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HARDY AND RELLICH TYPE INEQUALITIES WITH REMAINDERS

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Abstract. Hardy and Rellich type inequalities with an additional term are proved for compactly supported smooth functions on open subsets of the Euclidean space. We obtain one-dimensional Hardy type inequalities and their multidimensional analogues in convex domains with the finite inradius. We use Bessel functions and the Lamb constant. The statements proved are a generalization for the case of arbitrary $p \geqslant 2$ of the corresponding inequality proved by F. G. Avkhadiev, K.-J. Wirths (2011) for $p = 2$. Also we establish Rellich type inequalities on arbitrary domains, regular sets, on domains with θ-cone condition and on convex domains.

Keywords: Hardy inequality; Rellich type inequality; Bessel function; Lamb constant; distance function; Laplace operator

MSC 2020: 26D10, 26D15

1. Introduction

Let Ω be a domain in the Euclidean space $\mathbb{R}^n$ and let $C_0^1(\Omega)$ denote the space of continuously differentiable functions $f\colon \Omega \to \mathbb{R}$, which vanish on the boundary $\partial\Omega$ of the domain.

It is known that if Ω is convex, then the Hardy inequality

$$\int_\Omega |\nabla f(x)|^2 \,\mathrm{d}x \geqslant \frac{1}{4}\int_\Omega \frac{|f(x)|^2}{\delta(x)^2}\,\mathrm{d}x \quad \forall\, f \in H_0^1(\Omega) \tag{1.1}$$

is valid, where $H_0^1(\Omega)$ is the closure of the family $C_0^1(\Omega)$ with the finite Dirichlet integral and $\delta(x)$ is the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$ of Ω, i.e.,

$$\delta = \delta(x) = \inf_{y\in\partial\Omega} |x-y|.$$

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Note that the constant $\frac{1}{4}$ is sharp for any convex subdomain of $\mathbb{R}^n$, see [14], [16], [22], [23]. There are many improvements and modifications of the inequality (1.1), see [3], [4], [8]–[12], [14]–[34]. For instance, in [11], Avkhadiev and Wirths proved that the generalization of (1.1) for all $f \in C_0^1(\Omega)$

$$\int_\Omega \frac{|\nabla f(x)|^2}{\delta(x)^{s-1}}\,\mathrm{d}x \geqslant h\int_\Omega \frac{|f(x)|^2}{\delta(x)^{s+1}}\,\mathrm{d}x + \frac{\lambda^2}{\delta_0^q}\int_\Omega \frac{|f(x)|^2}{\delta(x)^{s-q+1}}\,\mathrm{d}x \tag{1.2}$$

holds with two sharp constants

$$h = \frac{s^2-\nu^2q^2}{4} \geqslant 0 \quad \text{and} \quad \lambda = \frac{q}{2}\lambda_\nu(2s/q) > 0,$$

where s and q are positive numbers, $\nu \in [0, s/q]$ and $z = \lambda_\nu(s)$ is the Lamb constant defined as the positive root of the equation $sJ_\nu(z) + 2zJ'_\nu(z) = 0$ for the Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu}k!\,\Gamma(k+1+\nu)}, \quad \nu \geqslant 0,$$

and Ω is an n-dimensional convex domain with finite inradius δ_0 defined as

$$\delta_0 = \delta_0(\Omega) = \sup_{x\in\Omega}\delta(x).$$

The inequality (1.2) is a bridge between Hardy's inequality of the classical form and sharp estimates of the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian under the Dirichlet boundary condition for n-dimensional convex domains Ω (for details, see [11] and references therein).

Note that the papers [4], [11], [17], [18]–[30], [33], [34] are also devoted to Hardy type inequalities with additional nonnegative terms. Hardy inequalities with remainders were first obtained by Maz'ya (see [24]) in the case, where Ω is a half-space. Brezis and Marcus in their paper [14] established such inequalities in the case when Ω is bounded and the constant in the inequalities depends on the diameter D. M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and A. Laptev in [20] proved inequalities with remainders in terms of the volume of Ω. Let us note that in a number of papers (see [1], [2], [17], [33]) L_p-inequalities are proved.

The aim of this paper is to obtain L_p-analogues of (1.2). For instance, we proved that the following theorem holds.

Theorem 1.1. *Let Ω be an n-dimensional convex domain of finite inradius δ_0 and let $\lambda_\nu(2(p-2)/q)$ be the Lamb constant. If $p > 2$, $q > 0$ and $\nu \in [0, (p-2)/q]$, and $f \in C_0^1(\Omega)$ such that $\nabla f(x)\delta^{1/p}(x) \in L^p(\Omega)$, then*

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x \geqslant d_{s,\nu}\int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1}}\,\mathrm{d}x + \frac{h_{s,\nu}}{\delta_0^q}\int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1-q}}\,\mathrm{d}x$$

and, if $k > 1$ is a positive integer, $p = 2k$ and $\nu \geqslant (p-2)/q$, then

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x + d_{s,\nu} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1}}\,\mathrm{d}x \geqslant \frac{h_{s,\nu}}{\delta_0^q} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1-q}}\,\mathrm{d}x,$$

where

$$d_{s,\nu} = \Big(\frac{p-2}{p}\Big)^{p-2} \frac{|(p-2)^2 - \nu^2 q^2|}{p^2} \quad \text{and} \quad h_{s,\nu} = \Big(\frac{p-2}{p}\Big)^p \frac{q^2 \lambda_\nu^2 (2(p-2)/q)}{(p-2)^2}.$$

Using Theorem 1.1 with $\nu = (p-2)/q$, we get the following corollary.

Corollary 1.1. *Let Ω be an open proper convex subset of the Euclidean space $\mathbb{R}^n$ with a finite inner radius δ_0. If $p > 2$, $q > 0$ and $f \in C_0^1(\Omega)$ such that $\nabla f(x)\delta^{1/p}(x) \in L^p(\Omega)$, then*

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x \geqslant \Big(\frac{p-2}{p}\Big)^p \frac{q^2 j^2_{[(p-2)/q]-1}}{(p-2)^2 \delta_0^q} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1-q}}\,\mathrm{d}x,$$

where $j_{[(p-2)/q]-1}$ is the first positive zero of the Bessel function $J_{[(p-2)/q]-1}$ of order $[(p-2)/q]-1$.

Taking into account the known facts (see [11] for more information)

$$J_{1/2} = \sqrt{\frac{2}{\pi}}\frac{\sin x}{\sqrt{x}}, \quad J_{-1/2} = \sqrt{\frac{2}{\pi}}\frac{\cos x}{\sqrt{x}}$$

and $j_{-1/2} = \pi/2$, $j_{1/2} = \pi$, we have the following assertion.

Corollary 1.2. *Let Ω be an open proper convex subset of the Euclidean space $\mathbb{R}^n$ with a finite inner radius δ_0. If $p > 2$ and $f \in C_0^1(\Omega)$ is such that $\nabla f(x)\delta^{1/p}(x) \in L^p(\Omega)$, then*

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x \geqslant \Big(\frac{2\pi}{3}\Big)^2 \Big(\frac{p-2}{p}\Big)^p \frac{1}{\delta_0^{2(p-2)/3}} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{(p+1)/3}}\,\mathrm{d}x$$

and

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x \geqslant \pi^2 \Big(\frac{p-2}{p}\Big)^p \frac{1}{\delta_0^{2(p-2)}} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{3-p}}\,\mathrm{d}x.$$

Since $qj_{[(p-2)/q]-1} \to p-2$ as $q \to 0$, Theorem 1.1 presents the known inequality, see [2]

$$\int_\Omega |\nabla f(x)|^p \delta(x)\,\mathrm{d}x \geqslant \Big(\frac{p-2}{p}\Big)^p \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1}}\,\mathrm{d}x$$

as a limit case when $q \to 0$. Note, that the sharpness of the constant $((p-2)/p)^p$ for any convex domain Ω was proved by Avkhadiev and Shafigullin, see [9].

