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n-gr-COHERENT RINGS AND GORENSTEIN GRADED MODULES

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Abstract. Let R be a graded ring and $n \geqslant 1$ be an integer. We introduce and study the notions of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules by using the notion of special finitely presented graded modules. On n-gr-coherent rings, we investigate the relationships between Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules. Among other results, we prove that any graded module in R-gr (or gr-R) admits a Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat) cover and preenvelope, respectively.

Keywords: n-gr-coherent ring; Gorenstein n-FP-gr-injective module; Gorenstein n-gr-flat module; cover; (pre)envelope

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1. Introduction

In the 1990s, Enochs, Jenda and Torrecillas introduced the concepts of Gorenstein injective and Gorenstein flat modules over arbitrary rings, see [14], [16]. In 2008, Mao and Ding introduced a special case of the Gorenstein injective modules that they called Gorenstein FP-injective modules, which were renamed by Ding and by Gillespie as injective, see [21]. These Gorenstein FP-injective modules are stronger than the Gorenstein injective modules, and in general an FP-injective module is not necessarily Gorenstein FP-injective, see [26], Proposition 2.7. For this reason, Gao and Wang introduced and studied in [19] another notion called *Gorenstein* FP*-injective modules* which is weaker than the usual Gorenstein injective modules. Furthermore, all FP-injective modules are in the class of Gorenstein FP-injective modules (see Section 2 for the definitions of these notions).

Here we deal with the graded aspect of some extensions of these notions. As is known, graded rings and modules are classical notions in algebra which build their values and strengths from their connection with algebraic geometry (see for instance [28], [29], [30]). Several authors have investegated the graded aspect of some

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notions in relative homological algebra. For example, Asensio, López Ramos and Torrecillas in [6], [5] introduced the notions of Gorenstein gr-projective, gr-injective and gr-flat modules. In recent years, the Gorenstein homological theory for graded rings has become an important area of research, see for instance [9], [18]. The notions of FP-gr-injective modules was introduced in [8], and in [34] the homological behavior of the FP-gr-injective modules on gr-coherent rings was investigated. Along the same lines, it is natural to generalize the notion of "FP-gr-injective modules and gr-flat modules" to "n-FP-gr-injective modules and n-gr-flat modules". This done by Zhao, Gao and Huang in [35] based on the notion of special finitely presented graded modules which they defined via projective resolutions of n-presented graded modules. Recently, in 2017, Mao gave a definition of Ding gr-injective modules via FP-gr-injective modules, see [25]. Under this definition, these Ding gr-injective modules are stronger than the Gorenstein gr-injective modules, and an FP-gr-injective module is not necessarily Ding gr-injective in general, see [25], Corollary 3.7. So, for any $n \geqslant 1$, we study the consequences of extending the notion of n-FP-gr-injective and n-gr-flat modules to that of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules, respectively. Then for any $n \geqslant 1$ here by using n-FP-gr-injective modules and n-gr-flat modules, we introduce the concept of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules. Under this definition, Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules are weaker than the usual Gorenstein gr-injective and Gorenstein gr-flat modules, respectively. Also, for any $n \geqslant 1$, all gr-injective, n-FP-gr-injective modules and gr-flat, n-gr-flat modules are Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat, respectively, and in general, Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat R-modules need not be n-FP-gr-injective and n-gr-flat, unless in certain cases, see Proposition 3.18.

The paper is organized as follows:

In Section 2, some fundamental concepts and some preliminary results are stated.

In Section 3, we introduce Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules for an integer $n \geqslant 1$ and then we give some characterizations of these modules. Among other results, we prove that, for an exact sequence $0 \to A \to B \to C \to 0$ of graded left R-modules, if A and B are Gorenstein n-FP-gr-injective, then C is Gorenstein n-FP-gr-injective if and only if every n-presented module in R-gr with $\text{gr-pd}_R(U) < \infty$ is $(n+1)$-presented, and it follows that $({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is a hereditary cotorsion pair if and only if every n-presented module in R-gr with $\text{gr-pd}_R(U) < \infty$ is $(n+1)$-presented and every $M \in ({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp}$ has an exact left $(\text{gr-}\mathcal{FI}_n)$-resolution, where $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ and $\text{gr-}\mathcal{FI}_n$ denote the classes of Gorenstein n-FP-gr-injective and n-FP-gr-injective modules in R-gr, respectively. Also, for a graded left (or right) R-module M over a left n-gr-coherent ring R: M is Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat) if and only if M^* is Gorenstein

n-gr-flat (or Gorenstein n-FP-gr-injective). Furthermore, the class of Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat) modules are closed under direct limits (or direct products). In this section, examples are given in order to show that Gorenstein m-FP-gr-injectivity (or Gorenstein m-gr-flatness) does not imply Gorenstein n-FP-gr-injectivity (or Gorenstein n-gr-flatness) for any $m > n$. Also, examples are given showing that Gorenstein n-FP-gr-injectivity does not imply gr-injectivity. In this paper, gr-$\mathcal{I}$ denotes the classes of gr-injective modules in R-gr and gr-$\mathcal{F}$, gr-$\mathcal{F}_n$ and $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ denotes the classes of gr-flat, n-gr-flat and Gorenstein n-gr-flat modules in gr-R, respectively.

In Section 4, it is shown that the class of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules cover and preenvelop on n-gr-coherent rings. We also establish some equivalent characterizations of n-gr-coherent rings in terms of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules.

2. Preliminaries

Throughout this paper, all rings considered are associative with identity element and the R-modules are unital. By R-Mod and Mod-R we will denote the category of all left R-modules and right R-modules, respectively.

In this section, some fundamental concepts and notations are stated.

Let n be a nonnegative integer and M a left R-module. Then, a module M is said to be *Gorenstein injective* (or *Gorenstein flat*) (see [14], [16]) if there is an exact sequence

$$\ldots \to I_1 \to I_0 \to I^0 \to I^1 \to \ldots$$

of injective (or flat) left R-modules with $M = \ker(I^0 \to I^1)$ such that $\mathrm{Hom}_R(U, -)$ (or $U \otimes_R -$) leaves the sequence exact whenever U is an injective left (or right) R-module.

A module M is said to be *n-presented* (see [12], [13]) if there is an exact sequence

$$F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to U \to 0$$

of left R-modules, where each F_i is finitely generated free, and a ring R is called *left n-coherent* if every n-presented left R-module is $(n+1)$-presented. A module M is said to be n-FP-*injective* (see [11]) if $\mathrm{Ext}^n_R(U, M) = 0$ for any n-presented left R-module U. In the case, where $n = 1$, n-FP-injective modules are nothing but the well-known FP-injective modules. A right module N is called *n-flat* if $\mathrm{Tor}^R_n(N, U) = 0$ for any n-presented left R-module U.

A module M is said to be *Gorenstein* FP*-injective* (see [26]) if there is an exact sequence

$$\mathbf{E} = \ldots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots$$

of injective left modules with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_R(U, \mathbf{E})$ is an exact sequence whenever U is an FP-injective left R-module. Then, in [19], Gao and Wang introduced another concept of Gorenstein FP-injective modules as follows: a module M is said to be Gorenstein FP-injective (see [19]) if there is an exact sequence

$$\mathbf{E} = \ldots \to E_1 \to E_0 \to E^0 \to E^1 \to \ldots$$

of FP-injective left modules with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_R(P, \mathbf{E})$ is an exact sequence whenever P is a finitely presented module with $\mathrm{pd}_R(P) < \infty$.

Let G be a multiplicative group with a neutral element e. A graded ring R is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus, Re is a subring of R, $1 \in Re$ and R_σ is an Re-bimodule for every $\sigma \in G$. A *graded left* (or *right*) R*-module* is a left (or right) R-module M endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each M_σ is a subgroup of the additive group of M such that $R_\sigma M_\tau \subseteq M_{\sigma\tau}$ for all $\sigma, \tau \in G$. For any graded left R-modules M and N, the set $\mathrm{Hom}_{R\text{-gr}}(M, N) := \{f\colon M \to N \mid f \text{ is } R\text{-linear and } f(M_\sigma) \subseteq N_\sigma \text{ for any } \sigma \in G\}$, which is the group of all morphisms from M to N in the class R-gr of all graded left R-modules (gr-R will denote the class of all graded right R-modules). It is well known that R-gr is a Grothendieck category. An R-linear map $f\colon M \to N$ is said to be a *graded morphism of degree* τ with $\tau \in G$ if $f(M_\sigma) \subseteq N_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree σ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type G. We will denote by $\mathrm{Ext}^i_{R\text{-gr}}$ and EXT^i_R the right derived functors of $\mathrm{Hom}_{R\text{-gr}}$ and HOM_R, respectively. Given a graded left R-module M, the *graded character module* of M is defined as $M^* := \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the rational numbers field and $\mathbb{Z}$ is the integers ring. It is easy to see that $M^* = \bigoplus_{\sigma \in G} \mathrm{HOM}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let M be a graded right R-module and N a graded left R-module. The abelian group $M \otimes_R N$ may be graded by putting $(M \otimes_R N)_\sigma$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \otimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the *graded tensor product* of M and N.

