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SOME BOUNDS FOR THE ANNIHILATORS OF LOCAL COHOMOLOGY AND EXT MODULES

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Abstract. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and t be a nonnegative integer. Let M and N be two finitely generated R-modules. In certain cases, we give some bounds under inclusion for the annihilators of $\mathrm{Ext}^t_R(M,N)$ and $\mathrm{H}^t_{\mathfrak{a}}(M)$ in terms of minimal primary decomposition of the zero submodule of M, which are independent of the choice of minimal primary decomposition. Then, by using those bounds, we compute the annihilators of local cohomology and Ext modules in certain cases.

Keywords: local cohomology module; Ext module; annihilator; primary decomposition

MSC 2020: 13D45, 13D07

1. Introduction

Throughout the paper, R is a commutative Noetherian ring with nonzero identity. The ith local cohomology of an R-module M with respect to an ideal $\mathfrak{a}$ was defined by Grothendieck as follows:

$$\mathrm{H}^i_{\mathfrak{a}}(M) = \varinjlim_n \mathrm{Ext}^i_R(R/\mathfrak{a}^n, M),$$

see [6], [7], [10] for more details.

In this section, we assume M is a nonzero finitely generated R-module, N is a Gorenstein R-module, $0 = M_1 \cap \ldots \cap M_n$ is a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$ and $\mathfrak{a}$ is an ideal of R. We refer the reader to [12], Section 6 for basic properties of primary decomposition of modules and to [13], [14] for more details about the Gorenstein modules (see also the paragraph before Lemma 2.4).

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We denote for an R-module M $\sup\{i \in \mathbb{N}_0\colon \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$ by $\mathrm{cd}_R(\mathfrak{a}, M)$. Assume $d = \dim_R(M) < \infty$. By Grothendieck's Vanishing Theorem, $\mathrm{cd}_R(\mathfrak{a}, M) \leqslant d$. When $\mathrm{cd}_R(\mathfrak{a}, M) = d$, then we have

$$\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}_i)=d} M_i\Bigr). \tag{1.1}$$

This equality is proved by Lynch (see [11], Theorem 2.4) whenever R is a complete local ring and $M = R$. In [5], Theorem 2.6, Bahmanpour et al. proved that $\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))$ whenever $\mathfrak{a} = \mathfrak{m}$ and R is a complete local ring, where $T_R(\mathfrak{a}, M)$ denotes the largest submodule N of M such that $\mathrm{cd}_R(\mathfrak{a}, N) < \mathrm{cd}_R(\mathfrak{a}, M)$. Then Bahmanpour in [4], Theorem 3.2, extended the result of Lynch for the R-module M. Next, Atazadeh et al. in [2], Proposition 3.8 proved this equality whenever R is a local ring (not necessarily complete) and finally in [1], Corollary 2.7, they extended it to the nonlocal case. (Note that $T_R(\mathfrak{a}, M) = \bigcap_{\mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}_i)=\mathrm{cd}_R(\mathfrak{a}, M)} M_i$ (see [2], Remark 2.5) also, if $(R, \mathfrak{m})$ is a complete local ring and $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then by the Lichtenbaum-Hartshorne Vanishing Theorem, $\mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = d$ if and only if $\dim_R(R/\mathfrak{p}) = d$ and $\sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}$.)

In the second section (see Theorem 2.5 and Remark 2.6) for an arbitrary integer t we give a bound for the annihilator of $\mathrm{Ext}^t_R(M, N)$ in terms of minimal primary decomposition of the zero submodule of M. More precisely, we show that

$$\mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \Delta(t)} M_i\Bigr) \subseteq \mathrm{Ann}_R(\mathrm{Ext}^t_R(M, N)) \subseteq \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i\Bigr), \tag{1.2}$$

where $\Delta(t) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Supp}_R(N)\colon \mathrm{ht}_R(\mathfrak{p}) \leqslant t\}$, $\Sigma(t) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) \cap \mathrm{Supp}_R(N)\colon \mathrm{ht}_R(\mathfrak{p}) = t\}$ and $\mathrm{MinAss}_R(M)$ denotes the set of minimal elements of $\mathrm{Ass}_R(M)$. If $t = \mathrm{grade}(\mathrm{Ann}_R(M), N) < \infty$, then the above index sets are equal and we can compute the annihilator of $\mathrm{Ext}^t_R(M, N)$. Note that in general, for an arbitrary integer t there is not a subset Σ of $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(\mathrm{Ext}^t_R(M, N)) = \mathrm{Ann}_R\bigl(M/\bigcap_{\mathfrak{p}_i \in \Sigma} M_i\bigr)$, see Example 2.7.