This paper organization is as follows. In the first section, we prove one-dimensional inequalities. Using these one-dimensional inequalities, we get Hardy type inequalities in the multidimensional case. Note that we use Avkhadiev's method to get multidimensional inequalities from a corresponding one-dimensional inequality, see [1], [2] and [8] for more details. We prove Hardy type inequalities in convex domains with the finite inradius.

The last part is devoted to Rellich type inequalities with remainders. Rellich inequalities on arbitrary domains, on regular sets, on domains with θ-cone condition and on convex domains are proved. We refer to [5]–[7], [12], [13], [17] and [31] for more information about Rellich type inequalities.

We especially want to highlight a remarkable book by Balinsky, Evans and Lewis (see [12]), which collected the most beautiful results on multidimensional inequalities of Hardy and Rellich type.

For example, in [13], Barbatis obtained, that for a convex bounded domain Ω and all $f \in C_0^\infty(\Omega)$ the Rellich type inequality

$$\int_\Omega |\Delta f(x)|^2\,\mathrm{d}x \geqslant \frac{9}{16}\int_\Omega \frac{|f(x)|^2}{\delta(x)^4} + Kn(n+2)\Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|}\Big)^{4/n}\int_\Omega |f(x)|^2\,\mathrm{d}x \tag{1.3}$$

holds, where $K = \frac{11}{48}$. Here, Δ stands for the Laplace operator, $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in the Euclidean space $\mathbb{R}^n$ and $|\Omega|$ is the volume of the set Ω.

This result was improved by Evans and Lewis in [17] for all $n \geqslant 4$. Namely, they proved that the constant $K \approx 1.25$ for all $n \geqslant 4$. We show that $K \geqslant 0.417322$ for all $n \geqslant 2$. Therefore, our result improves the bound given by (1.3) for all $n \geqslant 2$.

2. One-dimensional estimates

Suppose that $q \in (0,\infty)$, $s \in (0,\infty)$ and $\nu \geqslant 0$. Denote by J_ν the Bessel function of order ν

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k!\,\Gamma(k+1+\nu)}$$

and put for all $x \in [0,1]$ that $F_{\nu,s,q}(x) = x^{s/2} J_\nu(\lambda(2s/q)x^{q/2})$.

Following [10] and [11] the Lamb constant is called the *first positive root* $z = \lambda_\nu(s)$ of the equation

$$sJ_\nu(z) + 2zJ'_\nu(z) = 0, \tag{2.1}$$

where $\nu \geqslant 0$ and s are fixed parameters.

It is easily shown that

$$F'_{\nu,s,q}(x) = \frac{s}{2}x^{(s/2)-1}J_\nu(\lambda(2s/q)x^{q/2}) + qx^{(q/2)+(s/2)-1}\lambda_\nu(2s/q)J'_\nu(\lambda_\nu(2s/q)x^{q/2}),$$
$$F'_{\nu,s,q}(1) = 0, \quad F_{\nu,s,q}(x) > 0, \quad x \in (0,1], \quad \text{and} \quad F'_{\nu,s,q}(x) > 0, \quad x \in (0,1).$$

Moreover, as is known, the function $y = F_{\nu,s,q}(x)$ is a solution of the differential equation

$$x^2y'' + (1-s)xy' + \Big(\frac{s^2 - \nu^2q^2}{4} + \frac{q^2\lambda_\nu^2(2s/q)}{4x^{-q}}\Big)y = 0 \tag{2.2}$$

and the equality

$$\lim_{x\to 0} \frac{xF'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)} = \frac{s+\nu q}{2} \tag{2.3}$$

holds. See [10], [11] for more information.

The following lemma holds.

Lemma 2.1. *Let $\lambda_\nu(2s/q)$ be the Lamb constant. Suppose that $p \geqslant 2$, $s > 0$, $q \in (0,\infty)$ and $\nu \geqslant 0$, and f is a positive nondecreasing absolutely continuous function in $[0,1]$ such that $f(0) = 0$ and*

$$\lim_{x\to 0} \frac{f^p(x)}{x^{s-1}} \frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)} = 0, \tag{2.4}$$

then

$$\int_0^1 \frac{f^{p-2}(x)f'^2(x)}{x^{s-1}}\,\mathrm{d}x \geqslant \frac{s^2-\nu^2q^2}{p^2}\int_0^1 \frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x + \frac{q^2\lambda_\nu^2(2s/q)}{p^2}\int_0^1 \frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x. \tag{2.5}$$

P r o o f. Clearly,

$$\begin{aligned} 0 \leqslant P &:= \int_0^1 \frac{f^{p-2}(x)}{x^{s-1}}\Big(f'(x) - \frac{2}{p}\frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)}f(x)\Big)^2\,\mathrm{d}x \\ &= \int_0^1 \frac{f^{p-2}(x)f'^2(x)}{x^{s-1}}\,\mathrm{d}x - \frac{4}{p^2}\int_0^1 \frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)x^{s-1}}\,\mathrm{d}f^p(x) \\ &\quad + \frac{4}{p^2}\int_0^1 \frac{f^p(x)}{x^{s-1}}\frac{F'^2_{\nu,s,q}(x)}{F^2_{\nu,s,q}(x)}\,\mathrm{d}x. \end{aligned}$$

Integrating by parts, one easily obtains

$$P=\int_0^1\frac{f^{p-2}(x)f'^2(x)}{x^{s-1}}\,\mathrm{d}x-\lim_{x\to0}\frac{f^p(x)}{x^{s-1}}\frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)}$$
$$+\frac{4}{p^2}\int_0^1 f^p(x)\frac{x^2F''_{\nu,s,q}(x)+(1-s)xF'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)x^{s+1}}\,\mathrm{d}x.$$

Using the asymptotic behavior (2.4) and the differential equation (2.2) we have

$$\int_0^1\frac{f^{p-2}(x)f'^2(x)}{x^{s-1}}\,\mathrm{d}x\geqslant\frac{4}{p^2}\int_0^1 f^p(x)\Big(\frac{s^2-\nu^2q^2}{4x^{s+1}}+\frac{q^2\lambda_\nu^2(2s/q)}{4x^{s-q+1}}\Big)\,\mathrm{d}x.$$

This completes the proof of Lemma 2.1. □

Remark. If k is a positive integer and $p=2k$, then in Lemma 2.1 we can assume that f is any absolutely continuous function.

Further we put

$$c_s=\frac{|s^2-\nu^2q^2|^{p/2}}{p^p}\quad\text{and}\quad\mu_s=c_s\frac{p}{2}\frac{q^2\lambda_\nu^2(2s/q)}{|s^2-\nu^2q^2|},$$

where $\lambda_\nu(2s/q)$ is the Lamb constant defined as the first positive root of the equation (2.1).

Lemma 2.2. *Let $\lambda_\nu(2s/q)$ be the Lamb constant. Suppose that $p\geqslant2$, $s>0$ and $q\in(0,\infty)$, and f is an absolutely continuous function in $[0,1]$ such that $f(0)=0$ and $f'(x)x^{(p-s-1)/p}\in L^p[0,1]$. If $\nu\in[0,s/q]$ then the inequality*

$$\int_0^1\frac{|f'(x)|^p}{x^{s-p+1}}\,\mathrm{d}x\geqslant c_s\int_0^1\frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x+\mu_s\int_0^1\frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x$$

holds, and if $\nu\geqslant(s-1)/q$, k is a positive integer and $p=2k$, then

$$\int_0^1\frac{f'^p(x)}{x^{s-p+1}}\,\mathrm{d}x+c_s(p-1)\int_0^1\frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x\geqslant\mu_s\int_0^1\frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x.$$

Proof. For an absolutely continuous function $f\colon[0,1]\to\mathbb{R}$ with the property $f(0)=0$ and $f'(x)x^{(p-s-1)/p}\in L^p[0,1]$ we have

$$|f(x)|^p\leqslant\Big(\int_0^x|f'(t)|\,\mathrm{d}t\Big)^p\leqslant\Big(\int_0^x t^{(s-p+1)/(p-1)}\,\mathrm{d}t\Big)^{p-1}\int_0^x\frac{|f'(t)|^p}{t^{s-p+1}}\,\mathrm{d}t$$
$$=\Big(\frac{p-1}{s}\Big)^{p-1}x^s\int_0^x\frac{|f'(t)|^p}{t^{s-p+1}}\,\mathrm{d}t.$$

Using the last estimate and (2.3) we get

$$\lim_{x\to0}\frac{|f(x)|^p}{x^{s-1}}\frac{F'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)}=0.$$

Consequently, we can use Lemma 2.1. Let us consider two cases.