If M is a graded left R-module and $\sigma \in G$, then $M(\sigma)$ is the graded left R-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the σ*-suspension* of M. We may regard the σ-suspension as an isomorphism of categories $T_\sigma\colon R\text{-gr} \to R\text{-gr}$, given on objects as $T_\sigma(M) = M(\sigma)$ for any $M \in R\text{-gr}$. The forgetful functor $U\colon R\text{-gr} \to R\text{-Mod}$ associates to M, the underlying ungraded R-module. This functor has a right adjoint F which associate to $M \in R\text{-Mod}$ the graded R-module $F(M) = \bigoplus_{\sigma\in G} ({}^\sigma M)$, where each ${}^\sigma M$ is a copy of M written $\{{}^\sigma x\colon x \in M\}$ with R-module structure defined by $r{*}{}^\tau x = {}^{\sigma\tau}(rx)$ for each $r \in R_\sigma$. If $f\colon M \to N$ is R-linear, then $F(f)\colon F(M) \to F(N)$ is a graded morphism given by $F(f)({}^\sigma x) = {}^\sigma f(x)$.

The injective (or flat) objects of R-gr(or gr-R) will be called gr-*injective* (or gr-*flat*) modules, because M is gr-injective (or gr-flat) if and only if it is a injective (or flat) graded module. By gr-$\mathrm{pd}_R(M)$ and gr-$\mathrm{fd}_R(M)$ we will denote the gr-projective and gr-flat dimension of a graded module M, respectively. A graded left (or right) module M is said to be *Gorenstein* gr-*injective* (or *Gorenstein* gr-*flat*) (see [5], [6], [9]) if there is an exact sequence

$$\ldots \to I_1 \to I_0 \to I^0 \to I^1 \to \ldots$$

of gr-injective (or gr-flat) left (or right) modules with $M = \ker(I^0 \to I^1)$ such that $\mathrm{Hom}_{R\text{-gr}}(E, -)$ (or $- \otimes_R E$) leaves the sequence exact whenever E is a gr-injective R-module. The gr-injective envelope of M is denoted by $E^g(M)$. A graded left module M is said to be *Ding* gr-*injective* (see [25]) if there is an exact sequence $\ldots \to I_1 \to I_0 \to I^0 \to I^1 \to \ldots$ of gr-injective left modules, with $M = \ker(I^0 \to I^1)$ such that $\mathrm{Hom}_{R\text{-gr}}(E, -)$ leaves the sequence exact whenever E is an FP-gr-injective left R-module.

Definition 2.1 ([35], Definition 3.1)**.** Let $n \geqslant 0$ be an integer. Then, a graded left module U is called *n-presented* if there exists an exact sequence $F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to U \to 0$ in R-gr, where each F_i is a finitely generated free left R-module.

Set $K_{n-1} = \mathrm{Im}(F_{n-1} \to F_{n-2})$ and $K_n = \mathrm{Im}(F_n \to F_{n-1})$. Then we get a short exact sequence $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ in R-gr, where F_{n-1} is a finitely generated free module. The modules K_n and K_{n-1} will be called *special finitely* gr-*generated* and special finitely gr-presented, respectively. The sequence $(\Delta)\colon 0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ in R-gr will be called a *special short exact sequence.*

Moreover, a short exact sequence $0 \to A \to B \to C \to 0$ in R-gr is called *special* gr-*pure* if the induced sequence

$$0 \to \mathrm{HOM}_R(K_{n-1}, A) \to \mathrm{HOM}_R(K_{n-1}, B) \to \mathrm{HOM}_R(K_{n-1}, C) \to 0$$

is exact for every special finitely gr-presented module K_{n-1}. In this case A is said to be *special* gr-*pure* in B.

Analogously to the classical case, a graded ring R is called *left* n-gr-*coherent* if each n-presented module in R-gr is $(n+1)$-presented.

Ungraded n-presented modules have been used by many authors in order to extend some homological notions. For example, in [10], let R be an associative ring and M be a left R-module. Then module M is called FP_n*-injective* if $\mathrm{Ext}^1_R(L, M) = 0$ for all n-presented left R-modules L. In 2018, Zhao, Gao and Huang in [35] showed that if we similarly use the derived functor EXT^1 to define the FP_n-gr-injective and FP_∞-gr-injective modules, then they are just the FP_n-injective and FP_∞-injective objects in the class of graded modules, respectively. If L is an n-presented graded left R-module with $n \geqslant 2$, then $\mathrm{EXT}^1_R(L, M) = \mathrm{Ext}^1_R(L, M)$ for any graded R-module M. For this reason, they introduced the concept of n-FP-gr-injective modules as follows: A graded left R-module M is called n-FP-gr-*injective* (see [35]) if $\mathrm{EXT}^n_R(N, M) = 0$ for any finitely n-presented graded left R-module N. If $n = 1$, then M is FP-gr-*injective*. A graded right R-module M is called n-gr-*flat* (see [35]) if $\mathrm{Tor}^n_R(M, N) = 0$ for any finitely n-presented graded left R-module N.

If U is an n-presented graded left R-module and $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ is a special short exact sequence in R-gr with respect to U, then $\mathrm{EXT}^n_R(U, M) \cong \mathrm{EXT}^1_R(K_{n-1}, M)$ for any graded left R-module M and $\mathrm{Tor}^R_n(M, U) \cong \mathrm{Tor}^R_1(M, K_{n-1})$ for any graded right R-module M. The n-FP-gr-injective dimension of a graded left R-module M, denoted by $n\text{-FP-gr-id}_R(M)$, is defined to be the least integer k such that $\mathrm{EXT}^{k+1}_R(K_{n-1}, M) = 0$ for any special gr-presented module K_{n-1} in R-gr. The n-gr-flat dimension of a graded right R-module M, denoted by $n\text{-gr-fd}_R(M)$, is defined to be the least integer k such that $\mathrm{Tor}^R_{k+1}(M, K_{n-1}) = 0$ for any special gr-presented module K_{n-1} in R-gr. Also,

$$\mathrm{l}.n\text{-FP-gr-}\dim(R) = \sup\{n\text{-FP-gr-id}_R(M)\colon\ M \text{ is a graded left module}\}$$

and

$$\mathrm{r}.n\text{-gr-}\dim(R) = \sup\{n\text{-gr-fd}_R(M)\colon\ M \text{ is a graded right module}\}.$$

3. Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules

In this section, we introduce and study Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules which are defined as follows:

Definition 3.1. Let R be a graded ring and let $n \geqslant 1$ an integer. Then, a module M in R-gr is called *Gorenstein* n-FP-gr-*injective* if there exists an exact sequence of n-FP-gr-injective modules in R-gr of this form:

$$\mathbf{A} = \ldots \to A_1 \to A_0 \to A^0 \to A^1 \to \ldots$$

with $M = \ker(A^0 \to A^1)$ such that $\mathrm{HOM}_R(K_{n-1}, \mathbf{A})$ is an exact sequence whenever K_{n-1} is a special gr-presented module in R-gr with gr-pd$_R(K_{n-1}) < \infty$.

The class of Gorenstein n-FP-gr-injective will be denoted $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$. A module N in gr-R is called *Gorenstein* n-gr-*flat* if there exists the following exact sequence of n-gr-flat modules in gr-R of this form:

$$\mathbf{F} = \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

with $N = \ker(F^0 \to F^1)$ such that $\mathbf{F} \otimes_R K_{n-1}$ is an exact sequence whenever K_{n-1} is a special gr-presented module in R-gr with gr-fd$_R(K_{n-1}) < \infty$.

The class of Gorenstein n-FP-gr-flat will be denoted $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$.

In the ungraded case, the R-modules A_i and A^i (or F_i and F^i) as in the definition above are called n-FP-*injective* (or n-*flat*). Also, R-modules M and N are called *Gorenstein* n-FP-*injective* and *Gorenstein* n-*flat*, respectively, and K_{n-1} is a special presented left module with respect to any n-presented left R-module U.

Remark 3.2. Let R be a graded ring. Then:

(1) gr-$\mathcal{I} \subseteq$ gr-$\mathcal{FI}_1 \subseteq$ gr-$\mathcal{FI}_2 \subseteq \ldots \subseteq$ gr-$\mathcal{FI}_n \subseteq \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$. But, Gorenstein n-FP-gr-injective R-modules need not be gr-injective, see Example 3.3 (1). Also, gr-$\mathcal{F} \subseteq$ gr-$\mathcal{F}_1 \subseteq$ gr-$\mathcal{F}_2 \subseteq \ldots \subseteq$ gr-$\mathcal{F}_n \subseteq \mathcal{G}_{\text{gr-}\mathcal{F}_n}$. In general, every Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat) R-module is not n-FP-gr-injective (or n-gr-flat), except in a certain state, see Proposition 3.18.