In the third section, we consider the annihilators of local cohomology modules. By using the above bound on the annihilators of Ext modules, when $(R, \mathfrak{m})$ is a local ring, we show in Theorem 3.2 that

$$\mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \Delta'(t)} M_i\Bigr) \subseteq \mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M)) \subseteq \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \Sigma'(t)} M_i\Bigr), \tag{1.3}$$

where $\Delta'(t) = \{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon \dim_R(R/\mathfrak{p}) \geqslant t\}$ and $\Sigma'(t) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M)\colon \dim_R(R/\mathfrak{p}) = t\}$. Next, whenever R is not necessarily local, in Theorem 3.4, we give a bound for the annihilator of the top local cohomology module $\mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}_R(\mathfrak{a},M)}(M)$, which implies equality (1.1) when $d = \mathrm{cd}_R(\mathfrak{a}, M)$. Finally, for each t, in Theorem 3.6, we provide a bound for the annihilator of $\mathrm{H}_{\mathfrak{a}}^t(M)$ when M is Cohen-Macaulay, and also we compute its annihilator at $t = \mathrm{grade}(\mathfrak{a}, M)$. All the given bounds are independent of the choice of minimal primary decomposition. We adopt the convention that the intersection of empty family of submodules of an R-module M is M.

2. Bounds for the annihilators of Ext-modules

Assume M, N are finitely generated R-modules such that N is a Gorenstein module, and $0 = M_1 \cap \ldots \cap M_n$ is a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$. We refer the reader to [12], Section 6 for basic properties and unexplained terminologies about the primary decomposition of modules and to [13], [14] for more details about the Gorenstein modules. In this section (see Theorem 2.5) for each integer t we give a bound for the annihilator of $\mathrm{Ext}_R^t(M, N)$ in terms of minimal primary decomposition of the zero submodule of M, which is independent of the choice of minimal primary decomposition. As an application, in the case, where $t = \mathrm{grade}(\mathrm{Ann}_R(M), N)$, we compute the annihilator of $\mathrm{Ext}_R^t(M, N)$. More precisely, for $t = \mathrm{grade}(\mathrm{Ann}_R(M), N)$ we have

$$\mathrm{Ann}_R(\mathrm{Ext}_R^t(M, N)) = \mathrm{Ann}_R\Big(M/\bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i\Big),$$

where $\Sigma(t) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) \cap \mathrm{Supp}_R(N)\colon\ \mathrm{ht}_R(\mathfrak{p}) = t\}$, see Theorem 2.5 and Remark 2.6. Note that in general, for an arbitrary integer t there is not a subset Σ of $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(\mathrm{Ext}_R^t(M, N)) = \mathrm{Ann}_R\big(M/\bigcap_{\mathfrak{p}_i \in \Sigma} M_i\big)$, see Example 2.7. These results will be used in the third section to compute the annihilators of local cohomology modules.

Before proving these results, we need some lemmas.

Lemma 2.1 ([12], Theorem 6.8). *Let M be a nonzero finitely generated R-module. Let $\mathrm{Ass}_R(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, and $0 = M_1 \cap \ldots \cap M_n$ be a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$. Assume Φ is a subset of $\mathrm{Ass}_R(M)$ and $N = \bigcap_{\mathfrak{p}_i \in \Phi} M_i$. Then*

$$\mathrm{Ass}_R(M/N) = \Phi \quad \textit{and} \quad \mathrm{Ass}_R(N) = \mathrm{Ass}_R(M) \setminus \Phi.$$

Assume N is a submodule of an R-module M. For any multiplicatively closed subset S of R we denote the contraction of $S^{-1}N$ under the canonical map $M \to S^{-1}M$ by $S_M(N)$. Assume $\Sigma \subseteq \operatorname{Ass}_R(M)$. We say that Σ is an isolated subset of $\operatorname{Ass}_R(M)$ if it satisfies the following condition: if $\mathfrak{q} \in \operatorname{Ass}_R(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$, then $\mathfrak{q} \in \Sigma$.

The following lemma is well-known, but we prove it for the readers' convenience.