Case 1: $\nu \in [0, s/q]$. Without loss of generality it can be assumed that f is a positive and nondecreasing function. Indeed, if

$$g(x) = \int_0^x |f'(t)|\,\mathrm{d}t,$$

where $f(x) = \int_0^x f'(t)\,\mathrm{d}t$, and the inequality

$$\int_a^b g^p(x)w(x)\,\mathrm{d}x \leqslant C_1 \int_a^b g'^p(x)v(x)\,\mathrm{d}x \tag{2.6}$$

holds then since

$$|f(x)| \leqslant \int_0^x |f'(t)|\,\mathrm{d}t = g(x), \quad g'(x) = |f'(x)|,$$

we have

$$\begin{aligned}\int_a^b |f(x)|^p w(x)\,\mathrm{d}x &\leqslant \int_a^b g^p(x)w(x)\,\mathrm{d}x \leqslant C_1 \int_a^b g'^p(x)v(x)\,\mathrm{d}x\\ &= C_1 \int_a^b |f'(x)|^p v(x)\,\mathrm{d}x.\end{aligned}$$

Using the inequality (2.5) and the elementary inequality (see [19])

$$a^{p_1} b^{p_2} \leqslant \Big(\frac{p_1 a + p_2 b}{p_1 + p_2}\Big)^{p_1 + p_2}$$

to the quantities

$$a = \frac{f^p(x)}{x^{s+1}}, \quad b = \frac{p^p}{(s^2 - \nu^2 q^2)^{p/2}} \frac{f'^p(x)}{x^{s-p+1}}, \quad p_1 = 1 - \frac{2}{p} \quad \text{and} \quad p_2 = \frac{2}{p},$$

we obtain

$$\frac{p^p}{(s^2 - \nu^2 q^2)^{p/2}} \int_0^1 \frac{f'^p(x)}{x^{s-p+1}}\,\mathrm{d}x \geqslant \int_0^1 \frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x + \frac{p}{2}\frac{q^2 \lambda_\nu^2(2s/q)}{s^2 - \nu^2 q^2} \int_0^1 \frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x.$$

Case 2: $\nu \geqslant s/q$. By Lemma 2.1 and Remark, we have

$$\frac{p^2}{\nu^2 q^2 - s^2} \int_0^1 \frac{f^{p-2}(x) f'^2(x)}{x^{s-1}}\,\mathrm{d}x + \int_0^1 \frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x \geqslant \frac{q^2 \lambda_\nu^2(2s/q)}{\nu^2 q^2 - s^2} \int_0^1 \frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x.$$

Applying the inequality (2.5) and the elementary inequality (see [19])

$$a^{p_1}b^{p_2} \leqslant \Bigl(\frac{p_1a+p_2b}{p_1+p_2}\Bigr)^{p_1+p_2}$$

to the quantities

$$a=\frac{f^p(x)}{x^{s+1}}, \quad b=\frac{p^p}{(\nu^2q^2-s^2)^{p/2}}\frac{f'^p(x)}{x^{s-p+1}}, \quad p_1=1-\frac{2}{p} \quad \text{and} \quad p_2=\frac{2}{p},$$

we have

$$\frac{p^p}{(\nu^2q^2-s^2)^{p/2}}\int_0^1\frac{f'^p(x)}{x^{s-p+1}}\,\mathrm{d}x+(p-1)\int_0^1\frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x \geqslant \frac{p}{2}\frac{q^2\lambda_\nu^2(2s/q)}{\nu^2q^2-s^2}\int_0^1\frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x.$$

This completes the proof of Lemma 2.2. □

Theorem 2.1. *Suppose that* $0<b-a<\infty$, $\delta(x)=\max\{x-a,b-x\}$, $p\in[2,\infty)$, $s\in(0,\infty)$ *and* $q\in(0,\infty)$. *Let* $f\colon [a,b]\to\mathbb{R}$ *be an absolutely continuous function such that* $f(a)=f(b)=0$ *and* $f'(x)/\delta^{(s-p+1)/p}(x)\in L^p[a,b]$. *If* $\nu\in[0,s/q]$ *then the inequality*

$$\int_a^b\frac{|f'(x)|^p}{\delta(x)^{s-p+1}}\,\mathrm{d}x \geqslant c_s\int_a^b\frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x+\frac{\mu_s}{\delta_0^q}\int_a^b\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x \tag{2.7}$$

is valid, and if $\nu\geqslant s/q$, k *is a positive integer and* $p=2k$, *then*

$$\int_a^b\frac{f'^p(x)}{\delta(x)^{s-p+1}}\,\mathrm{d}x+c_s(p-1)\int_a^b\frac{f^p(x)}{\delta(x)^{s+1}}\,\mathrm{d}x \geqslant \frac{\mu_s}{\delta_0^q}\int_a^b\frac{f^p(x)}{\delta(x)^{s-q+1}}\,\mathrm{d}x, \tag{2.8}$$

where $\delta_0=\frac{1}{2}(b-a)$.

P r o o f. By the change $x=\varrho t$ of variables for any constant $\varrho>0$ the inequality of Lemma 2.2 implies that

$$\int_0^\varrho\frac{|f'(x)|^p}{x^{s-p+1}}\,\mathrm{d}x \geqslant c_s\int_0^\varrho\frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x+\frac{\mu_s}{\varrho^q}\int_0^\varrho\frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x.$$

Now apply the last inequality to the functions $u(t)=f(t+a)$ and $u(t)=f(b-t)$ with $\varrho=\delta_0=\frac{1}{2}(b-a)$. We have

$$\int_{\delta_0}^b\frac{|f'(x)|^p}{(b-x)^{s-p+1}}\,\mathrm{d}x \geqslant c_s\int_{\delta_0}^b\frac{|f(x)|^p}{(b-x)^{s+1}}\,\mathrm{d}x+\frac{\mu_s}{\delta_0^q}\int_{\delta_0}^b\frac{|f(x)|^p}{(b-x)^{s-q+1}}\,\mathrm{d}x \tag{2.9}$$

and

$$\int_a^{\delta_0}\frac{|f'(x)|^p}{(x-a)^{s-p+1}}\,\mathrm{d}x \geqslant c_s\int_a^{\delta_0}\frac{|f(x)|^p}{(x-a)^{s+1}}\,\mathrm{d}x+\frac{\mu_s}{\delta_0^q}\int_a^{\delta_0}\frac{|f(x)|^p}{(x-a)^{s-q+1}}\,\mathrm{d}x. \tag{2.10}$$

Summing up (2.9) and (2.10), we get (2.7). The inequality (2.8) is proved similarly. This completes the proof of Theorem 2.1. □

Let now

$$a_{s,\nu} = \frac{|(s-1)^2 - \nu^2 q^2|(s-1)^{p-2}}{2^{3-p}p^{p-1}} \quad \text{and} \quad b_{s,\nu} = \frac{(s-1)^{p-2}q^2\lambda_\nu^2(2s/q)}{2^{3-p}p^{p-1}},$$

where $\lambda_\nu(2s/q)$ is the Lamb constant.

