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NECESSARY AND SUFFICIENT CONDITIONS FOR THE TWO-WEIGHT WEAK TYPE MAXIMAL INEQUALITY IN ORLICZ CLASS

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Abstract. We collect known and prove new necessary and sufficient conditions for the weighted weak type maximal inequality of the form

$$\Phi_1(\lambda)\varrho(\{x\in X\colon\ M_\mu f(x)>\lambda\})\leqslant c\int_X \Phi_2(c|f(x)|)\sigma(x)\,\mathrm{d}\mu(x),$$

which extends some known results.

Keywords: weight; weak type inequality; Hardy-Littlewood maximal function; Orlicz class

MSC 2020: 42B25, 46E30

1. Introduction

As an important part of harmonic analysis, weighted theory has attracted much attention for a long time. One fundamental result in the weighted theory for Hardy-Littlewood maximal functions is attributed to Muckenhoupt, see [8]. He stated that the weighted weak type inequality of the form

$$\varrho(\{x\in\mathbb{R}^n\colon\ Mf(x)>\lambda\})\leqslant\frac{c}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p\sigma(x)\,\mathrm{d}x,\quad 1<p<\infty, \tag{1.1}$$

holds, if and only if $(\varrho,\sigma)\in A_p$, namely

$$\biggl\{\frac{1}{|Q|}\int_Q\varrho(x)\,\mathrm{d}x\biggr\}\biggl\{\frac{1}{|Q|}\int_Q(\sigma(x))^{-1/(p-1)}\,\mathrm{d}x\biggr\}^{p-1}\leqslant c\quad\forall\,Q.$$

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Muckenhoupt's result reveals a deep connection between the boundedness of Hardy-Littlewood maximal function on different function spaces and the weight functions.

It is an important and interesting topic of the weighted theory to consider the generalized form of Muckenhoupt's result in various function spaces. There are many excellent related works in Orlicz spaces, here we only list a few which we are interested in. Gallardo in [3] introduced the A_Φ-condition and used it to characterize the two-weight weak type inequality of Hardy-Littlewood maximal functions in Orlicz spaces. Bagby in [1], and Bloom and Kerman in [2] considered separately a one-weight extra-weak type inequality and a one-weight weak type inequality for Hardy-Littlewood maximal functions in Orlicz spaces. Gogatishvili and Kokilashvili in [4] obtained some necessary and sufficient conditions of a four-weight weak type maximal inequality. Pick has made the most progress (see [9], Theorem 1), he showed without the Δ_2-condition the following theorem.

Theorem A. *Let (Φ_1,Ψ_1) and (Φ_2,Ψ_2) be two pairs of complementary N-functions, ϱ and σ be weight functions. Then the following statements are equivalent:*

(i) *there is a constant $c \geqslant 1$ such that two-weight weak type inequality*

$$\Phi_1(\lambda)\varrho(\{x\in X\colon\ M_\mu f(x)>\lambda\}) \leqslant c\int_X \Phi_2(c|f(x)|)\sigma(x)\,\mathrm{d}\mu(x) \tag{1.2}$$

holds for any μ-measurable function f and arbitrary $\lambda>0$;

(ii) *there is a constant $c_1 \geqslant 1$ such that the modified two-weight Jensen inequality*

$$\Phi_1(|f|_B)\varrho(B) \leqslant c_1\int_B \Phi_2(c_1|f(x)|)\sigma(x)\,\mathrm{d}\mu(x) \tag{1.3}$$

holds for any μ-measurable function f and any ball B;

(iii) *$(\varrho,\sigma)\in A_{\Phi_1,\Phi_2}$: there are constants $c_2 \geqslant 1$ and $\varepsilon>0$ such that the inequality*

$$\Phi_1\Big(\frac{\varepsilon}{\lambda\mu B}\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\Big)\varrho(B) \leqslant c_2\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \tag{1.4}$$

holds for arbitrary $\lambda>0$ and any ball B.

The symbols appeareding in Theorem A will be illustrated in the next section. For the strong type inequality of Hardy-Littlewood maximal functions in weighted Orlicz spaces, one can refer to [4], [5], [9].