Lemma 2.2 ([3], Theorem 4.10, Exercise 4.23). *Let M be a finitely generated R-module, and N a proper submodule of M. Let $N = \bigcap\limits_{i=1}^{n} N_i$ be a minimal primary decomposition of N in M with $\operatorname{Ass}_R(M/N_i) = \mathfrak{p}_i$ for all $1 \leqslant i \leqslant n$. Assume Σ is an isolated subset of $\operatorname{Ass}_R(M/N)$. Then*

$$\bigcap_{\mathfrak{p}_i \in \Sigma} N_i = S_M(N),$$

where $S = R \setminus \bigcup\limits_{\mathfrak{p} \in \Sigma} \mathfrak{p}$. In particular, $\bigcap\limits_{\mathfrak{p}_i \in \Sigma} N_i$ is independent of the choice of minimal primary decomposition of N in M.

Proof. Assume $\Sigma \subseteq \operatorname{Ass}_R(M/N)$ is an isolated subset of $\operatorname{Ass}_R(M/N)$ and $S = R \setminus \bigcup\limits_{\mathfrak{p} \in \Sigma} \mathfrak{p}$. If $S^{-1}\Big(M/\bigcap\limits_{\mathfrak{p}_i \in \operatorname{Ass}_R(M/N) \setminus \Sigma} N_i\Big) \neq 0$, then there exists

$$\mathfrak{q} \in \operatorname{Ass}_R\Bigg(M/\bigcap_{\mathfrak{p}_i \in \operatorname{Ass}_R(M/N) \setminus \Sigma} N_i\Bigg) = \operatorname{Ass}_R(M/N) \setminus \Sigma$$

such that $\mathfrak{q} \cap S = \emptyset$. Since $\mathfrak{q} \cap S = \emptyset$, by the Prime Avoidance Theorem, $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Sigma$. But Σ is an isolated subset of $\operatorname{Ass}_R(M/N)$ and so $\mathfrak{q} \in \Sigma$, which is a contradiction. Hence, $S^{-1}\Big(\bigcap\limits_{\mathfrak{p}_i \in \operatorname{Ass}_R(M/N) \setminus \Sigma} N_i\Big) = S^{-1}M$. It follows that $S^{-1}N = \bigcap\limits_{\mathfrak{p}_i \in \Sigma} S^{-1}N_i$. Contracting both sides under the canonical map $M \to S^{-1}M$ we obtain $(S^{-1}N)^c = \bigcap\limits_{\mathfrak{p}_i \in \Sigma} (S^{-1}N_i)^c$. Now, assume $\mathfrak{p}_i \in \Sigma$. It is clear that $N_i \subseteq (S^{-1}N_i)^c$. Conversely, if $m \in (S^{-1}N_i)^c$, then $m/1 \sim n/s$ for some $n \in N_i$ and $s \in S$. Hence, $tsm = tn \in N_i$ for some $t \in S$. Since N_i is a $\mathfrak{p}_i$-primary submodule of M and $ts \notin \mathfrak{p}_i$, we have $m \in N_i$. Therefore, $N_i = (S^{-1}N_i)^c$, and hence $(S^{-1}N)^c = \bigcap\limits_{\mathfrak{p}_i \in \Sigma} N_i$. This completes the proof. $\square$

Remark 2.3. Let the situation and notations be as in the above lemma. Assume, in addition that $\Sigma = \emptyset$, and we consider the above lemma in this special case separately. It is clear that Σ is an isolated subset of $\operatorname{Ass}_R(M/N)$ and $\bigcap\limits_{\mathfrak{p}_i \in \Sigma} N_i = M$ because the intersection of the empty family of subsets of a set M is M. On the other hand, we have $S = R \setminus \bigcup_{\mathfrak{p}\in\Sigma} \mathfrak{p} = R$. Since $0 \in S$, we obtain $S^{-1}(N) = S^{-1}(M) = 0$, and so the contraction of $S^{-1}(N)$ under the map $M \to S^{-1}(M)$ is M. Therefore, we have $S_M(N) = M = \bigcap_{\mathfrak{p}_i\in\Sigma} N_i$ in this case.

Let $(R, \mathfrak{m})$ be a local ring. A nonzero finitely generated R-module G is said to be *Gorenstein* if

$$\operatorname{depth}_R(G) = \dim_R(G) = \operatorname{inj}\dim_R(G) = \operatorname{depth}_R(R) = \dim_R(R)$$

(so R is Cohen-Macaulay) or equivalently, $\operatorname{Ext}^i_R(R/\mathfrak{m}, G)$ is nonzero only at $i = \dim_R(G)$, see [13], Theorem 3.11. More generally, if R is not necessarily local, a nonzero finitely generated R-module G is said to be Gorenstein if $G_\mathfrak{p}$ is a Gorenstein $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Supp}_R(G)$, see [13], Corollary 3.7. When $(R, \mathfrak{m})$ is a complete local ring, then Gorenstein modules under isomorphism are the nonempty finite direct sums of the canonical module, see [14], Corollary 2.7.