Lemma 2.3. *Suppose that $p \geqslant 2$, $s > 0$, $q \in (0,\infty)$, and f is an absolutely continuous function in $[0,1]$ such that $f(0) = 0$ and $f'(x)x^{(1-2s/p)(1-1/p)} \in L^p[0,1]$. If $\nu \in [0, s/q]$, then*

$$\int_0^1 \frac{|f'(x)|^p}{x^{(1-2s/p)(1-p)}}\,\mathrm{d}x \geqslant a_{s,\nu}\int_0^1 \frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x + b_{s,\nu}\int_0^1 \frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x,$$

and if $\nu \geqslant s/q$, k is a positive integer and $p = 2k$, then

$$\int_0^1 \frac{|f'(x)|^p}{x^{(1-2s/p)(1-p)}}\,\mathrm{d}x + a_{s,\nu}\int_0^1 \frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x \geqslant b_{s,\nu}\int_0^1 \frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x.$$

P r o o f. Note that for an absolutely continuous function $f\colon [0,1] \to \mathbb{R}$ with the property $f(0) = 0$ and $f'(x)x^{(1-2s/p)(1-1/p)} \in L^p[0,1]$ we have

$$\begin{aligned}|f(x)|^p &\leqslant \Big(\int_0^x |f'(t)|\,\mathrm{d}t\Big)^p \leqslant \Big(\int_0^x t^{2s/p-1}\,\mathrm{d}t\Big)^{p-1}\int_0^x \frac{|f'(t)|^p}{t^{(1-2s/p)(1-p)}}\,\mathrm{d}t \\ &= \Big(\frac{p}{2s}\Big)^{p-1} x^{2s(p-1)/p}\int_0^x \frac{|f'(t)|^p}{t^{(1-2s/p)(1-p)}}\,\mathrm{d}t.\end{aligned}$$

Consequently, the condition (2.5) holds. Combining the following Opial type inequality (see [32], page 312)

$$\int_0^1 \frac{|f(x)|^{p-2}|f'(x)|^2}{x^{s-1}}\,\mathrm{d}x \leqslant \frac{p^{p-3}}{2^{p-3}(s-1)^{p-2}}\int_0^1 \frac{|f'(x)|^p}{x^{(1-2s/p)(1-p)}}\,\mathrm{d}x$$

and Lemma 2.1, we get

$$\begin{aligned}&\frac{p^{p-3}}{2^{p-3}s^{p-2}}\int_0^1 \frac{f'^p(x)}{x^{(p-2s)(1/p-1)}}\,\mathrm{d}x \\ &\qquad \geqslant \frac{s^2-\nu^2q^2}{p^2}\int_0^1 \frac{f^p(x)}{x^{s+1}}\,\mathrm{d}x + \frac{q^2\lambda_\nu^2(2s/q)}{p^2}\int_0^1 \frac{f^p(x)}{x^{s-q+1}}\,\mathrm{d}x.\end{aligned}$$

If $\nu \in [0, s/q]$, then for all absolutely continuous functions

$$\begin{aligned}&\frac{p^{p-3}}{2^{p-3}s^{p-2}}\int_0^1 \frac{|f'(x)|^p}{x^{(p-2s)(1/p-1)}}\,\mathrm{d}x \\ &\qquad \geqslant \frac{s^2-\nu^2q^2}{p^2}\int_0^1 \frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x + \frac{q^2\lambda_\nu^2(2s/q)}{p^2}\int_0^1 \frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x\end{aligned}$$

and if $\nu \geqslant s/q$, k is a positive integer and $p = 2k$, then

$$\frac{p^{p-3}}{2^{p-3}s^{p-2}} \int_0^1 \frac{|f'(x)|^p}{x^{(p-2s)(1/p-1)}}\,\mathrm{d}x + \frac{\nu^2 q^2 - s^2}{p^2} \int_0^1 \frac{|f(x)|^p}{x^{s+1}}\,\mathrm{d}x \geqslant \frac{q^2\lambda_\nu^2(2s/q)}{p^2} \int_0^1 \frac{|f(x)|^p}{x^{s-q+1}}\,\mathrm{d}x.$$

This completes the proof of Lemma 2.3. □

The application of Lemma 2.3 yields:

Theorem 2.2. *Suppose that* $0 < b-a < \infty$, $\delta(x) = \max\{x-a, b-x\}$, $p \in [2,\infty)$, $s \in (0,\infty)$, $q \in (0,\infty)$ *and* $\nu \in [0, s/q]$. *Let* $f\colon [a,b] \to \mathbb{R}$ *be an absolutely continuous function such that* $f(a) = f(b) = 0$ *and* $f'(x)/\delta^{(1-2s/p)(1/p-1)}(x) \in L^p[a,b]$. *Then the inequality*

$$\delta_0^{s(1-2/p)} \int_a^b \frac{|f'(x)|^p}{\delta(x)^{(1-2s/p)(1-p)}}\,\mathrm{d}x \geqslant a_{s,q} \int_a^b \frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x + \frac{b_{s,\nu}}{\delta_0^q} \int_a^b \frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x \tag{2.11}$$

is valid, and if $\nu \geqslant (s-1)/q$, k *is a positive integer and* $p = 2k$, *then*

$$\delta_0^{s(1-2/p)} \int_a^b \frac{|f'(x)|^p}{\delta(x)^{(1-2s/p)(1-p)}}\,\mathrm{d}x + a_{s,\nu} \int_a^b \frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x \geqslant \frac{b_{s,\nu}}{\delta_0^q} \int_a^b \frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x,$$

where $\delta_0 = \frac{1}{2}(b-a)$.

Suppose that

$$d_{s,\nu} = \Big(\frac{p-2}{p}\Big)^{p-2} \frac{|(p-2)^2 - \nu^2 q^2|}{p^2} \quad \text{and} \quad h_{s,\nu} = \Big(\frac{p-2}{p}\Big)^{p-2} \frac{q^2\lambda_\nu^2(2(p-2)/q)}{p^2},$$

where $\lambda_\nu(2(p-2)/q)$ is the Lamb constant.

Lemma 2.4. *Suppose that* $p > 2$, $q > 0$ *and* f *is an absolutely continuous function such that* $f(0) = 0$ *and* $f'(x)x^{1/p} \in L^p[0,1]$. *If* $\nu \in [0, (p-2)/q]$ *then the Hardy type inequality*

$$\int_0^1 |f'(x)|^p x\,\mathrm{d}x \geqslant d_{s,\nu} \int_0^1 \frac{|f(x)|^p}{x^{p-1}}\,\mathrm{d}x + h_{s,\nu} \int_0^1 \frac{|f(x)|^p}{x^{p-1-q}}\,\mathrm{d}x$$

holds, and if $\nu \geqslant (s-1)/q$, k *is a positive integer and* $p = 2k$, *then*

$$\int_0^1 |f'(x)|^p x\,\mathrm{d}x + d_{s,\nu} \int_0^1 \frac{|f(x)|^p}{x^{p-1}}\,\mathrm{d}x \geqslant h_{s,\nu} \int_0^1 \frac{|f(x)|^p}{x^{p-1-q}}\,\mathrm{d}x.$$

P r o o f. Since

$$|f(x)|^p = \Big(\int_0^x |f'(t)|\,\mathrm{d}t\Big)^p \leqslant \Big(\int_0^x t^{-1/(p-1)}\,\mathrm{d}t\Big)^{p-1} \int_0^x |f'(t)|^p t\,\mathrm{d}t$$
$$= \Big(\frac{p}{p-1}\Big)^{p-1} x^p \int_0^x |f'(t)|^p t\,\mathrm{d}t$$

and

$$\lim_{x\to 0} \frac{xF'_{\nu,s,q}(x)}{F_{\nu,s,q}(x)} = \frac{s+\nu}{2},$$

the condition (2.5) holds. By Lemma 2.1 and the Opial type inequality (see [32], page 313)

$$\int_0^1 \frac{|f(x)|^{p-2}|f'(x)|^2}{x^{p-3}}\,\mathrm{d}x \leqslant \Big(\frac{p}{p-2}\Big)^{p-2} \int_0^1 |f'(x)|^p x\,\mathrm{d}x,$$

we obtain

$$\Big(\frac{p}{p-2}\Big)^{p-2} \int_0^1 |f'(x)|^p x\,\mathrm{d}x$$
$$\geqslant \frac{(p-2)^2-\nu^2q^2}{p^2} \int_0^1 \frac{|f(x)|^p}{x^{p-1}}\,\mathrm{d}x + \frac{q^2\lambda_\nu^2(2(p-2)/q)}{p^2} \int_0^1 \frac{|f(x)|^p}{x^{p-1-q}}\,\mathrm{d}x.$$