(2) $\mathcal{G}_{\text{gr-}\mathcal{FI}_1} \subseteq \mathcal{G}_{\text{gr-}\mathcal{FI}_2} \subseteq \ldots \subseteq \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ and $\mathcal{G}_{\text{gr-}\mathcal{F}_1} \subseteq \mathcal{G}_{\text{gr-}\mathcal{F}_2} \subseteq \ldots \subseteq \mathcal{G}_{\text{gr-}\mathcal{F}_n}$. But for any integers $m > n$, Gorenstein m-FP-gr-injective (or Gorenstein m-gr-flat) R-modules need not be Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat), see Example 3.3 (2), (3).

(3) In Definition 3.1, it is clear that $\ker(A_i \to A_{i-1})$ and $\ker(A^i \to A^{i+1})$ are Gorenstein n-FP-gr-injective, and $\ker(F_i \to F_{i-1})$, $\ker(F^i \to F^{i+1})$ are Gorenstein n-gr-flat for any $i \geqslant 1$.

It is known that the trivial extension of a commutative ring A by an A-module M, $R = A \ltimes M$, is a $\mathbb{Z}_2$-graded ring, see [2], [3].

Example 3.3.

(1) Let K be a field with characteristic $p \neq 0$ and let $G = \bigcup_{k\geqslant 1} G_k$, where G_k is the cyclic group with generator a_k, the order of a_k is p^k and $a_k = a_{k+1}^p$. Let $R = K[G]$. Then, by Remark 3.2, $R[H]$ is Gorenstein n-FP-gr-injective for every group H, since by [8], Example (iii), $R[H]$ is n-FP-gr-injective but it is not gr-injective.

(2) Let A be a field, E a nonzero A-vector space and $R = A \ltimes E$ be a trivial extension of A by E. If $\dim_A E = 1$, then by Remark 3.2, every R-module in R-gr is Gorenstein n-FP-gr-injective, see [1], Corollary 2.2. If E is an A-vector space with infinite rank, then by [24], Theorem 3.4, every 2-presented module in R-gr is projective. So, every module in R-gr is 2-FP-gr-injective and hence, every module in R-gr is Gorenstein 2-FP-gr-injective. If every module in R-gr is Gorenstein 1-FP-gr-injective, then R is gr-regular, a contradiction.

(3) Let $R = k[X]$, where k is a field. Then, by Theorem 3.16, every graded right R-module is Gorenstein 2-gr-flat, and there is a graded right R-module that is not Gorenstein 1-gr-flat, since $\mathrm{l.FP\text{-}gr\text{-}dim}(R) \leqslant 1$, see Proposition 3.18 and [35], Example 3.6.

We start with the result which proves that the behaviour of Gorenstein n-FP-gr-injective (or Gorenstein n-gr-flat) modules in short exact sequences is the same as the one of the classical homological notions.

Proposition 3.4. *Let R be a graded ring. Then:*

(1) *For every short exact sequence $0 \to A \to B \to C \to 0$ in R-gr, B is Gorenstein n-FP-gr-injective if A and C are Gorenstein n-FP-gr-injective.*

(2) *For every short exact sequence $0 \to A \to B \to C \to 0$ in gr-R, B is Gorenstein n-gr-flat if A and C are Gorenstein n-gr-flat.*

P r o o f. (1) By Definition 3.1, there is an exact sequence $\ldots \to A_1 \to A_0 \to A^0 \to A^1 \to \ldots$ of n-FP-gr-injective modules in R-gr, where $A = \mathrm{Ker}(A^0 \to A^1)$, $K_i' = \mathrm{Ker}(A_i \to A_{i-1})$ and $(K^i)' = \mathrm{Ker}(A^i \to A^{i+1})$. Also, there is an exact sequence $\ldots \to C_1 \to C_0 \to C^0 \to C^1 \to \ldots$ of n-FP-gr-injective modules in R-gr, where $C = \mathrm{Ker}(C^0 \to C^1)$, $K_i'' = \mathrm{Ker}(C_i \to C_{i-1})$ and $(K^i)'' = \mathrm{Ker}(C^i \to C^{i+1})$. For any n-presented graded left module P, $\mathrm{EXT}_R^n(P, A_i \oplus C_i) = \mathrm{EXT}_R^n(P, A_i) \oplus \mathrm{EXT}_R^n(P, C_i) = 0$, then $A_i \oplus C_i$ is n-FP-gr-injective for any $i \geqslant 0$. Similarly, $A^i \oplus C^i$ is n-FP-gr-injective for any $i \geqslant 0$. Therefore, there is an exact sequence

$$\mathscr{Y} = \ldots \to A_1 \oplus C_1 \to A_0 \oplus C_0 \to A^0 \oplus C^0 \to A^1 \oplus C^1 \to \ldots$$

of n-FP-gr-injective modules in R-gr, where $B = \mathrm{Ker}(A^0 \oplus C^0 \to A^1 \oplus C^1)$, $K_i = K_i' \oplus K_i'' = \mathrm{Ker}(A_i \oplus C_i \to A_{i-1} \oplus C_{i-1})$ and $K^i = (K^i)' \oplus (K^i)'' = \mathrm{Ker}(A^i \oplus C^i \to A^{i+1} \oplus C^{i+1})$. Let K_{n-1} be a special gr-presented module in R-gr with $\mathrm{gr\text{-}pd}_R(K_{n-1}) < \infty$. Then $\mathrm{EXT}_R^1(K_{n-1}, B) = 0$, and also we have: $\mathrm{EXT}_R^1(K_{n-1}, K_i) = \mathrm{EXT}_R^1(K_{n-1}, K_i' \oplus K_i'') = 0$. Similarly, $\mathrm{EXT}_R^1(K_{n-1}, K^i) = 0$. Consequently, $\mathrm{HOM}_R(K_{n-1}, \mathscr{Y})$ is exact and so B is Gorenstein n-FP-gr-injective.

(2) By Definition 3.1, there is an exact sequence $\ldots \to A_1 \to A_0 \to A^0 \to A^1 \to \ldots$ of n-gr-flat modules in gr-R, where $A = \mathrm{Ker}(A^0 \to A^1)$, $K_i' = \mathrm{Ker}(A_i \to A_{i-1})$ and

$(K^i)' = \mathrm{Ker}(A^i \to A^{i+1})$. Also, there is an exact sequence $\ldots \to C_1 \to C_0 \to C^0 \to C^1 \to \ldots$ of n-gr-flat modules in gr-R, where $C = \mathrm{Ker}(C^0 \to C^1)$, $K_i'' = \mathrm{Ker}(C_i \to C_{i-1})$ and $(K^i)'' = \mathrm{Ker}(C^i \to C^{i+1})$. Similarly to (1), there is an exact sequence

$$\mathscr{Y} = \ldots \to A_1 \oplus C_1 \to A_0 \oplus C_0 \to A^0 \oplus C^0 \to A^1 \oplus C^1 \to \ldots$$

of n-gr-flat modules in gr-R, where $B = \mathrm{Ker}(A^0 \oplus C^0 \to A^1 \oplus C^1)$, and if K_{n-1} is a special gr-presented module in R-gr with $\text{gr-fd}_R(K_{n-1}) < \infty$, then $\mathscr{Y} \otimes_R K_{n-1}$ is exact and so B is Gorenstein n-gr-flat. □

Transfer results of n-FP-injective and Gorenstein n-FP-injective modules with respect to the functor F are given in the following result.

Proposition 3.5. *Let R be a ring graded by a group G.*

(1) *If M is an n-FP-injective left R-module, then $F(M)$ is n-FP-gr-injective.*

(2) *If M is a Gorenstein n-FP-injective left R-module, then $F(M)$ is Gorenstein n-FP-gr-injective.*

P r o o f. (1) If $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ is special short exact sequence in R-gr with respect to an n-presented graded left R-module U, then similar to the proof of [34], Lemma 2.3, $0 = \mathrm{Ext}^1_{R\text{-gr}}(K_{n-1}, F(M)(\sigma)) = \mathrm{Ext}^n_{R\text{-gr}}(U, F(M)(\sigma))$, and hence by [35], Proposition 3.10, $F(M)$ is n-FP-gr-injective.