The following property of Gorenstein modules is needed in the proof of the main theorem of this section.

Lemma 2.4. *Let G be a Gorenstein R-module and $\mathfrak{p}$ a prime ideal of R. Then $\mathfrak{p} \in \operatorname{Supp}_R(G)$ if and only if $G \neq \mathfrak{p}G$.*

Proof. Assume $\mathfrak{p} \in \operatorname{Supp}_R(G)$. Hence, $G_\mathfrak{p} \neq 0$ and consequently, $G_\mathfrak{p} \neq \mathfrak{p}R_\mathfrak{p}G_\mathfrak{p}$ by Nakayama's Lemma. It follows that $G \neq \mathfrak{p}G$. Conversely, assume $G \neq \mathfrak{p}G$. Thus, there exists $\mathfrak{q} \in \operatorname{Supp}_R(G)$ such that $G_\mathfrak{q} \neq \mathfrak{p}R_q G_\mathfrak{q}$. Therefore, $\mathfrak{p} \subseteq \mathfrak{q}$, and hence [13], Corollary 4.14, implies that $\mathfrak{p} \in \operatorname{Supp}_R(G)$. $\square$

Now we are ready to state and prove the main theorem of this section which provides bound for the annihilators of Ext modules. Local version of this theorem (see Remark 2.6 equation (2.1)) will be used to compute the annihilators of local cohomology modules in the next section.

Theorem 2.5. *Let M, N be nonzero finitely generated R-modules and let $0 = M_1 \cap \ldots \cap M_n$ be a minimal primary decomposition of the zero submodule of M with $\operatorname{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$. Let $t \in \mathbb{N}_0$ and set $\Delta(t) = \{\mathfrak{p} \in \operatorname{Ass}_R(M)\colon \operatorname{grade}(\mathfrak{p}, N) \leqslant t\}$, $\Sigma(t) = \{\mathfrak{p} \in \operatorname{MinAss}_R(M)\colon \operatorname{grade}(\mathfrak{p}, N) = t\}$, $S^t = R \setminus \bigcup_{\mathfrak{p}\in\Delta(t)} \mathfrak{p}$, and $T^t = R \setminus \bigcup_{\mathfrak{p}\in\Sigma(t)} \mathfrak{p}$. Then*

(i) $\bigcap_{\mathfrak{p}_i\in\Delta(t)} M_i = S^t_M(0)$ *and* $\bigcap_{\mathfrak{p}_i\in\Sigma(t)} M_i = T^t_M(0)$. *In particular,* $\bigcap_{\mathfrak{p}_i\in\Delta(t)} M_i$ *and* $\bigcap_{\mathfrak{p}_i\in\Sigma(t)} M_i$ *are independent of the choice of minimal primary decomposition of the zero submodule of M.*

(ii) $S^t_M(0)$ *is the largest submodule L of M such that $\operatorname{Ext}^i_R(L, N) = 0$ for all $i \leqslant t$.*

(iii) *There is the inclusion*

$$\operatorname{Ann}_R(M/S^t_M(0)) \subseteq \operatorname{Ann}_R(\operatorname{Ext}^t_R(M,N)).$$

If, in addition, N is a Gorenstein module, then

$$\operatorname{Ann}_R(\operatorname{Ext}^t_R(M,N)) \subseteq \operatorname{Ann}_R(M/T^t_M(0)).$$

(iv) *If N is a Gorenstein module such that $\operatorname{Supp}_R(M)\cap\operatorname{Supp}_R(N)\neq\emptyset$ and $t = \operatorname{grade}(\operatorname{Ann}_R(M),N)$, then $\Delta(t)=\Sigma(t)$ and*

$$\operatorname{Ann}_R(\operatorname{Ext}^t_R(M,N)) = \operatorname{Ann}_R(M/T^t_M(0)).$$

Proof. Set $S = S^t_M(0)$ and $T = T^t_M(0)$.

(i) Since $\Delta(t)$ and $\Sigma(t)$ are isolated subsets of $\operatorname{Ass}_R(M)$, (i) is an immediate consequence of Lemma 2.2.