The purpose of this paper is to make a further study of the necessary and sufficient conditions for the weighted inequality (1.2) to hold. We recall a basic property of the A_p weight: $(\varrho,\sigma)\in A_p$ if and only if $(\sigma^{-1/(p-1)},\varrho^{-1/(p-1)})\in A_{p'}$, where

$p' = p/(p-1)$. It follows from Muckenhoupt's result that the weighted inequality (1.1) holds, if and only if the weighted inequality

$$\int_{\{x\in\mathbb{R}^n\colon\, Mf(x)>\lambda\}} \sigma^{-1/(p-1)}(x)\,\mathrm{d}x \leqslant \frac{c}{\lambda^{p'}} \int_{\mathbb{R}^n} |f(x)|^{p'} \varrho^{-1/(p-1)}(x)\,\mathrm{d}x \tag{1.5}$$

holds. Based on the equivalence of (1.1) and (1.5), we try our best to find a generalized form of (1.5) in Orlicz spaces, which is equivalent to (1.2).

The main result of this paper is stated as follows:

Theorem 1.1. *Let* (Φ_1,Ψ_1) *and* (Φ_2,Ψ_2) *be two pairs of complementary* N-*functions,* ϱ *and* σ *be weight functions. Then the following statements are equivalent:*

(i) *there is a constant* $c\geqslant 1$ *such that the two-weight weak type inequality*

$$\Phi_1(\lambda)\varrho(\{x\in X\colon\, M_\mu f(x)>\lambda\}) \leqslant c\int_X \Phi_2(c|f(x)|)\sigma(x)\,\mathrm{d}\mu(x)$$

holds for any μ-*measurable function* f *and arbitrary* $\lambda>0$;

(ii) *there is a constant* $c_1\geqslant 1$ *such that the inequality*

$$\int_{\{x\in X\colon\, M_\mu f(x)>\lambda\}} \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c_1\int_X \Psi_1\Big(c_1\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x) \tag{1.6}$$

holds for any μ-*measurable function* f *and arbitrary* $\lambda>0$;

(iii) *there is a constant* $c_2\geqslant 1$ *such that the inequality*

$$\int_B \Psi_2\Big(\frac{|f|_B}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c_2\int_B \Psi_1\Big(c_2\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x) \tag{1.7}$$

holds for any μ-*measurable function* f *and any ball* B;

(iv) *there are constants* $c_3\geqslant 1$ *and* $\varepsilon>0$ *such that the inequality*

$$\int_B \Psi_2\Big(\varepsilon\frac{\Phi_1(\lambda)}{\lambda}\,\frac{\varrho(B)}{\mu B\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c_3\Phi_1(\lambda)\varrho(B) \tag{1.8}$$

holds for arbitrary $\lambda>0$ *and any ball* B.

Our result presents some new necessary and sufficient conditions for the weighted inequality (1.2) to hold, which extends some known results. In particular, a generalized form of (1.5) in Orlicz spaces is obtained, see Theorem 1.1 (ii).

2. Preliminaries

Let (X,μ) be a complete measure space with a quasimetric d, we assume that each ball $B=B(x,r)=\{y\colon\ d(x,y)<r\}$ is μ-measurable, $x,y\in X$, $r>0$, and μ is a doubling measure with respect to d, there is a constant $c>0$ such that $\mu B(x,2r)\leqslant c\mu B(x,r)$ for all $x\in X$ and $r>0$. We also assume that the space (X,d,μ) possesses the Besicovitch property: Let E be a bounded subset of X and $\mathcal{F}$ be a family of balls covering E such that, for every $x\in E$, there is a ball $B_x=B(x,r_x)\in\mathcal{F}$. Then there exist an at most countable family $\{B_i\}=\{B(x_i,r_{x_i})\}$ and a constant $c>0$, independent of E and $\{B_i\}$, such that:

(i) $E\subset\bigcup\limits_i B_i$;

(ii) $\sum\limits_i \chi_{B_i}\leqslant c$, where χ_{B_i} is the characteristic function of B_i.