This completes the proof of Lemma 2.4. □

The application of Lemma 2.4 yields:

Theorem 2.3. *Suppose that $0 < b-a < \infty$, $\delta(x) = \max\{x-a, b-x\}$ and $f\colon [a,b] \to \mathbb{R}$ is an absolutely continuous function such that $f(a) = f(b) = 0$ and $f'(x)\delta^{1/p}(x) \in L^p[a,b]$. If $p > 2$, $q > 0$ and $\nu \in [0,(p-2)/q]$, then*

$$\int_a^b |f'(x)|^p \delta(x)\,\mathrm{d}x \geqslant d_{s,\nu} \int_a^b \frac{|f(x)|^p}{\delta(x)^{p-1}}\,\mathrm{d}x + \frac{h_{s,\nu}}{\delta_0^q} \int_a^b \frac{|f(x)|^p}{\delta(x)^{p-1-q}}\,\mathrm{d}x$$

and, if $k > 1$ is a positive integer, $p = 2k$ and $\nu \geqslant (p-2)/q$, then

$$\int_a^b |f'(x)|^p \delta(x)\,\mathrm{d}x + d_{s,\nu} \int_a^b \frac{|f(x)|^p}{\delta(x)^{p-1}}\,\mathrm{d}x \geqslant \frac{h_{s,\nu}}{\delta_0^q} \int_a^b \frac{|f(x)|^p}{\delta(x)^{p-1-q}}\,\mathrm{d}x,$$

where $\delta_0 = \frac{1}{2}(b-a)$.

3. Sharpness of constants

Note that both the constants in the inequality of Lemma 2.2 are sharp, when $\nu \geqslant 0$ and $p = 2$ (see [11] for more information). At the same time, we know nothing about the sharpness of constants in the case $p > 2$.

In the next lemma we prove that the constant

$$\Big(\frac{p-2}{p}\Big)^{p-2} \frac{(p-2)^2 - \nu^2 q^2}{p^2}$$

in Lemma 2.4 is sharp in the case $\nu = 0$.

Lemma 3.1. *If $p > 2$ and $q > 0$ then for any $\varepsilon > 0$ there exists a function f_ε that satisfies the conditions of Lemma 2.4 and the inequality*

$$\int_0^1 |f_\varepsilon'(x)|^p x \, \mathrm{d}x \leqslant \frac{((p-2)^2 + 4\varepsilon)^{p/2}}{p^p} \int_0^1 \frac{|f_\varepsilon(x)|^p}{x^{p-1}} \, \mathrm{d}x \\ + \frac{(p-2)^{p-2}}{p^p} \frac{q^2 \lambda_0^2 (2(p-2)/q)}{p^p} \int_0^1 \frac{|f_\varepsilon(x)|^p}{x^{p-1-q}} \, \mathrm{d}x.$$

P r o o f. Let $\varepsilon > 0$ and $f_\varepsilon(x) = t^{(p-2+\varepsilon/(p-1))/p}$. Without loss of generality we suppose that $\varepsilon \leqslant 1$. Straightforward computations give that

$$\int_0^1 |f_\varepsilon'(x)|^p x \, \mathrm{d}x = \Big(p - 2 + \frac{\varepsilon}{p-1}\Big)^p \frac{p-1}{p^p \varepsilon} < ((p-2)^2 + 4\varepsilon)^{p/2} \frac{p-1}{p^p \varepsilon} \\ = \frac{((p-2)^2 + 4\varepsilon)^{p/2}}{p^p} \int_0^1 \frac{|f_\varepsilon(x)|^p}{x^{p-1}} \, \mathrm{d}x,$$

which implies Lemma 3.1. □

4. Hardy inequalities in convex domains

Let Ω be an open proper convex subset of the Euclidean space $\mathbb{R}^n$ with a finite inner radius

$$\delta_0 = \delta_0(\Omega) = \sup_{x \in \Omega} \delta(x),$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function to the boundary of domain.

By $C_0^1(\Omega)$ we denote the family of continuously differentiable functions $f\colon \Omega \to \mathbb{R}$ with compact supports lying in Ω.

Recall that

$$a_{s,\nu}=\frac{\left|(s-1)^2-\nu^2q^2\right|(s-1)^{p-2}}{2^{3-p}p^{p-1}},\quad b_{s,\nu}=\frac{(s-1)^{p-2}q^2\lambda_\nu^2(2s/q)}{2^{3-p}p^{p-1}},$$
$$c_s=\frac{|s^2-\nu^2q^2|^{p/2}}{p^p},\quad \mu_s=c_s\frac{p}{2}\frac{q^2\lambda_\nu^2(2s/q)}{|s^2-\nu^2q^2|},$$

where $\lambda_\nu(2s/q)$ is the Lamb constant defined as the first positive root of the equation (2.1).

The following theorem holds.

Theorem 4.1. *Let Ω be an open proper convex subset of the Euclidean space $\mathbb{R}^n$ with a finite inner radius δ_0 and let $\lambda_\nu(2s/q)$ be the Lamb constant. Suppose that $p\in[2,\infty)$, $s\in(0,\infty)$ and $q\in(0,\infty)$, and $f\in C_0^1(\Omega)$ is such that $\nabla f(x)/\delta^{(s-p+1)/p}(x)\in L^p(\Omega)$. If $\nu\in[0,s/q]$ then the inequality*

$$\int_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{s-p+1}}\,\mathrm{d}x\geqslant c_s\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x+\frac{\mu_s}{\delta_0^q}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x$$

is valid, and if $\nu\geqslant s/q$, k is a positive integer and $p=2k$, then

$$\int_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{s-p+1}}\,\mathrm{d}x+c_s(p-1)\int_\Omega\frac{f^p(x)}{\delta(x)^{s+1}}\,\mathrm{d}x\geqslant\frac{\mu_s}{\delta_0^q}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x.$$

Theorem 4.2. *Let Ω be an open proper convex subset of the Euclidean space $\mathbb{R}^n$ with a finite inner radius δ_0. Suppose that $p\in[2,\infty)$, $s\in(0,\infty)$ and $q\in(0,\infty)$, and $f\in C_0^1(\Omega)$ is such that $\nabla f(x)/\delta^{(1-2s/p)(1/p-1)}(x)\in L^p(\Omega)$. Then the inequality*

$$\delta_0^{s(1-2/p)}\int_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{(1-2s/p)(1-p)}}\,\mathrm{d}x\geqslant a_{s,q}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x+\frac{b_{s,q}}{\delta_0^q}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x$$

is valid, and if $\nu\geqslant(s-1)/q$, k is a positive integer and $p=2k$, then

$$\delta_0^{s(1-2/p)}\int_\Omega\frac{|\nabla f(x)|^p}{\delta(x)^{(1-2s/p)(1-p)}}\,\mathrm{d}x+a_{s,\nu}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s+1}}\,\mathrm{d}x\geqslant\frac{b_{s,\nu}}{\delta_0^q}\int_\Omega\frac{|f(x)|^p}{\delta(x)^{s-q+1}}\,\mathrm{d}x.$$

P r o o f of Theorem 1.1, Theorem 4.1 and Theorem 4.2. To prove the case of $n\geqslant 2$ we use the method of Avkhadiev (see [1], [2], [8]). This method allows to get a multidimensional inequality from a corresponding one-dimensional inequality. We give a brief description of this method.

Let Λ be an arbitrary open domain in $\mathbb{R}^n$, $n \geqslant 2$. Using approximation of the open set Λ by cubes, Avkhadiev showed that it suffices to prove inequalities only for sets of the form

$$K(S) = \{x \in \Lambda_1 \colon \text{ there exist a point } y \in S \text{ such that } \delta(x, \Lambda) = |x - y|\},$$

where Λ_1 is some partition of the domain Λ, $k \in \{1, 2, \dots, n\}$, and S is an $(n-k)$-dimensional cube face.