(ii) By Lemma 2.1, in view of [7], Proposition 1.2.10, we have

$$\begin{aligned}\operatorname{grade}(\operatorname{Ann}_R(S),N) &= \operatorname{grade}\big(\sqrt{\operatorname{Ann}_R(S)},N\big) = \operatorname{grade}\Bigg(\bigcap_{\mathfrak{p}\in\operatorname{Ass}_R(S)}\mathfrak{p},N\Bigg)\\ &= \min_{\mathfrak{p}\in\operatorname{Ass}_R(S)}\operatorname{grade}(\mathfrak{p},N) = \min_{\mathfrak{p}\in\operatorname{Ass}_R(M)\setminus\Delta(t)}\operatorname{grade}(\mathfrak{p},N) > t.\end{aligned}$$

Since $\operatorname{grade}(\operatorname{Ann}_R(S),N) > t$, we have $\operatorname{Ext}^i_R(S,N) = 0$ for all $i\leqslant t$ by [7], Proposition 1.2.10 (e). Also, we note that if $\Delta(t) = \operatorname{Ass}_R(M)$, then $S=0$ and $\operatorname{grade}(\operatorname{Ann}_R(S),N) = \operatorname{grade}(R,N)=\infty$. Now, assume L is a submodule of M such that $\operatorname{Ext}^i_R(L,N)=0$ for all $i\leqslant t$. Suppose for the sake of contradiction that $L\nsubseteq S$. Then

$$0\neq L/(L\cap S)\cong (L+S)/S\subseteq M/S.$$

Thus, $\emptyset\neq\operatorname{Ass}_R(L/(L\cap S))\subseteq\Delta(t)$. Hence, there exists $\mathfrak{p}\in\operatorname{Ass}_R(L/(L\cap S))\subseteq V(\operatorname{Ann}_R(L))$ such that $\operatorname{grade}(\mathfrak{p},N)\leqslant t$. But this is impossible, because by our assumption, $\operatorname{grade}(\operatorname{Ann}_R(L),N)>t$; see again [7], Proposition 1.2.10 (e). Hence, $L\subseteq S$ and the proof of (ii) is completed.

(iii) Since $\operatorname{Ext}^t_R(S,N)=0$, the exact sequence $0\to S\to M\to M/S\to 0$ induces the epimorphism $\operatorname{Ext}^t_R(M/S,N)\to\operatorname{Ext}^t_R(M,N)$. It follows that

$$\operatorname{Ann}_R(M/S)\subseteq\operatorname{Ann}_R(\operatorname{Ext}^t_R(M/S,N))\subseteq\operatorname{Ann}_R(\operatorname{Ext}^t_R(M,N))$$

and hence the first inclusion in (iii) holds. To prove the second inclusion in (iii), assume that N is a Gorenstein module. If $\Sigma(t)=\emptyset$, then $T=M$ by Remark 2.3 and there is nothing to prove. Hence, suppose that $\Sigma(t)\neq\emptyset$, $\mathfrak{p}_i\in\Sigma(t)$ and $y\in\operatorname{Ann}_R(\operatorname{Ext}^t_R(M,N))$. Since $\operatorname{grade}(\mathfrak{p}_i,N)=t<\infty$, we have $\mathfrak{p}_iN\neq N$ and so, by Lemma 2.4, $\mathfrak{p}_i\in\operatorname{Supp}_R(N)$. Hence, $N_{\mathfrak{p}_i}$ is a Gorenstein $R_{\mathfrak{p}_i}$-module, see [13], Corollary 3.7. Because N is Cohen-Macaulay, we have $\dim_{R_{\mathfrak{p}_i}}(N_{\mathfrak{p}_i}) = \operatorname{grade}(\mathfrak{p}_i,N)=t$

and so by [13], Theorem 4.12, we have $\dim_{R_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i}) = \dim_{R_{\mathfrak{p}_i}}(N_{\mathfrak{p}_i}) = t$. We proved that $N_{\mathfrak{p}_i}$ is a Gorenstein $R_{\mathfrak{p}_i}$-module of dimension t, and hence, in view of the faithfully flatness of completion, we can deduce that $\widehat{N_{\mathfrak{p}_i}}$ is also a Gorenstein $\widehat{R_{\mathfrak{p}_i}}$-module of dimension t. Hence, $\widehat{N_{\mathfrak{p}_i}} \cong \omega^{\alpha}_{\widehat{R_{\mathfrak{p}_i}}}$ for some $\alpha \in \mathbb{N}$ (see [14], Corollary 2.7), where $\omega_{\widehat{R_{\mathfrak{p}_i}}}$ denotes the canonical module of $\widehat{R_{\mathfrak{p}_i}}$. Since $\widehat{R_{\mathfrak{p}_i}}$ is a Cohen-Macaulay complete local ring of dimension t, by the Local Duality Theorem (see [6], Theorem 11.2.8 and Remarks 10.2.2 (ii)) we have