We should remark that the Besicovitch property still holds for a unbounded subset E, if $\sup\{r_x, x\in E\}<\infty$.

Let f be a μ-measurable locally integrable function on X, set

$$(f)_B=\frac{1}{\mu B}\int_B f(x)\,\mathrm{d}\mu.$$

The maximal operator of f is defined by

$$M_\mu f(x)=\sup_{x\in B}\frac{1}{\mu B}\int_B |f(y)|\,\mathrm{d}\mu(y),\quad x\in X,$$

where the supremum is taken over all balls B containing x.

An almost everywhere positive μ-measurable locally integrable function is called a *weight function*. Let ϱ be a weight function, we set

$$\varrho(B)=\int_B \varrho(x)\,\mathrm{d}\mu(x).$$

Let Φ be an N-function, then the function given by $\Psi(t)=\sup\{st-\Phi(s)\}$ is called the *complementary function* of Φ. Here $\Phi(s)$ is an N-function if and only if $\Psi(t)$ is an N-function. We call (Φ,Ψ) a *pair of complementary* an N-functions, which satisfy the Young inequality

$$st\leqslant\Phi(s)+\Psi(t),$$

see [10].

Let (Φ,Ψ) be a pair of complementary N-functions, then $\Phi(t)/t$ and $\Psi(t)/t$ are continuous and increasing on $(0,\infty)$ and satisfy

$$\Psi\Big(\frac{\Phi(t)}{t}\Big)\leqslant\Phi(t)\leqslant\Psi\Big(\frac{\Phi(t)}{t}\Big),\quad \Phi\Big(\frac{\Psi(t)}{t}\Big)\leqslant\Psi(t)\leqslant\Phi\Big(2\frac{\Psi(t)}{t}\Big)\tag{2.1}$$

for all $t>0$, see [9].

Throughout this paper, we use c and c_i to denote positive constants. They may denote different constants at different occurrences.

3. Proof of Theorem 1.1

Proof. The equivalence between (i) and (iv) can be obtained by Theorem 2.2.3 of [6]. We show that (ii) $\Leftrightarrow$ (iii) and each of them is equivalent to the modified two-weight Jensen inequality (1.3); we then complete the proof by Theorem A, see Theorem 1 of [9].

(ii) $\Rightarrow$ (iii) Since $B \subset \{x \in X\colon\ M_\mu(2f\chi_B)(x) > |f|_B\}$, then the inequality (1.7) can be obtained directly from (1.6).

(iii) $\Rightarrow$ (ii) Without loss of generality, we may assume that $|f|_B > 0$. By the Young inequality, we have

$$\begin{aligned}\Phi_1(|f|_B)\varrho(B) &= \frac{1}{\mu B}\int_B |f(x)|\frac{\Phi_1(|f|_B)\varrho(B)}{|f|_B}\,\mathrm{d}\mu(x)\\ &= \frac{1}{\mu B}\int_B |f(x)|\Big(\int_B \frac{\Phi_1(|f|_B)\varrho(y)}{|f|_B}\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x)\\ &= \frac{1}{2c_2}\int_B 2c_2^2|f(x)|\frac{(\mu B)^{-1}\int_B \Phi_1(|f|_B)\varrho(y)/(c_2|f|_B)\,\mathrm{d}\mu(y)}{\sigma(x)}\sigma(x)\,\mathrm{d}\mu(x)\\ &\leqslant \frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|)\sigma(x)\,\mathrm{d}\mu(x)\\ &\quad + \frac{1}{2c_2}\int_B \Psi_2\Big(\frac{(\mu B)^{-1}\int_B \Phi_1(|f|_B)\varrho(y)/(c_2|f|_B)\,\mathrm{d}\mu(y)}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x).\end{aligned}$$