While calculating integrals over the set $K(S)$, we have to employ either spherical or cylindric or Cartesian coordinates which allows us to pass to the corresponding iterated integral and to prove only one-dimensional inequalities. For a convex domain the situation is simple and one-dimensional inequalities are extended straightforwardly to the spatial case. This completes the proofs of Theorem 1.1, Theorem 4.1 and Theorem 4.2. $\square$

5. Rellich inequalities in domains

5.1. Inequalities on arbitrary domains. Let Ω be an open subset of the Euclidean space $\mathbb{R}^n$ and let $C_0^1(\Omega)$ be the family of continuously differentiable functions $f \colon \Omega \to \mathbb{R}$ with compact supports lying in Ω. Denote the unit sphere in $\mathbb{R}^n$ by $\mathbb{S}^{n-1}$. For each $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$, put

$$\begin{gathered}
\tau_\nu(x) = \min\{s > 0 \colon x + s\nu \notin \Omega\}, \quad \varrho_\nu(x) = \min(\tau_\nu(x), \tau_{-\nu}(x)), \\
D_\nu(x) = \tau_\nu(x) + \tau_{-\nu}(x), \quad \Omega_x = \{y \in \Omega \colon x + t(y - x) \in \Omega \ \forall\, t \in [0,1]\}, \\
\delta(x) = \inf_{\nu \in \mathbb{S}^{n-1}} \tau_\nu(x) = \operatorname{dist}(x, \partial\Omega), \quad D(\Omega) := \sup_{x \in \Omega,\, \nu \in \mathbb{S}^{n-1}} D_\nu(x).
\end{gathered}$$

The volume of Ω_x is denoted by $|\Omega_x|$. Clearly, $\Omega_x \subset \Omega$. In [15], Davies introduced the mean distance function $\varrho(x)$ by the formula

$$\varrho(x) := \int_{\mathbb{S}^{n-1}} \frac{1}{\varrho_\nu(x)^2} \, \mathrm{d}\omega(\nu),$$

where $\mathrm{d}\omega(\nu)$ is the normalized measure on $\mathbb{S}^{n-1}$, i.e., $\int_{\mathbb{S}^{n-1}} \mathrm{d}\omega(\nu) = 1$.

For a general $s \in (0, \infty)$, there is the analogue (see [12], [17], [33])

$$\varrho(x, s) := \int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}\omega(\nu)}{\varrho_\nu(x)^s}.$$

Further assume that $s > 0$, $q > 0$, $\nu \in [0, s/q]$ and $\lambda_\nu(2s/q)$ is the Lamb constant. In this article we suppose that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, $D(\Omega)$ is the diameter of Ω,

$$B(n, s) = \frac{\Gamma(\frac{1}{2}(s+1))\Gamma(\frac{1}{2}n)}{\sqrt{\pi}\Gamma(\frac{1}{2}(n+s))}, \quad c(s) := \frac{s^2 - \nu^2 q^2}{4} \quad \text{and} \quad \mu(s) := \frac{q^2 \lambda_\nu^2(2s/q)}{4}.$$

The following theorem holds.

Theorem 5.1. *Let Ω be a domain in $\mathbb{R}^n$. Suppose that $s > 0$, $q > 0$ and $\nu \in [0, s/q]$. If $s \geqslant 1$ and $s + 3 \geqslant q$, then for all $f \in C_0^2(\Omega)$*

$$\frac{1}{n(n+2)} \int_\Omega \left[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^n \left| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \right|^2 \right] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \varrho(x, s+3) + \frac{\mu(s+2) 2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \, \mathrm{d}x.$$

If $s \geqslant 1$ and $s + 3 \leqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{n(n+2)} \int_\Omega \left[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^n \left| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \right|^2 \right] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \varrho(x, s+3) + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \, \mathrm{d}x.$$

If $0 < s \leqslant 1$ and $s + 3 \geqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{3}{n(n+2)} \frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega |\Delta u(x)|^2 \, \mathrm{d}x$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \varrho(x, s+3) + \frac{\mu(s+2) 2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \, \mathrm{d}x.$$

If $0 < s \leqslant 1$ and $s + 3 \leqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{3}{n(n+2)} \frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega |\Delta u(x)|^2 \, \mathrm{d}x$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \varrho(x, s+3) + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \, \mathrm{d}x.$$

P r o o f. It follows from Theorem 2.1 for $p = 2$ that for all $f \in C_0^1(a, b)$

$$\int_a^b \frac{|f'(x)|^2}{\delta(x)^{s-1}}\,\mathrm{d}x \geqslant c(s) \int_a^b \frac{|f(x)|^2}{\delta(x)^{s+1}}\,\mathrm{d}x + \frac{\mu(s)}{\delta_0^q} \int_a^b \frac{|f(x)|^2}{\delta(x)^{s-q+1}}\,\mathrm{d}x. \tag{5.1}$$

Note that this inequality was proved by Avkhadiev and Wirths in [11]. Applying (5.1) to the function f defined by $f(x) = u'(x)$, we get

$$\int_a^b \frac{|u''(x)|^2}{\delta(x)^{s-1}}\,\mathrm{d}x \geqslant c(s) \int_a^b \frac{|u'(x)|^2}{\delta(x)^{s+1}}\,\mathrm{d}x.$$

Therefore, by (5.1) we have

$$\int_a^b \frac{|u''(x)|^2}{\delta(x)^{s-1}}\,\mathrm{d}x \geqslant c(s)\Big(c(s+2) \int_a^b \frac{|u(x)|^2}{\delta(x)^{s+3}}\,\mathrm{d}x + \frac{\mu(s+2)}{\delta_0^q} \int_a^b \frac{|u(x)|^2}{\delta(x)^{s-q+3}}\,\mathrm{d}x\Big)$$

for all $u \in C_0^2(a, b)$.

Consequently,

$$\int_{a_\nu}^{b_\nu} \frac{|\partial_\nu^2 u(x)|^2}{\varrho_\nu(x)^{s-1}}\,\mathrm{d}x \geqslant c(s)\Big(c(s+2) \int_{a_\nu}^{b_\nu} \frac{|u(x)|^2}{\varrho_\nu(x)^{s+3}}\,\mathrm{d}x + \frac{\mu(s+2)}{\delta_0^q} \int_{a_\nu}^{b_\nu} \frac{|u(x)|^2}{\varrho_\nu(x)^{s-q+3}}\,\mathrm{d}x\Big),$$

where $\partial_\nu^2 u, \nu \in \mathbb{S}^{n-1}$ denotes the second derivative of u in the direction of ν, (a_ν, b_ν) is the interval of intersection of Ω with the ray in the direction ν and $\delta_0 = \frac{1}{2}(b_\nu - a_\nu)$.

Integrating both sides of the last inequality with respect to the normalized surface measure $\mathrm{d}\omega(\nu)$ on $\mathbb{S}^{n-1}$, we get

$$\begin{aligned} &\int_\Omega \int_{\mathbb{S}^{n-1}} \frac{|\partial_\nu^2 u(x)|^2}{\varrho_\nu(x)^{s-1}}\,\mathrm{d}\omega(\nu)\,\mathrm{d}x \\ &\quad\geqslant c(s) \int_\Omega \int_{\mathbb{S}^{n-1}} \frac{c(s+2)}{\varrho_\nu(x)^{s+3}} + \Big(\frac{2}{D_\nu(x)}\Big)^q \frac{\mu(s+2)}{\varrho_\nu(x)^{s-q+3}}\,\mathrm{d}\omega(\nu)|u(x)|^2\,\mathrm{d}x. \end{aligned} \tag{5.2}$$

Let us consider four cases.