$$\begin{aligned}
\operatorname{Ann}_{R_{\mathfrak{p}_i}}(\operatorname{Ext}^t_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}, N_{\mathfrak{p}_i})) &= R_{\mathfrak{p}_i} \cap \operatorname{Ann}_{\widehat{R_{\mathfrak{p}_i}}}(\operatorname{Ext}^t_{\widehat{R_{\mathfrak{p}_i}}}(\widehat{M_{\mathfrak{p}_i}}, \widehat{N_{\mathfrak{p}_i}})) \\
&= R_{\mathfrak{p}_i} \cap \operatorname{Ann}_{\widehat{R_{\mathfrak{p}_i}}}(\operatorname{Ext}^t_{\widehat{R_{\mathfrak{p}_i}}}(\widehat{M_{\mathfrak{p}_i}}, \omega_{\widehat{R_{\mathfrak{p}_i}}})) \\
&= R_{\mathfrak{p}_i} \cap \operatorname{Ann}_{\widehat{R_{\mathfrak{p}_i}}}(\Gamma_{\widehat{\mathfrak{p}_i R_{\mathfrak{p}_i}}}(\widehat{M_{\mathfrak{p}_i}})) \\
&= \operatorname{Ann}_{R_{\mathfrak{p}_i}}(\Gamma_{\mathfrak{p}_i R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})) = \operatorname{Ann}_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}).
\end{aligned}$$

(Note that since $\mathfrak{p}_i$ is a minimal element of $\operatorname{Ass}_R(M)$, it follows that $\dim_{R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) = 0$ and hence, $\Gamma_{\mathfrak{p}_i R_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}) = M_{\mathfrak{p}_i}$.)

Now, if $1 \leqslant j \neq i \leqslant n$, then $(M/M_j)_{\mathfrak{p}_i} = 0$, because $\operatorname{Ass}_R(M/M_j) = \{\mathfrak{p}_j\}$ and $\mathfrak{p}_i$ is a minimal element of $\operatorname{Ass}_R(M)$. Thus, $(M_j)_{\mathfrak{p}_i} = M_{\mathfrak{p}_i}$ for all $1 \leqslant j \neq i \leqslant n$ and so $\left(\bigcap_{j=1}^{n} M_j\right)_{\mathfrak{p}_i} \cong (M_i)_{\mathfrak{p}_i}$. Since $M_{\mathfrak{p}_i} \cong (M/0)_{\mathfrak{p}_i} \cong \left(M/\bigcap_{j=1}^{n} M_j\right)_{\mathfrak{p}_i} \cong (M/M_i)_{\mathfrak{p}_i}$, we have $y/1 \in (\operatorname{Ann}_R(M/M_i))_{\mathfrak{p}_i}$, and hence $y/1 \sim z/s$ for some $z \in \operatorname{Ann}_R(M/M_i)$, $s \in R \backslash \mathfrak{p}_i$. Thus, $rsy = rz \in \operatorname{Ann}_R(M/M_i)$ for some $r \in R \setminus \mathfrak{p}_i$. Hence, $rsyM \subseteq M_i$. Since M_i is a $\mathfrak{p}_i$-primary submodule of M, it follows from $rs \notin \mathfrak{p}_i$ that $yM \subseteq M_i$. Because $\mathfrak{p}_i$ is an arbitrary element of $\Sigma(t)$, $yM \subseteq \bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i$, and by part (i), this implies that $yM \subseteq T$. Thus, $\operatorname{Ann}_R(\operatorname{Ext}^t_R(M, N)) \subseteq \operatorname{Ann}_R(M/T)$.