(2) Let M be a Gorenstein n-FP-injective left R-module. Then, there exists an exact sequence of n-FP-injective left modules:

$$\mathbf{B} = \ldots \to B_1 \to B_0 \to B^0 \to B^1 \to \ldots$$

with $M = \ker(B^0 \to B^1)$ such that $\mathrm{Hom}_R(K'_{n-1}, \mathbf{B})$ is an exact sequence whenever K'_{n-1} is a special finitely presented module in R-gr with $\mathrm{pd}_R(K'_{n-1}) < \infty$. By (1), $F(B_i)$ and $F(B^i)$ are n-FP-gr-injective for any $i \geqslant 0$. Since the functor F is exact, we get the following exact sequence:

$$\mathbf{F}(\mathbf{B}) = \ldots \to F(B_1) \to F(B_0) \to F(B^0) \to F(B^1) \to \ldots$$

of n-FP-gr-injective left R-modules with $F(M) = \ker(F(B^0) \to F(B^1))$. If K_{n-1} is a special gr-presented left module with $\text{gr-pd}_R(K_{n-1}) < \infty$, then $U(K_{n-1})$ is finitely presented with $\mathrm{pd}_R(U(K_{n-1})) < \infty$. By hypothesis, $\mathrm{Hom}_R(U(K_{n-1}), \mathbf{B})$ is exact. Therefore, from $\mathrm{Hom}_R(U(K_{n-1}), \mathbf{B}) = \mathrm{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B}))$, it follows that $\mathrm{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B}))$ is exact and consequently the isomorphism

$$\mathrm{HOM}_R(K_{n-1}, \mathbf{F}(\mathbf{B})) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(K_{n-1}, \mathbf{F}(\mathbf{B}))_\sigma \cong \bigoplus_{\sigma \in G} \mathrm{Hom}_{R\text{-gr}}(K_{n-1}, \mathbf{F}(\mathbf{B})(\sigma))$$

implies that $F(M)$ is Gorenstein n-FP-gr-injective. □

Now, we give a characterization of a graded ring R on which n-presented modules in R-gr with gr-pd$_R(U) < \infty$ (or gr-fd$_R(U) < \infty$) are $(n+1)$-presented. For this, we need the following lemma.

Lemma 3.6. *Assume that every n-presented module in R-gr with gr-fd$_R(U) < \infty$ is $(n+1)$-presented. Then for any $t \geqslant 1$:*

(1) $\mathrm{EXT}^t_R(K_{n-1}, M) = 0$ *for any Gorenstein n-FP-gr-injective left R-module M and any special gr-presented left R-module K_{n-1} with gr-pd$_R(K_{n-1}) < \infty$.*

(2) $\mathrm{Tor}^R_t(M, K_{n-1}) = 0$ *for any Gorenstein n-gr-flat right R-module M and any special gr-presented left R-module K_{n-1} with gr-fd$_R(K_{n-1}) < \infty$.*

Proof. (1) Assume that K_{n-1} is a special gr-presented module in R-gr with gr-pd$_R(K_{n-1}) \leqslant m$ respect to any n-presented module U in R-gr. If M is a Gorenstein n-FP-gr-injective left R-module, then, there is a left n-FP-gr-injective resolution of M in R-gr. So, we have:

$$0 \to N \to E_{m-1} \to \ldots \to E_0 \to M \to 0,$$

where every E_j is n-FP-gr-injective for every $0 \leqslant j \leqslant m-1$. Since gr-fd$_R(U) < \infty$, U is $(n+1)$-presented, and so $\mathrm{EXT}^{i+1}_R(K_{n-1}, E_j) = 0$ for any $i \geqslant 0$. Hence, $\mathrm{EXT}^{i+1}_R(K_{n-1}, M) \cong \mathrm{EXT}^{m+i+1}_R(K_{n-1}, N)$, and since gr-pd$_R(K_{n-1}) \leqslant m$, it follows that, $\mathrm{EXT}^{i+1}_R(K_{n-1}, M) = 0$ for any $i \geqslant 0$.

(2) Let K_{n-1} be a special gr-presented module in R-gr with gr-fd$_R(K_{n-1}) \leqslant m$ for any n-presented module U in R-gr. If M is a Gorenstein n-gr-flat right R-module, then there is a right n-gr-flat resolution of M in gr-R of the form:

$$0 \to M \to F^0 \to \ldots \to F^{m-1} \to N \to 0,$$

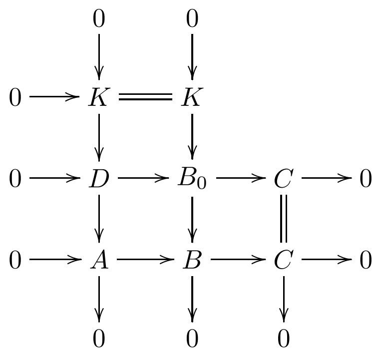
where every F^j is n-gr-flat for every $0 \leqslant j \leqslant m-1$. Since U is $(n+1)$-presented, we have $\mathrm{Tor}^R_{i+1}(F^j, K_{n-1}) = 0$ for any $i \geqslant 0$. If gr-fd$_R(K_{n-1}) \leqslant m$, then $\mathrm{Tor}^R_{i+1}(M, K_{n-1}) \cong \mathrm{Tor}^R_{m+i+1}(N, K_{n-1}) = 0$, and so $\mathrm{Tor}^R_{i+1}(M, K_{n-1}) = 0$ for any $i \geqslant 0$. □

Theorem 3.7. *Let R be a graded ring. Then the following statements are equivalent:*

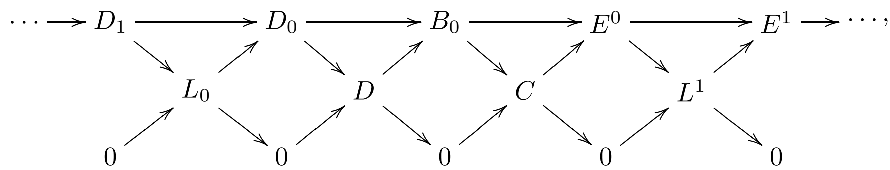
(1) *Every n-presented module in R-gr with gr-pd$_R(U) < \infty$ is $(n+1)$-presented.*

(2) *For every short exact sequence $0 \to A \to B \to C \to 0$ in R-gr, C is Gorenstein n-FP-gr-injective if A and B are Gorenstein n-FP-gr-injective.*

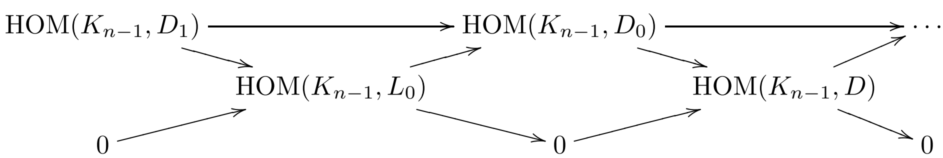
Proof. (1) ⇒ (2) If B is a Gorenstein n-FP-gr-injective module in R-gr, then by Definition 3.1 and Remark 3.2, there is an exact sequence $0 \to K \to B_0 \to B \to 0$ in R-gr, where B_0 is n-FP-gr-injective and K is Gorenstein n-FP-gr-injective. Consider that the following commutative diagram with exact rows exists:



By Proposition 3.4 (1), D is Gorenstein n-FP-gr-injective, and so we have a commutative diagram in R-gr:



where D_i and B_0 are n-FP-gr-injective, E^i is gr-injective, $C = \mathrm{Ker}(E^0 \to E^1)$, $D = \mathrm{Ker}(B_0 \to C)$, $L_i = \mathrm{Ker}(D_i \to D_{i-1})$ and $L^i = \mathrm{Ker}(E^i \to E^{i+1})$. By Remark 3.2, E^i and L_i are Gorenstein n-FP-gr-injective and hence by Lemma 3.6 (1), $\mathrm{EXT}^t_R(K_{n-1}, L_i) = \mathrm{EXT}^t_R(K_{n-1}, D) = 0$ for any special gr-presented K_{n-1} module in R-gr with $\text{gr-pd}_R(K_{n-1}) < \infty$ and any $t \geqslant 1$. Therefore, we have the following exact commutative diagram:



Hence, C is Gorenstein n-FP-gr-injective.