Case 1: $s \geqslant 1$ and $s + 3 \geqslant q$. Obviously, for any $\nu \in \mathbb{S}^{n-1}$

$$\frac{1}{\varrho_\nu(x)^{s-1}} \leqslant \frac{1}{\delta(x)^{s-1}}, \qquad \frac{1}{\varrho_\nu(x)^{s-q+3}} \geqslant \frac{2^{s-q+3}}{D(\Omega)^{s-q+3}}. \tag{5.3}$$

Moreover, in his paper [33], Tidblom proved that

$$\int_{\mathbb{S}^{n-1}} \Big(\frac{2}{D_\nu(x)}\Big)^q \,\mathrm{d}\omega(\nu) \geqslant \Big(\frac{n|\Omega_x|}{|\mathbb{S}^{n-1}|}\Big)^{-q/n}. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), we obtain

$$\int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) \frac{\mathrm{d}x}{\delta(x)^{s-1}} \geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2)\varrho(x, s+3) + \frac{\mu(s+2)2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

Case 2: $s \geqslant 1$ and $s+3 < q$. Clearly, for any $\nu \in \mathbb{S}^{n-1}$

$$\frac{1}{\varrho_\nu(x)^{s-1}} \leqslant \frac{1}{\delta(x)^{s-1}}, \quad \frac{1}{\varrho_\nu(x)^{s-q+3}} \geqslant \frac{1}{\delta(x)^{s-q+3}}.$$

Similarly, we have

$$\int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) \frac{\mathrm{d}x}{\delta(x)^{s-1}} \geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2)\varrho(x, s+3) + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

Case 3: $0 < s < 1$ and $s+3 \geqslant q$. Evidently, for any $\nu \in \mathbb{S}^{n-1}$

$$\frac{1}{\varrho_\nu(x)^{s-1}} \leqslant \frac{2^{s-1}}{D(\Omega)^{s-1}}, \quad \frac{1}{\varrho_\nu(x)^{s-q+3}} \geqslant \frac{2^{s-q+3}}{D(\Omega)^{s-q+3}}.$$

In the same way, we obtain

$$\frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) \,\mathrm{d}x \geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2)\varrho(x, s+3) + \frac{\mu(s+2)2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

Case 4: $0 < s < 1$ and $s+3 < q$. It is clear that for any $\nu \in \mathbb{S}^{n-1}$

$$\frac{1}{\varrho_\nu(x)^{s-1}} \leqslant \frac{2^{s-1}}{D(\Omega)^{s-1}}, \quad \frac{1}{\varrho_\nu(x)^{s-q+3}} \geqslant \frac{1}{\delta(x)^{s-q+3}}.$$

As before, we get

$$\frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) \,\mathrm{d}x \geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2)\varrho(x, s+3) + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

In [17], Evans and Lewis proved that if Ω is a domain in $\mathbb{R}^n$, then for all $u \in C^2(\mathbb{R}^n)$

$$\int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) = \frac{1}{n(n+2)} \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^{n} \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big],$$

and for all $u \in C_0^2(\Omega)$

$$\int_\Omega \int_{\mathbb{S}^{n-1}} |\partial_\nu^2 u(x)|^2 \,\mathrm{d}\omega(\nu) \,\mathrm{d}x = \frac{3}{n(n+2)} \int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x.$$

This completes the proof of Theorem 5.1. □

5.2. Inequalities on regular sets. Recall that

$$\frac{1}{\varrho(x)^2} := \int_{\mathbb{S}^{n-1}} \frac{1}{\varrho_\nu(x)^2} \,\mathrm{d}\omega(\nu),$$

where $\mathrm{d}\omega(\nu)$ is the normalized measure on $\mathbb{S}^{n-1}$.

We say that a domain $\Omega \subset \mathbb{R}^n$ is regular if there exists a finite constant $m(\Omega) > 0$ such that

$$\delta(x) \leqslant \varrho(x) \leqslant m(\Omega)\delta(x) \quad \forall\, x \in \Omega.$$

We denote the regularity constant for the domain Ω by $m(\Omega)$ (see [15], [34] for information).

In [15], [34], sufficient conditions for regularity are obtained. For example, Davies in [15] got the following sufficient condition:
The region $\Omega \subseteq \mathbb{R}^n$ is regular if there exists a constant $m(\Omega)$ such that

$$|\{y \in \Omega\colon\ |y - a| < r\}| \geqslant 2m(\Omega) r^2$$

for all $a \in \partial\Omega$ and all $r > 0$.

In [34], Tukhvatullina proved a sufficient condition of regularity for multidimensional domains in $\mathbb{R}^n$, $n \geqslant 2$. Some examples of regular domains were considered in [34]. In particular, concentric circles with radii R_1 and R_2, when $R_2 \geqslant R_1/5$, and balls with removed spherical sector are examples of regular domains.

Let us remember that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, and $D(\Omega)$ is the diameter of Ω,

$$\Omega_x = \{y \in \Omega\colon\ x + t(y - x) \in \Omega \quad \forall\, t \in [0,1]\},$$
$$B(n,s) = \frac{\Gamma(\frac{1}{2}(s+1))\Gamma(\frac{1}{2}n)}{\sqrt{\pi}\Gamma(\frac{1}{2}(n+s))},$$
$$c(s) := \frac{s^2 - \nu^2 q^2}{4} \quad \text{and} \quad \mu(s) := \frac{q^2 \lambda_\nu^2(2s/q)}{4}.$$

Since the function $f(t) = t^{s/2}$ is convex when $s \geqslant 2$ and $t > 0$, we can use Jensen's inequality to get

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\varrho_\nu(x)^s} \,\mathrm{d}\omega(\nu) \geqslant \Big(\int_{\mathbb{S}^{n-1}} \frac{1}{\varrho_\nu(x)^2} \,\mathrm{d}\omega(\nu) \Big)^{s/2}.$$

Consequently, for regular domains we have

$$\frac{1}{\varrho(x,s)^s} := \int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}\omega(\nu)}{\varrho_\nu(x)^s} \geqslant \frac{1}{m(\Omega)^s \delta(x)^s}.$$

It is obvious that $|\Omega_x| \leqslant |\Omega|$. Therefore, the following theorem holds.

Theorem 5.2. *Let Ω be a regular domain in $\mathbb{R}^n$ and let $m(\Omega)$ be the regularity constant for the domain Ω. Suppose that $s \geqslant 2$, $q > 0$ and $\nu \in [0, s/q]$. If $s + 3 \geqslant q$, then for all $f \in C_0^2(\Omega)$*

$$\frac{1}{n(n+2)} \int_\Omega \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^n \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(\frac{c(s+2)}{m(\Omega)^{s+3} \delta(x)^{s+3}} + \frac{\mu(s+2) 2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

If $s + 3 \leqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{n(n+2)} \int_\Omega \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^n \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(\frac{c(s+2)}{m(\Omega)^{s+3} \delta(x)^{s+3}} + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

Example 1. *Let Ω_0 be concentric circles with radii R_1 and R_2, where $R_2 \geqslant R_1/5$. It is proved in [34] that $m(\Omega_0) = 2\sqrt{12}$. Consequently, if $s \geqslant 2$ and $s + 3 \leqslant q$, then for all $f \in C_0^2(\Omega)$*

$$\frac{1}{8} \int_\Omega \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^2 \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(\frac{c(s+2)}{48^{(s+3)/2} \delta(x)^{s+3}} + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{1}{R_1^2 - R_2^2} \Big)^{q/2} \Big) \,\mathrm{d}x.$$

Example 2. *Let Ω_1 be a ball with the spherical sector removed. Let R be the radius of the ball. Consider the cone that corresponds to the removed spherical sector. By α we denote the cone angle, i.e., the angle between the rim of the cap and the direction to the middle of the cap as seen from the sphere center. In* [34] *it was shown that*

$$m(\Omega_1) = \frac{2\sqrt{7}}{\sin\frac{1}{4}\alpha}.$$

Consequently, if $s \geqslant 2$ and $s+3 \leqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{15}\int_\Omega \Big[|\Delta u(x)|^2 + 2\sum_{i,j=1}^{3}\Big|\frac{\partial^2 u(x)}{\partial x_i \partial y_j}\Big|^2\Big]\frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s)\int_\Omega |u(x)|^2\Big(\Big(\frac{\sin\frac{1}{4}\alpha}{2\sqrt{7}}\Big)^{s+3}\frac{c(s+2)}{\delta(x)^{s+3}} + \frac{\mu(s+2)}{\delta(x)^{s-q+3}}\Big(\frac{1}{R^3\cos^2\frac{1}{4}\alpha}\Big)^{q/3}\Big)\,\mathrm{d}x.$$

Above we use that $|\mathbb{S}^2| = 4\pi$ and $|\Omega_1| = \frac{4}{3}\pi R^3\cos\frac{1}{4}\alpha$.