(iv) Assume N is a Gorenstein module such that $\operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N) \neq \emptyset$. Thus, $N/(\operatorname{Ann}_R(M))N \neq 0$, and so $t = \operatorname{grade}(\operatorname{Ann}_R(M), N) < \infty$, see [7], Definition 1.2.6. It is clear that $\Sigma(t) \subseteq \Delta(t)$. To prove the reverse inclusion, let $\mathfrak{p} \in \Delta(t)$. Since $\operatorname{Ann}_R(M) \subseteq \mathfrak{p}$, we obtain $\operatorname{grade}(\operatorname{Ann}_R(M), N) \leqslant \operatorname{grade}(\mathfrak{p}, N)$ and consequently $\operatorname{grade}(\operatorname{Ann}_R(M), N) = \operatorname{grade}(\mathfrak{p}, N)$. Now, let $\mathfrak{q} \in \operatorname{Supp}_R(M)$ be such that $\mathfrak{q} \subseteq \mathfrak{p}$. It follows from $\operatorname{grade}(\mathfrak{p}, N) = t < \infty$ that $\mathfrak{p} \in \operatorname{Supp}_R(N)$, and so $\mathfrak{q} \in \operatorname{Supp}_R(N)$ by [13], Corollary 4.14. Hence, by [7], Theorem 2.1.3 (b) and [13], Theorem 4.12, we have

$$\begin{aligned}
t &= \operatorname{grade}(\operatorname{Ann}_R(M), N) \leqslant \operatorname{grade}(\mathfrak{q}, N) = \dim_{R_{\mathfrak{q}}}(N_{\mathfrak{q}}) \\
&= \dim_{(R_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}}}(N_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} = \dim_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) - \dim_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/(\mathfrak{q}R_{\mathfrak{p}})N_{\mathfrak{p}}) \\
&= \operatorname{grade}(\mathfrak{p}, N) - \dim_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) = t - \dim_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}).
\end{aligned}$$

Therefore, $\dim_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) = 0$ or equivalently $\mathfrak{q} = \mathfrak{p}$. Hence, $\mathfrak{p} \in \operatorname{MinAss}_R(M)$ and consequently $\Delta(t) \subseteq \Sigma(t)$. □

For an integer t and an R-module M, we denote, respectively, the sets $\{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon \dim_R(R/\mathfrak{p}) = t\}$ and $\{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon \dim_R(R/\mathfrak{p}) \geqslant t\}$ by $\mathrm{Ass}_R^t(M)$ and $\mathrm{Ass}_R^{\geqslant t}(M)$. Similarly, $\mathrm{MinAss}_R^t(M)$ and $\mathrm{MinAss}_R^{\geqslant t}(M)$ are defined as above by replacing $\mathrm{Ass}_R(M)$ by $\mathrm{MinAss}_R(M)$. Also, when $\dim_R(M) < \infty$, the set of prime ideals in $\mathrm{Ass}_R(M)$ of the highest possible dimension $\{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon \dim_R(R/\mathfrak{p}) = \dim_R(M)\}$ is denoted by $\mathrm{Assh}_R(M)$.

Remark 2.6. Let the situation and notations be as in the above theorem. Let N be a Gorenstein R-module and $\mathfrak{p}$ a prime ideal of R. Then $\mathrm{grade}(\mathfrak{p}, N) = t < \infty$ if and only if $N \neq \mathfrak{p}N$ or equivalently, $\mathfrak{p} \in \mathrm{Supp}_R(N)$. Also, if $\mathfrak{p} \in \mathrm{Supp}_R(N)$, then $N_{\mathfrak{p}}$ is a Gorenstein module on the local ring $R_{\mathfrak{p}}$ and in view of [13], Theorem 4.12, we have

$$\mathrm{grade}(\mathfrak{p}, N) = \dim_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = \mathrm{ht}_R(\mathfrak{p}).$$

Hence,

$$\Delta(t) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) \cap \mathrm{Supp}_R(N)\colon\ \mathrm{ht}_R(\mathfrak{p}) \leqslant t\},$$
$$\Sigma(t) = \{\mathfrak{p} \in \mathrm{MinAss}_R(M) \cap \mathrm{Supp}_R(N)\colon\ \mathrm{ht}_R(\mathfrak{p}) = t\}.$$

In the remainder of this remark, assume in addition that R is a local ring of dimension d. Then $\mathrm{ht}_R(\mathfrak{p}) = d - \dim_R(R/\mathfrak{p})$ and $\mathrm{Supp}_R(N) = \mathrm{Spec}(R)$. Thus, the above theorem states that

$$\begin{aligned}\mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geqslant d-t}(M)} M_i\Bigr) &\subseteq \mathrm{Ann}_R(\mathrm{Ext}_R^t(M,N)) \\ &\subseteq \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathrm{MinAss}_R^{d-t}(M)} M_i\Bigr).\end{aligned} \tag{2.1}$$