It follows from (1.7) and (2.1) that

$$\begin{aligned}\Phi_1(|f|_B)\varrho(B) &\leqslant \frac{1}{2c_2}\int_B \Phi_2(2c_2^2|f(x)|)\sigma(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_B \Psi_1\Big(\frac{\Phi_1(|f|_B)}{|f|_B}\Big)\varrho(x)\,\mathrm{d}\mu(x)\\ &\leqslant \frac{1}{2}c_3\int_B \Phi_2(c_3|f(x)|)\sigma(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_B \Phi_1(|f|_B)\varrho(x)\,\mathrm{d}\mu(x)\\ &= \frac{1}{2}c_3\int_B \Phi_2(c_3|f(x)|)\sigma(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\Phi_1(|f|_B)\varrho(B),\end{aligned}$$

where $c_3 = \max\{c_2^{-1}, 2c_2^2\}$, from which we obtain that the modified two-weight Jensen inequality (1.3) holds. For each natural number n, set

$$M_\mu^n f(x) = \sup \frac{1}{\mu B}\int_B |f(y)|\,\mathrm{d}\mu(y),$$

where the supremum is taken over all balls B in X, containing x with a radius $r \leqslant n$. For any ball B satisfying

$$\lambda \leqslant \frac{1}{\mu B}\int_B |f(y)|\,\mathrm{d}\mu(y),$$

then by Theorem A, the Young inequality and (1.4), we have

$$
\begin{aligned}
&\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\\
&\qquad \leqslant \frac{1}{\lambda\mu B}\int_B |f(y)|\Big(\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\Big)\,\mathrm{d}\mu(y)\\
&\qquad = \frac{1}{2c}\int_B \frac{2c}{\varepsilon}\frac{|f(y)|}{\varrho(y)}\frac{\varepsilon}{\lambda\mu B}\Big(\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\Big)\varrho(y)\,\mathrm{d}\mu(y)\\
&\qquad \leqslant \frac{1}{2c}\int_B \Psi_1\Big(\frac{2c}{\varepsilon}\frac{|f(y)|}{\varrho(y)}\Big)\varrho(y)\,\mathrm{d}\mu(y)\\
&\qquad\quad + \frac{1}{2c}\int_B \Phi_1\Big(\frac{\varepsilon}{\lambda\mu B}\Big(\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\Big)\Big)\varrho(y)\,\mathrm{d}\mu(y)\\
&\qquad \leqslant \frac{1}{2}c_4\int_B \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x) + \frac{1}{2}\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x),
\end{aligned}
$$

where $c_4 = \max\{c^{-1}, 2c\varepsilon^{-1}\}$. We then obtain

$$
\int_B \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c_4\int_B \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x). \tag{3.1}
$$

For each $x \in \{x \in X\colon\ M_\mu^n f(x) > \lambda\}$, there exists a ball B' ($x \in B'$, $0 < r_{B'} \leqslant n$) such that

$$
\int_{B'} |f(y)|\,\mathrm{d}\mu(y) > \lambda\mu(B').
$$

According to the Besicovitch property, we can take at most countable balls from the ball family $\{B'\}$ such that

$$
\{x \in X\colon\ M_\mu^n f(x) > \lambda\} \subset \bigcup_i B_i, \quad \sum_i \chi_{B_i} \leqslant c'. \tag{3.2}
$$

Then by (3.1) and (3.2), we obtain

$$
\begin{aligned}
&\int_{\{x\in X\colon\ M_\mu^n f(x)>\lambda\}} \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x)\\
&\qquad \leqslant \sum_i \int_{B_i} \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c_4\sum_i \int_{B_i} \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x)\\
&\qquad = c_4\sum_i \int_X \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\chi_{B_i}\,\mathrm{d}\mu(x) \leqslant c'c_4\int_X \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x).
\end{aligned}
$$

Let $n \to \infty$, then we have

$$
\int_{\{x\in X\colon\ M_\mu f(x)>\lambda\}} \Psi_2\Big(\frac{\lambda}{\sigma(x)}\Big)\sigma(x)\,\mathrm{d}\mu(x) \leqslant c'c_4\int_X \Psi_1\Big(c_4\frac{|f(x)|}{\varrho(x)}\Big)\varrho(x)\,\mathrm{d}\mu(x).
$$

□

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