(2) ⇒ (1) Let U be an n-presented graded left R-module with $\text{gr-pd}_R(U) < \infty$, and let $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ be a special short exact sequence in R-gr with respect to U, where K_n is a special gr-generated module. We show that K_n is special gr-presented. Let M be a Gorenstein n-FP-injective module and $0 \to M \to E \to L \to 0$ an exact sequence in R-Mod, where E is injective. Then $0 \to F(M) \to$

$F(E) \to F(L) \to 0$ is exact, where $F(M)$ and $F(E)$ are Gorenstein n-FP-gr-injective in R-gr by Proposition 3.5. So by (2), we deduce that $F(L)$ is Gorenstein n-FP-gr-injective. We have:

$$0 = \operatorname{Ext}^1_{R\text{-gr}}(F_{n-1}, F(M)) \to \operatorname{Ext}^1_{R\text{-gr}}(K_n, F(M)) \to \operatorname{Ext}^2_{R\text{-gr}}(K_{n-1}, F(M)) \to 0.$$

So, $\operatorname{Ext}^1_{R\text{-gr}}(K_n, F(M)) \cong \operatorname{Ext}^2_{R\text{-gr}}(K_{n-1}, F(M))$. On the other hand,

$$0 = \operatorname{Ext}^1_{R\text{-gr}}(K_{n-1}, F(E)) \to \operatorname{Ext}^1_{R\text{-gr}}(K_{n-1}, F(L)) \to \operatorname{Ext}^2_{R\text{-gr}}(K_{n-1}, F(M)) \to 0.$$

Hence, $\operatorname{Ext}^1_{R\text{-gr}}(K_{n-1}, F(L)) \cong \operatorname{Ext}^2_{R\text{-gr}}(K_{n-1}, F(M))$. Since $F(L)$ is Gorenstein n-FP-gr injective, we get $0 = \operatorname{EXT}^1_R(K_{n-1}, F(L))_\sigma \cong \operatorname{Ext}^1_{R\text{-gr}}(K_{n-1}, F(L)(\sigma))$ for any $\sigma \in G$. This implies that $\operatorname{Ext}^1_{R\text{-gr}}(K_{n-1}, F(L)) = 0$ and consequently $\operatorname{Ext}^1_{R\text{-gr}}(K_n, F(M)) = 0$. So, the following commutative diagram exists:

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Hom}_{R\text{-gr}}(K_n, F(M)) & \to & \operatorname{Hom}_{R\text{-gr}}(K_n, F(E)) & \to & \operatorname{Hom}_{R\text{-gr}}(K_n, F(L)) & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \operatorname{Hom}_R(K_n, M) & \longrightarrow & \operatorname{Hom}_R(K_n, E) & \longrightarrow & \operatorname{Hom}_R(K_n, L) & &
\end{array}$$

So, $\operatorname{Ext}^1_{R\text{-gr}}(K_n, F(M)) \cong \operatorname{Ext}^1_R(K_n, M) = 0$ for any Gorenstein n-FP-injective left R-module M. Since every FP-injective left module is Gorenstein n-FP-injective, $\operatorname{Ext}^1_{R\text{-gr}}(K_n, F(N)) \cong \operatorname{Ext}^1_R(K_n, N) = 0$ for any FP-injective left module N and so K_n is 1-presented. Therefore, U is $(n+1)$-presented in R-gr. □

Corollary 3.8. *Let every n-presented module in R-gr with $\text{gr-pd}_R(U) < \infty$ be $(n+1)$-presented. Then a module M in R-gr is Gorenstein n-FP-gr-injective if and only if every gr-pure submodule and any gr-pure epimorphic image of M are Gorenstein n-FP-gr-injective.*

Proof. ($\Rightarrow$) Let M be a Gorenstein n-FP-gr-injective module in R-gr. If the exact sequence $0 \to K \to M \to M/K \to 0$ is gr-pure, then by [8], Proposition 2.2, $\operatorname{EXT}^1_R(K_{n-1}, K) = 0$ for every special gr-presented module K_{n-1} in R-gr. So, we have $0 = \operatorname{EXT}^1_R(K_{n-1}, K) \cong \operatorname{EXT}^n_R(U, K)$ for any n-presented module U in R-gr. Thus, K is n-FP-gr-injective, and hence K is Gorenstein n-FP-gr-injective by Remark 3.2. Therefore, by Theorem 3.7, M/K is Gorenstein n-FP-gr-injective.

($\Leftarrow$) Assume that the exact sequence $0 \to K \to M \to L \to 0$ in R-gr is gr-pure, where L and K are Gorenstein n-FP-gr-injective. Then, by Proposition 3.4 (1), M is Gorenstein n-FP-gr-injective. □

The following definition is the graded version of [15], [4].

Definition 3.9. Let $\eth$ be a class of graded left R-module. Then:
(1) $\eth^\perp = \operatorname{KerExt}^1_{R\text{-gr}}(\eth, -) = \{C\colon \operatorname{Ext}^1_{R\text{-gr}}(L, C) = 0 \text{ for any } L \in \eth\}$.
(2) ${}^\perp\eth = \operatorname{KerExt}^1_{R\text{-gr}}(-, \eth) = \{C\colon \operatorname{Ext}^1_{R\text{-gr}}(C, L) = 0 \text{ for any } L \in \eth\}$.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of graded R-modules is called a *cotorsion theory* if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *hereditary* if whenever $0 \to F' \to F \to F'' \to 0$ is exact in R-gr with $F, F'' \in \mathcal{F}$ then F' is also in $\mathcal{F}$, or equivalently, if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in R-gr with $C, C' \in \mathcal{C}$, then C'' is also in $\mathcal{C}$.

Corollary 3.10. *Let R be a graded ring. Then the following statements are equivalent:*

(1) *$({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is a hereditary cotorsion pair.*

(2) *Every n-presented module in R-gr with* gr-pd$(U) < \infty$ *is $(n+1)$-presented and every $M \in ({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp}$ has an exact left* (gr-$\mathcal{FI}_n$)*-resolution.*

Proof. (1) $\Rightarrow$ (2) Let M be a Gorenstein n-FP-injective left R-module and $0 \to M \to E \to L \to 0$ an exact sequence in R-Mod, where E is injective. Then, $0 \to F(M) \to F(E) \to F(L) \to 0$ is exact in R-gr, where $F(M)$ and $F(E)$ are Gorenstein n-FP-gr-injective by Proposition 3.5. So by hypothesis, $F(L)$ is Gorenstein n-FP-gr-injective. If U is an n-presented graded left R-module with gr-pd$_R(U) < \infty$, then similar to the proof (2) $\Rightarrow$ (1) of Theorem 3.7, it follows that U is $(n+1)$-presented. Since $({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp} = \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ and every $N \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ has an left exact (gr-$\mathcal{FI}_n$)-resolution, then $M \in ({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp}$ as well.

(2) $\Rightarrow$ (1) Note that we have to show that $({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp} = \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$. If $M \in ({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n})^{\perp}$, then a n-FP-gr-injective resolution $\ldots \to A_3 \to A_1 \to A_0 \to M \to 0$ of M in R-gr exists. Also, we have an exact sequence $0 \to M \to E_0 \to E_1 \to \ldots$ in R-gr, where any E_i is gr-injective. So, there exists an exact sequence

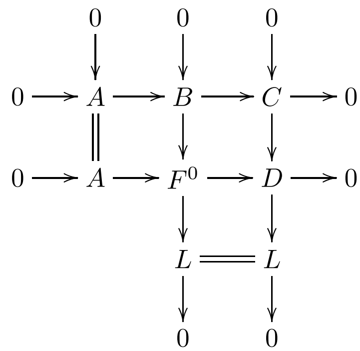
$$\mathscr{Y}\colon\ \ldots \to A_3 \to A_1 \to A_0 \to E_0 \to E_1 \to \ldots$$

of n-FP-gr-injective modules in R-gr with $M = \text{Ker}(E_0 \to E_1)$. Let $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ be a special short exact sequence in R-gr with gr-pd$_R(K_{n-1}) < \infty$. Then, by hypothesis, K_n is a gr-presented module with gr-pd$_R(K_n) < \infty$. So, by [32], Theorem 6.10, and by using the inductive presumption on gr-pd$_R(K_{n-1})$, we deduce that HOM$_R(K_{n-1}, \mathscr{Y})$ is exact. Thus, M is Gorenstein n-FP-gr-injecive and hence $M \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$.

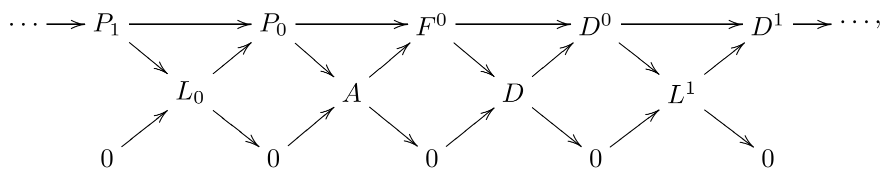
Now, if $0 \to A \to B \to C \to 0$ is a short exact sequence in R-gr, where $A, B \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$, then by Theorem 3.7, $C \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$. Hence, the pair $({}^{\perp}\mathcal{G}_{\text{gr-}\mathcal{FI}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is a hereditary cotorsion pair. □

Proposition 3.11. *Assume that every n-presented module in R-gr with* gr-fd$_R(U) < \infty$ *is $(n+1)$-presented. Then for every short exact sequence $0 \to A \to B \to C \to 0$ in* gr-R*, A is Gorenstein n-gr-flat if B and C are Gorenstein n-gr-flat.*

P r o o f. If B is a Gorenstein n-gr-flat module in gr-R, then by Definition 3.1 and Remark 3.2, there is an exact sequence $0 \to B \to F^0 \to L \to 0$ in gr-R, where F^0 is n-gr-flat and L is Gorenstein n-gr-flat. We have the following pushout diagram with exact rows:



By Proposition 3.4 (2), D is Gorenstein n-gr-flat, and so we have the following commutative diagram in gr-R:



where D^i and F^0 are n-gr-flat modules, P_i is gr-flat, $A = \mathrm{Ker}(F^0 \to D)$, $D = \mathrm{Ker}(D^0 \to D^1)$, $L_i = \mathrm{Ker}(P_i \to P_{i-1})$ and $L^i = \mathrm{Ker}(D^i \to D^{i+1})$. By Remark 3.2, P_i and L^i are Gorenstein n-gr-flat and hence by Lemma 3.6 (2), $\mathrm{Tor}_t^R(L^i, K_{n-1}) = \mathrm{Tor}_t^R(D, K_{n-1}) = 0$ for any special gr-presented module K_{n-1} in R-gr with gr-fd$_R(K_{n-1}) < \infty$ and any $t \geqslant 0$. So, similar to the proof (1) $\Rightarrow$ (2) of Theorem 3.7, it follows that $- \otimes_R K_{n-1}$ on the above horizontal sequence in diagram is exact and so A is Gorenstein n-gr-flat. □

Corollary 3.12. *Let every n-presented module in R-gr with gr-fd$_R(U) < \infty$ be $(n+1)$-presented. Then a module M in gr-R is Gorenstein n-gr-flat if and only if every gr-pure submodule and any gr-pure epimorphic image of M are Gorenstein n-gr-flat.*

P r o o f. ($\Rightarrow$) Let M be a Gorenstein n-gr-flat module in gr-R and K a gr-pure submodule in M. Then the exact sequence $0 \to K \to M \to M/K \to 0$ is gr-pure. So, if K_{n-1} is special gr-presented module in R-gr, then $\mathrm{Tor}_1^R(M/K, K_{n-1}) = 0$ and consequently by [20], Lemma 2.1, $\mathrm{Tor}_1^R(M/K, K_{n-1})^* \cong \mathrm{EXT}_R^1(K_{n-1}, (M/K)^*) = 0$.

Therefore, the exact sequence $0 \to (M/K)^* \to M^* \to K^* \to 0$ is special gr-pure in R-gr, and using [35], Proposition 3.10, we deduce that $(M/K)^*$ is n-FP-gr-injective. By [35], Proposition 3.8, M/K is n-gr-flat, and then Proposition 3.11 shows that K is Gorenstein n-gr-flat.

($\Leftarrow$) Let K be a gr-pure submodule in M. Then, the exact sequence $0 \to K \to M \to M/K \to 0$ is gr-pure. So, it follows by Proposition 3.4 (2), that M is Gorenstein n-gr-flat. □

Also, as for the classical injective (or flat) notion, the class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in R-gr (or $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in gr-R) is closed under direct products (or direct sums).

Proposition 3.13. *Let R be a graded ring. Then:*

(1) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in R-gr is closed under direct products.*

(2) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in gr-R is closed under direct sums.*

The next definition contains some general remarks about resolving classes of graded modules which will be useful in Sections 3 and 4. We use gr-$\mathscr{I}(R)$ to denote the class of finite injective graded left modules and the symbol gr-$\mathscr{F}(R)$ denotes the class of finite projective graded right modules (the graded version of [22]), 1.1. Resolving classes.

Definition 3.14. Let R be a graded ring and $\mathscr{X}$ a class of graded modules. Then:

(1) We call $\mathscr{X}$ gr-*injectively resolving* if gr-$\mathscr{I}(R) \subseteq \mathscr{X}$, and for every short exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathscr{X}$ the conditions $B \in \mathscr{X}$ and $C \in \mathscr{X}$ are equivalent.

(2) We call $\mathscr{X}$ gr-*projectively resolving* if gr-$\mathscr{F}(R) \subseteq \mathscr{X}$, and for every short exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathscr{X}$ the conditions $A \in \mathscr{X}$ and $B \in \mathscr{X}$ are equivalent.

By Definition 3.14, Propositions 3.4, 3.11, 3.13, Theorem 3.7 and the graded version of [22], Proposition 1.4, we have the following easy observations.

Proposition 3.15. *Assume that every n-presented module in R-gr with gr-$\text{fd}_R(U) < \infty$ is $(n+1)$-presented. Then:*

(1) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is gr-injectively resolving.*

(2) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is closed under direct summands.*

(3) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is gr-projectively resolving.*

(4) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is closed under direct summands.*

We know that, if R is a left n-gr-coherent ring, then every n-presented module in R-gr with gr-$\text{fd}_R(U) < \infty$ is $(n+1)$-presented. So in the following theorem according to previous results, we investigate the relationships between Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules on n-gr-coherent rings.

Theorem 3.16. *Let R be a left n-gr-coherent ring. Then:*

(1) *Module M in R-gr is Gorenstein n-FP-gr-injective if and only if M^* is Gorenstein n-gr-flat in gr-R.*

(2) *Module M in gr-R is Gorenstein n-gr-flat if and only if M^* is Gorenstein n-FP-gr-injective in R-gr.*

Proof. (1) ($\Rightarrow$) By Definition 3.1, there is an exact sequence $\ldots \to A_1 \to A_0 \to M \to 0$ in R-gr, where every A_i is n-FP-gr-injective, and by [35], Theorem 3.17, every $(A_i)^*$ is n-gr-flat in gr-R. So by [32], Lemma 3.53, there is an exact sequence $0 \to M^* \to (A_0)^* \to (A_1)^* \to \ldots$ in gr-R. Hence, we have:

$$\mathscr{X}\colon\ \ldots \to P_1 \to P_0 \to (A_0)^* \to (A_1)^* \to \ldots,$$

where P_i is gr-projective and n-gr-flat in gr-R by Remark 3.2 and also $M^* = \ker((A_0)^* \to (A_1)^*)$. Let $0 \to K_n \to F_{n-1} \to K_{n-1} \to 0$ be a special short exact sequence in R-gr with $\text{gr-fd}_R(K_{n-1}) < \infty$. Then K_n is a gr-presented module with $\text{gr-fd}_R(K_n) < \infty$, since R is n-gr-coherent. By [32], Theorem 6.10, and by using the inductive presumption on $\text{gr-fd}_R(K_{n-1})$, we deduce that $\mathscr{X} \otimes_R K_{n-1}$ is exact and then M^* is Gorenstein n-gr-flat.

($\Leftarrow$) Let M^* be a Gorenstein n-gr-flat module in gr-R. Then, by (2) ($\Rightarrow$), M^{**} is Gorenstein n-FP-gr-injective in R-gr. By [33], Proposition 2.3.5, M is gr-pure in M^{**}, and so by Corollary 3.8, M is Gorenstein n-FP-gr-injective.

(2) ($\Rightarrow$) By Definition 3.1, there is an exact sequence $0 \to M \to F^0 \to F^1 \to \ldots$ of n-gr-flat modules in gr-R. By [35], Proposition 3.8, $(F^i)^*$ is n-FP-gr-injective for any $i \geqslant 0$. So by [32], Lemma 3.53, there is an exact sequence $\ldots \to (F^1)^* \to (F^0)^* \to M^*$ in R-gr. For a module M^*, there is an exact sequence $0 \to M^* \to E_0 \to E_1 \to \ldots 0$ in R-gr, where E_i is gr-injective. Consider the following exact sequence:

$$\ldots \to (F^1)^* \to (F^0)^* \to E_0 \to E_1 \to \ldots$$

with $M^* = \ker(E_0 \to E_1)$. Hence, by analogy with the proof (2) $\Rightarrow$ (1) of Corollary 3.10, we obtain that M^* is Gorenstein n-FP-gr-injective.

($\Leftarrow$) Let M^* be a Gorenstein n-FP-gr-injective module in R-gr. Then, by (1) ($\Rightarrow$), M^{**} is Gorenstein n-gr-flat in gr-R. By [33], Proposition 2.3.5, M is gr-pure in M^{**}, and so by Corollary 3.12, M is Gorenstein n-gr-flat. □

Next, we are given other results of Gorenstein n-FP-gr-injective and n-gr-flat modules on n-gr-coherent rings.

Proposition 3.17. *Let R be a left n-gr-coherent ring. Then,*

(1) *the class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ in R-gr is closed under direct limits,*

(2) *the class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ in gr-R is closed under direct products.*

Proof. (1) Let $U \in R$-gr be an n-presented module and let $\{A_i\}_{i\in I}$ be a family of n-FP-gr-injective modules in R-gr. Then by [35], Theorem 3.17, $\varinjlim A_i$ is n-FP-gr-injective. So, if $\{M_i\}_{i\in I}$ is a family of Gorenstein n-FP-gr-injective modules in R-gr, then the following n-FP-gr-injective compelex:

$$\mathscr{Y}_i = \ldots \to (A_i)_1 \to (A_i)_0 \to (A_i)^0 \to (A_i)^1 \to \ldots,$$

where $M_i = \ker((A_i)^0 \to (A_i)^1)$, induces the following exact sequence of n-FP-gr-injective modules in R-gr:

$$\varinjlim \mathscr{Y}_i = \ldots \to \varinjlim (A_i)_1 \to \varinjlim (A_i)_0 \to \varinjlim (A_i)^0 \to \varinjlim (A_i)^1 \to \ldots,$$

where $\varinjlim M_i = \ker(\varinjlim (A_i)^0 \to \varinjlim (A_i)^1)$. Assume that K_{n-1} is special gr-presented module in R-gr with gr-pd$_R(K_{n-1}) < \infty$; then by [35], Proposition 3.13,

$$\mathrm{HOM}_R(K_{n-1}, \varinjlim \mathscr{Y}_i) \cong \varinjlim \mathrm{HOM}_R(K_{n-1}, \mathscr{Y}_i).$$

By hypothesis, $\mathrm{HOM}_R(K_{n-1}, \mathscr{Y}_i)$ is exact, and consequently $\varinjlim M_i$ is Gorenstein n-FP-gr-injective.

