{1/(2b)}q^{4/3+\varepsilon}) & \text{for } \chi \in B_\chi - A_\chi, \\ \frac{\varphi(q)}{q}\frac{L(b/a,\chi_0)}{L\Big(2b/a,\chi^{2b})}x^{1/a} + \frac{\varphi(q)}{q}\frac{L(a/b,\chi_0)}{L(2,\chi^{2b})}x^{1/b} \\ \qquad\qquad + O(x^{1/(2b)}q^{1/3+\varepsilon}) & \text{for } \chi \in A_\chi \cap B_\chi, \\ O(x^{1/(2b)}q^{4/3}) & \text{for } \chi \notin A_\chi \cup B_\chi. \end{cases} \tag{3.17}$$

Since $a < b < 2a$, $\frac{1}{4} < a/(2b) < \frac{1}{2}$ and $1 < b/a < 2$, then $(2b^2 - ab - a^2)/(2ab) < \frac{4}{3}$. Thus, $O(x^{1/(2b)}q^{4/3})$ dominates the bound of U_2 in (3.3) and Theorem 1.1 follows from (3.1), (3.16) and (3.17). □

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