5.3. Domains with θ-cone condition. The boundary $\partial\Omega$ is said to satisfy the θ-cone condition if every $x \in \partial\Omega$ is the vertex of the circular cone C_x of the semi angle θ which lies entirely in $\mathbb{R}^n \setminus \Omega$, see [12].

Assume that $|\mathbb{S}^{n-1}|$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$, $|\Omega|$ is the volume of the set Ω, $D(\Omega)$ is the diameter of Ω, $h = h(\frac{1}{2}\sin\theta)$,

$$B(n,s) = \frac{\Gamma(\frac{1}{2}(s+1))\Gamma(\frac{1}{2}n)}{\sqrt{\pi}\Gamma(\frac{1}{2}(n+s))},$$
$$c_s = \frac{((s-1)^2 - \nu^2 q^2)^{p/2}}{p^p} \quad \text{and} \quad \mu_s = \frac{p}{2}\frac{q^2\lambda_\nu^2(2(s-1)/q)}{(s-1)^2 - \nu^2 q^2}.$$

By $h(\alpha)$ we denote the solid angle subtended at the origin by a ball of radius $\alpha < 1$, whose centre is at the distance 1 from the origin. If $\partial\Omega$ satisfies the θ-cone condition, then for all $x \in \Omega$

$$\frac{1}{\varrho(x,s)^s} \geqslant \frac{h\left(\frac{1}{2}\sin\theta\right)}{2^s\delta(x)^s}.$$

For more information we refer to the book [12], page 86.

Using the last estimates and Theorem 5.1, we get the following theorem.

Theorem 5.3. *Let Ω be a domain in $\mathbb{R}^n$ and $\partial\Omega$ satisfy the θ-cone condition. Suppose that $s>0$, $q>0$ and $\nu\in[0,s/q]$. If $s\geqslant 1$ and $s+3\geqslant q$, then for all $f\in C_0^2(\Omega)$*

$$\frac{1}{n(n+2)}\int_\Omega\Big[|\Delta u(x)|^2+2\sum_{i,j=1}^n\Big|\frac{\partial^2u(x)}{\partial x_i\partial y_j}\Big|^2\Big]\frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s)\int_\Omega|u(x)|^2\Big(c(s+2)\frac{h\big(\frac12\sin\theta\big)}{2^{s+3}\delta(x)^{s+3}}+\frac{\mu(s+2)2^{s-q+3}}{D(\Omega)^{s-q+3}}\Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|}\Big)^{q/n}\Big)\,\mathrm{d}x.$$

If $s\geqslant 1$ and $s+3\leqslant q$, then for all $f\in C_0^2(\Omega)$

$$\frac{1}{n(n+2)}\int_\Omega\Big[|\Delta u(x)|^2+2\sum_{i,j=1}^n\Big|\frac{\partial^2u(x)}{\partial x_i\partial y_j}\Big|^2\Big]\frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s)\int_\Omega|u(x)|^2\Big(c(s+2)\frac{h\big(\frac12\sin\theta\big)}{2^{s+3}\delta(x)^{s+3}}+\frac{\mu(s+2)}{\delta(x)^{s-q+3}}\Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|}\Big)^{q/n}\Big)\,\mathrm{d}x.$$

If $0<s\leqslant 1$ and $s+3\geqslant q$, then for all $f\in C_0^2(\Omega)$

$$\frac{3}{n(n+2)}\frac{2^{s-1}}{D(\Omega)^{s-1}}\int_\Omega|\Delta u(x)|^2\,\mathrm{d}x$$
$$\geqslant c(s)\int_\Omega|u(x)|^2\Big(c(s+2)\frac{h(\frac12\sin\theta)}{2^{s+3}\delta(x)^{s+3}}+\frac{\mu(s+2)2^{s-q+3}}{D(\Omega)^{s-q+3}}\Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|}\Big)^{q/n}\Big)\,\mathrm{d}x.$$

If $0<s\leqslant 1$ and $s+3\leqslant q$, then for all $f\in C_0^2(\Omega)$

$$\frac{3}{n(n+2)}\frac{2^{s-1}}{D(\Omega)^{s-1}}\int_\Omega|\Delta u(x)|^2\,\mathrm{d}x$$
$$\geqslant c(s)\int_\Omega|u(x)|^2\Big(c(s+2)\frac{h(\frac12\sin\theta)}{2^{s+3}\delta(x)^{s+3}}+\frac{\mu(s+2)}{\delta(x)^{s-q+3}}\Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega_x|}\Big)^{q/n}\Big)\,\mathrm{d}x.$$

5.4. Inequalities in convex sets. Let Ω be a convex domain. It is known that for convex domains

$$\varrho(x;s)^{-s}:=\int_{\mathbb{S}^{n-1}}\frac{\mathrm{d}\omega(\nu)}{\varrho_\nu(x)^s}\geqslant\frac{B(n,s)}{\delta(x)^s},\quad\text{where } B(n,s)=\frac{\Gamma(\frac12(s+1))(\frac12 n)}{\sqrt{\pi}\Gamma(\frac12(n+s))}.$$

Recall that

$$c(s):=\frac{s^2-\nu^2q^2}{4}\quad\text{and}\quad\mu(s):=\frac{q^2\lambda_\nu^2(2s/q)}{4}.$$

Moreover, for the case of convex domains $|\Omega_x| = |\Omega|$ holds. Taking into account Theorem 5.1 we obtain the following theorem.

Theorem 5.4. *Let Ω be a convex domain in $\mathbb{R}^n$. Suppose that $s > 0$, $q > 0$ and $\nu \in [0, s/q]$. If $s \geqslant 1$ and $s + 3 \geqslant q$, then for all $f \in C_0^2(\Omega)$*

$$\frac{1}{n(n+2)} \int_\Omega \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^{n} \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \frac{B(n, s+3)}{\delta(x)^{s+3}} + \frac{\mu(s+2) 2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

If $s \geqslant 1$ and $s + 3 < q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{n(n+2)} \int_\Omega \Big[|\Delta u(x)|^2 + 2 \sum_{i,j=1}^{n} \Big| \frac{\partial^2 u(x)}{\partial x_i \partial y_j} \Big|^2 \Big] \frac{\mathrm{d}x}{\delta(x)^{s-1}}$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \frac{B(n, s+3)}{\delta(x)^{s+3}} + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

If $0 < s < 1$ and $s + 3 \geqslant q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{n(n+2)} \frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \frac{B(n, s+3)}{\delta(x)^{s+3}} + \frac{\mu(s+2) 2^{s-q+3}}{D(\Omega)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

If $0 < s < 1$ and $s + 3 < q$, then for all $f \in C_0^2(\Omega)$

$$\frac{1}{n(n+2)} \frac{2^{s-1}}{D(\Omega)^{s-1}} \int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x$$
$$\geqslant c(s) \int_\Omega |u(x)|^2 \Big(c(s+2) \frac{B(n, s+3)}{\delta(x)^{s+3}} + \frac{\mu(s+2)}{\delta(x)^{s-q+3}} \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{q/n} \Big) \,\mathrm{d}x.$$

Corollary 5.1. *Let Ω be a convex domain in $\mathbb{R}^n$. Then for all $f \in C_0^2(\Omega)$*

$$\int_\Omega |\Delta u(x)|^2 \,\mathrm{d}x \geqslant \frac{9}{16} \int_\Omega \frac{|u(x)|^2}{\delta(x)^4} + K n(n+2) \Big(\frac{|\mathbb{S}^{n-1}|}{n|\Omega|} \Big)^{4/n} \int_\Omega |u(x)|^2 \,\mathrm{d}x,$$

where $K = \frac{1}{3} \lambda_0^2(\frac{3}{2}) \approx 0.417322$.

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