(2) Let $U \in R$-gr be n-presented and let $\{F_i\}_{i\in I}$ be a family of n-gr-flat modules in gr-R. Then by [35], Theorem 3.17, $\prod_{i\in I} F_i$ is n-gr-flat. So, if $\{M_i\}$ is a family of Gorenstein n-gr-flat modules in gr-R, then the following n-gr-flat complex

$$\mathscr{X}_i = \ldots \to (F_i)_1 \to (F_i)_0 \to (F_i)^0 \to (F_i)^1 \to \ldots,$$

where $M_i = \ker((F_i)^0 \to (F_i)^1)$, induces the following exact sequence of n-gr-flat modules in gr-R:

$$\prod_{i\in I} \mathscr{X}_i = \ldots \to \prod_{i\in I} (F_i)_1 \to \prod_{i\in I} (F_i)_0 \to \prod_{i\in I} (F_i)^0 \to \prod_{i\in I} (F_i)^1 \to \ldots,$$

where $\prod_{i\in I} M_i = \ker\Big(\prod_{i\in I} (F_i)^0 \to \prod_{i\in I} (F_i)^1\Big)$. If K_{n-1} is special gr-presented, then

$$\Big(\prod_{i\in I} \mathscr{X}_i \bigotimes_R K_{n-1}\Big) \cong \prod_{i\in I} \Big(\mathscr{X}_i \bigotimes_R K_{n-1}\Big).$$

By hypothesis, $\mathscr{X}_i \bigotimes_R K_{n-1}$ is exact, and consequently $\prod_{i\in I} M_i$ is Gorenstein n-gr-flat. □

In the following proposition, we show that if R is n-gr-coherent, then every Gorenstein n-FP-gr-injective module in R-gr is n-FP-gr-injective if $\mathrm{l}.n\text{-FP-gr-}\dim(R) < \infty$, and every Gorenstein n-gr-flat module in gr-R is n-gr-flat if $\mathrm{r}.n\text{-gr-}\dim(R) < \infty$.

Proposition 3.18. *Let R be a left n-gr-coherent ring.*

(1) *If $\mathrm{l}.n\text{-FP-gr-}\dim(R) < \infty$, then every Gorenstein n-FP-gr-injective module in R-gr is n-FP-gr-injective.*

(2) *If $\mathrm{r}.n\text{-gr-}\dim(R) < \infty$, then every Gorenstein n-gr-flat module in gr-R is n-gr-flat.*

P r o o f. (1) Let $\mathrm{l}.n\text{-FP-gr-}\dim(R) \leqslant k$. If M is a Gorenstein n-FP-gr-injective module in R-gr, then there exists an exact sequence

$$0 \to N \to A_{k-1} \to A_{k-2} \to \ldots \to A_0 \to M \to 0$$

in R-gr, where every A_i is n-FP-gr-injective for any $0 \leqslant i \leqslant k-1$. Since R is n-gr-coherent for any $t \geqslant 1$, $\mathrm{EXT}^t_R(K_{n-1}, A_i) = 0$ for all special gr-presented left modules K_{n-1} with respect to every n-presented module U in R-gr. Let $L_i = \ker(A_i \to A_{i-1})$. Then we have

$$\begin{aligned}\mathrm{EXT}^{k+1}_R(K_{n-1}, N) &\cong \mathrm{EXT}^k_R(K_{n-1}, L_{k-2})\\ &\ \ \vdots\\ &\cong \mathrm{EXT}^2_R(K_{n-1}, L_0)\\ &\cong \mathrm{EXT}^1_R(K_{n-1}, M).\end{aligned}$$

Since $n\text{-FP-gr-id}_R(N) \leqslant k$, then $0 = \mathrm{EXT}^{k+1}_R(K_{n-1}, N) \cong \mathrm{EXT}^1_R(K_{n-1}, M) \cong \mathrm{EXT}^n_R(U, M)$ and consequently M is n-FP-gr-injective.

(2) The proof is similar to that of (1). □

4. Covers and preenvelopes by Gorenstein graded modules

For a graded ring R, let $\mathcal{F}$ be a class of graded left R-modules and M be a graded left R-module. Following [7], [35], we say that a graded morphism $f\colon F \to M$ is an *$\mathcal{F}$-precover* of M if $F \in \mathcal{F}$ and $\mathrm{Hom}_{R\text{-gr}}(F', F) \to \mathrm{Hom}_{R\text{-gr}}(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$.Moreover, if whenever a graded morphism $g\colon F \to F$ such that $fg = f$ is an automorphism of F, then $f\colon F \to M$ is called an *$\mathcal{F}$-cover* of M. The class $\mathcal{F}$ is called *(pre)covering* if each object in R-gr has an $\mathcal{F}$-(pre)cover. Dually, the notions of $\mathcal{F}$-preenvelopes, $\mathcal{F}$-envelopes and (pre)enveloping are defined.

In this section, by using duality pairs on n-gr-coherent rings, we show that the classes $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ (or $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$) or other signs are covering and preenveloping.

Definition 4.1 (The graded version of Definition 2.1 of [23]). Let R be a graded ring. Then, a *duality pair* over R is a pair $(\mathcal{M},\mathcal{C})$, where $\mathcal{M}$ is a class of graded left (or right) R-modules and $\mathcal{C}$ is a class of graded right (or left) R-modules, subject to the following conditions:

(1) For any graded module M, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{C}$.

(2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M},\mathcal{C})$ is called *(co)product-closed* if the class of $\mathcal{M}$ is closed under graded direct (co)products, and a duality pair $(\mathcal{M},\mathcal{C})$ is called *perfect* if it is coproduct-closed, $\mathcal{M}$ is closed under extensions and R belongs to $\mathcal{M}$.

Proposition 4.2. *If R is a left n-gr-coherent ring, then the pair $(\mathcal{G}_{\text{gr-}\mathcal{FI}_n}, \mathcal{G}_{\text{gr-}\mathcal{F}_n})$ is a duality pair.*

Proof. Let M be an R-module in R-gr. Then by Theorem 3.16 (1), $M \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ if and only if $M^* \in \mathcal{G}_{\text{gr-}\mathcal{F}_n}$. By Proposition 3.13 (2), any finite direct sum of Gorenstein n-gr-flat modules is Gorenstein n-gr-flat. Also, by Proposition 3.15 (4), $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is closed under direct summands. So, by Definition 4.1, the pair $(\mathcal{G}_{\text{gr-}\mathcal{FI}_n}, \mathcal{G}_{\text{gr-}\mathcal{F}_n})$ is a duality pair. $\square$

Proposition 4.3. *If R is a left n-gr-coherent ring, then the pair $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is a duality pair.*

Proof. Let M be an R-module in gr-R. Then by Theorem 3.16 (2), $M \in \mathcal{G}_{\text{gr-}\mathcal{F}_n}$ if and only if $M^* \in \mathcal{G}_{\text{gr-}\mathcal{FI}_n}$. By Proposition 3.13 (1), any finite direct sum of Gorenstein n-gr-FP-injective modules is Gorenstein n-FP-gr-injective and by Proposition 3.15 (2), $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is closed under direct summands. So, by Definition 4.1, the pair $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is a duality pair. $\square$

Theorem 4.4. *Let R be a left n-gr-coherent ring. Then:*

(1) *The class $\mathcal{G}_{\text{gr-}\mathcal{FI}_n}$ is covering and preenveloping.*

(2) *The class $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is covering and preenveloping.*

Proof. (1) Every direct limit of Gorenstein n-FP-gr-injective modules and every direct product of Gorenstein n-FP-gr-injective modules in R-gr are Gorenstein n-FP-gr-injective by Propositions 3.17 (1) and 3.13 (1), respectively. Also, by Corollary 3.8, the class of Gorenstein n-FP-gr-injective modules in R-gr is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 4.2 and [35], Theorem 4.2, we deduce that every R-module in R-gr has a Gorenstein n-FP-gr-injective cover and a Gorenstein n-FP-gr-injective preenvelope.

(2) Every direct sum of Gorenstein n-gr-flat modules and every direct product of Gorenstein n-gr-flat modules in gr-R are Gorenstein n-gr-flat by Propositions 3.13 (2) and 3.17 (2), respectively. Also, by Corollary 3.12, the class of Gorenstein n-gr-flat modules in gr-R is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. So, by Proposition 4.3 and [35], Theorem 4.2, we deduce that every R-module in gr-R has a Gorenstein n-gr-flat cover and a Gorenstein n-gr-flat preenvelope. $\square$

Now we give some equivalent characterizations for ${}_RR$ being Gorenstein n-FP-gr-injective in terms of the properties of Gorenstein n-FP-gr-injective and Gorenstein n-gr-flat modules.

Theorem 4.5. *Let R be a left n-gr-coherent ring. Then the following statements are equivalent:*

(1) *${}_RR$ is Gorenstein n-FP-gr-injective;*

(2) *every graded module in gr-R has a monic Gorenstein n-gr-flat preenvelope;*

(3) *every gr-injective module in gr-R is Gorenstein n-gr-flat;*

(4) *every n-FP-gr-injective module in gr-R is Gorenstein n-gr-flat;*

(5) *every flat module in R-gris Gorenstein n-FP-gr-injective;*

(6) *every graded module in R-gr has an epic Gorenstein n-FP-gr-injective cover.*

Moreover, if $\mathrm{l}.n\text{-FP-gr-}\dim(R) < \infty$, then the above conditions are also equivalent to:

(7) *every Gorenstein gr-flat module in R-gr is Gorenstein n-FP-gr-injective;*

(8) *every graded module in R-gr is Gorenstein n-FP-gr-injective;*

(9) *every Gorenstein gr-injective module in gr-R is Gorenstein n-gr-flat.*

Proof. (8) ⇒ (7), (7) ⇒ (5) and (9) ⇒ (3) are obvious.

(1) ⇒ (2) By Theorem 4.4 (2), every module M in gr-R has a Gorenstein n-gr-flat preenvelope $f\colon M \to F$. By Theorem 3.16 (1), R^* is Gorenstein n-gr-flat in gr-R, and so $\prod_{i\in I}^{\text{gr-}R} R^*$ is Gorenstein n-gr-flat by Proposition 3.17. On the other hand, $({}_RR)^*$ is a cogenerator in gr-R. Therefore, exact sequence of the form $0 \to M \xrightarrow{g} \prod_{i\in I}^{\text{gr-}R} R^*$ exists, and hence homomorphism $0 \to F \xrightarrow{h} \prod_{i\in I}^{\text{gr-}R} R^*$ such that $hf = g$ shows that f is monic.

(2) ⇒ (3) Let E be a gr-injective module in gr-R. Then E has a monic Gorenstein n-gr-flat preenvelope $f\colon E \to F$ by assumption. Therefore, the split exact sequence $0 \to E \to F \to F/E \to 0$ exists, and so E is direct summand of F. Hence, by Proposition 3.15, E is Gorenstein n-gr-flat.

(3) ⇒ (1) By (3), R^* is Gorenstein n-gr-flat in gr-R, since R^* is gr-injective. Therefore, R is Gorenstein n-FP-gr-injective in R-gr by Theorem 3.16 (1).

(3) ⇒ (4) Let M be an n-FP-gr-injective module in gr-R . Then by [35], Proposition 3.10, the exact sequence $0 \to M \to E^g(M) \to E^g(M)/M \to 0$ is special gr-pure. Since by (3), $E^g(M)$ is Gorenstein n-gr-flat, from Corollary 3.12, we deduce that M is Gorenstein n-gr-flat.

(4) ⇒ (5) Let F be a flat module in R-gr. Then, F^* is gr-injective in gr-R, so F^* is Gorenstein n-gr-flat by (4), and hence F is Gorenstein n-FP-gr-injective by Theorem 3.16 (1).

(5) ⇒ (6) By Theorem 4.4 (1), every module M in R-gr has a Gorenstein n-FP-gr-injective cover $f\colon A \to M$. On the other hand, there exists an exact sequence $\bigoplus_{\gamma\in S} R(\gamma) \to M \to 0$ for some $S \subseteq G$. Since $R(\gamma)$ is Gorenstein n-FP-gr-injective by assumption, we have that $\bigoplus_{\gamma\in S} R(\gamma)$ is Gorenstein n-FP-gr-injective by Proposition 3.17. Thus f is an epimorphism.

(6) ⇒ (1) By hypothesis, R has an epic Gorenstein n-FP-gr-injective cover $f\colon D \to R$ then we have a split exact sequence $0 \to \mathrm{Ker} f \to D \to R \to 0$, where D is a Gorenstein n-FP-gr-injective module in R-gr. So, by Proposition 3.15, R is Gorenstein n-FP-gr-injective in R-gr.

(1) ⇒ (8) Let M be a graded left R-module. Then there is an exact sequence $\ldots \to F_1 \to F_0 \to M \to 0$ in R-gr, where each F_i is gr-flat. If R is a Gorenstein n-FP-gr-injective module in R-gr, then by Proposition 3.18 (1), R is n-FP-gr-injective. Hence, by [35], Theorem 4.8, we deduce that every F_i is n-FP-gr-injective. Also, for module M, there is an exact sequence $0 \to M \to E_0 \to E_1 \to \ldots 0$ in R-gr, where every E_i is gr-injective. So, we have:

$$\ldots \to F_1 \to F_0 \to E_0 \to E_1 \to \ldots,$$

where F_i and E_i are n-FP-gr-injective and $M = \ker(E_0 \to E_1)$. Thus, similar to the proof (2) ⇒ (1) of Corollary 3.10, we get that M is Gorenstein n-FP-gr-injective.

(8) ⇒ (9) If M is a Gorenstein gr-injective module in gr-R, then M^* is in R-gr. So by hypothesis, M^* is Gorenstein n-FP-gr-injective, and hence by Theorem 3.16, it follows that M is Gorenstein n-gr-flat. □

Example 4.6. Let R be a commutative, Gorenstein Noetherian, complete, local ring, with $\mathfrak{m}$ its maximal ideal. Let $E = E(R/\mathfrak{m})$ be the R-injective hull of the residue field $R/\mathfrak{m}$ of R. By [31], Theorem A, $\lambda\text{-}\dim(R \ltimes E) = \dim R$, where $\dim R$ is the Krull dimension of R. We suppose that $\dim R = n$, then $(R \ltimes E)$ is n-gr-coherent. And if we take in [27], Theorem 4.2, $n = 1$ and $B = \{0\}$, we get $\mathrm{Hom}_R(E, E) = R$. Then, by [17], Corollary 4.37, $(R \ltimes E)$ is self gr-injective which implies that $(R \ltimes E)$ is a left n-FP-gr-injective module over itself. Hence, $R \ltimes E$ is *n-FC* graded ring (n-gr-coherent and n-FP-gr-injective), and then by Remark 3.2, $(R \ltimes E)$ is Gorenstein

n-FP-gr-injective. For example, consider the ring $R = K[[X_1, \dots, X_n]]$ of formal power series in n variables over a field K which is commutative, Gorenstein Noetherian, complete, local ring, with $\mathfrak{m} = (X_1, \dots, X_n)$ its maximal ideal. We obtain $\lambda\text{-}\dim(R \ltimes E(R/\mathfrak{m})) = n$, that is, $R \ltimes E(R/\mathfrak{m})$ is n-gr-coherent ring. Therefore, according to the above, $R \ltimes E(R/\mathfrak{m})$ is n-FC graded ring. Therefore, every left $R \ltimes E(R/\mathfrak{m})$-module is Gorenstein n-FP-gr-injective.

Proposition 4.7. *Let R be left n-gr-coherent. Then $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, (\mathcal{G}_{\text{gr-}\mathcal{F}_n})^{\perp})$ is a hereditary perfect cotorsion pair.*

Proof. Let $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ be a class of Gorenstein n-gr-flat modules in gr-R. Then, by Corolary 3.12, $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. On the other hand, $R \in \mathcal{G}_{\text{gr-}\mathcal{F}_n}$ by Remark 3.2, and $\mathcal{G}_{\text{gr-}\mathcal{F}_n}$ is closed under graded direct sums by Proposition 3.13. Therefore, it follows that the duality pair $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, \mathcal{G}_{\text{gr-}\mathcal{FI}_n})$ is perfect. Consequently by [35], Theorem 4.2, $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, (\mathcal{G}_{\text{gr-}\mathcal{F}_n})^{\perp})$ is a perfect cotorsion pair. Consider the short exact sequence $0 \to A \to B \to C \to 0$ in gr-R, where B and C are Gorenstein n-gr-flat. Then, by Proposition 3.11, A is Gorenstein n-gr-flat and hence the perfect cotorsion pair $(\mathcal{G}_{\text{gr-}\mathcal{F}_n}, (\mathcal{G}_{\text{gr-}\mathcal{F}_n})^{\perp})$ is hereditary. □

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