In particular, if $M \neq 0$, then

$$\mathrm{grade}(\mathrm{Ann}_R(M), N) = \dim_R(N) - \dim_R(N/(\mathrm{Ann}_R(M))N) = d - \dim_R(M)$$

and the equality in Theorem 2.5 (iv) can be rewriten as

$$\mathrm{Ann}_R(\mathrm{Ext}_R^{d-\dim_R(M)}(M,N)) = \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i\Bigr). \tag{2.2}$$

These results are needed in the proof of the main theorem of the next section (see Theorem 3.2) which provides some bounds for the annihilators of local cohomology modules.

We end this section by two examples showing how we can compute the above bounds for the annihilators of Ext modules. Moreover, these examples show that to improve the upper bound in (2.1) we cannot replace the index set $\mathrm{MinAss}_R^{d-t}(M)$

by the larger sets $\mathrm{MinAss}_R^{\geqslant d-t}(M)$, $\mathrm{Ass}_R^{d-t}(M)$ or $\mathrm{Ass}_R^{\geqslant d-t}(M)$ and also to improve the lower bound in (2.1) we cannot replace the index set $\mathrm{Ass}_R^{\geqslant d-t}(M)$ by the smaller set $\mathrm{Ass}_R^{d-t}(M)$. Also, in general, for an arbitrary integer t there is not a subset Σ of $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(\mathrm{Ext}_R^t(M,N)) = \mathrm{Ann}_R\Big(M/(\bigcap_{\mathfrak{p}_i\in\Sigma} M_i)\Big)$.

Let U be a subset of an R-module M. We use $\langle U\rangle$ to denote the submodule of M generated by U. If $U=\{m_1,\ldots,m_n\}$, then we show $\langle U\rangle$ by $\langle m_1,\ldots,m_n\rangle$.

Example 2.7. Let K be a field and let $R=K[[X,Y]]$ be the ring of formal power series over K in indeterminates X, Y.

Set $M=R/\langle X^2,XY\rangle$, $M_1=\langle X\rangle/\langle X^2,XY\rangle$ and $M_2=\langle X^2,Y\rangle/\langle X^2,XY\rangle$. Then $0=M_1\cap M_2$ is a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_1)=\{\mathfrak{p}_1=\langle X\rangle\}$ and $\mathrm{Ass}_R(M/M_2)=\{\mathfrak{p}_2=\langle X,Y\rangle\}$. So $\mathrm{Ass}_R(M)=\{\mathfrak{p}_1,\mathfrak{p}_2\}$ and $\mathrm{MinAss}_R(M)=\{\mathfrak{p}_1\}$. Hence, we have

$$\mathrm{Ass}_R^{\geqslant 2-t}(M)=\begin{cases}\emptyset & \text{if } t=0,\\ \{\mathfrak{p}_1\} & \text{if } t=1,\\ \{\mathfrak{p}_1,\mathfrak{p}_2\} & \text{if } t=2,\end{cases} \quad\text{and}\quad \mathrm{MinAss}_R^{2-t}(M)=\begin{cases}\emptyset & \text{if } t=0,2,\\ \{\mathfrak{p}_1\} & \text{if } t=1.\end{cases}$$

It follows that

$$\mathrm{Ann}_R\Bigg(M/\bigcap_{\mathfrak{p}_i\in\mathrm{Ass}_R^{\geqslant 2-t}(M)} M_i\Bigg)=\begin{cases}R & \text{if } t=0,\\ \langle X\rangle & \text{if } t=1,\\ \langle X^2,XY\rangle & \text{if } t=2,\end{cases}$$

and

$$\mathrm{Ann}_R\Bigg(M/\bigcap_{\mathfrak{p}_i\in\mathrm{MinAss}_R^{2-t}(M)} M_i\Bigg)=\begin{cases}R & \text{if } t=0,2,\\ \langle X\rangle & \text{if } t=1.\end{cases}$$

Hence, Remark 2.6 implies that

$$\mathrm{Hom}_R(M,R)=0,\quad \mathrm{Ann}_R(\mathrm{Ext}_R^1(M,R))=\langle X\rangle$$

and

$$\langle X^2,XY\rangle\subseteq \mathrm{Ann}_R(\mathrm{Ext}_R^2(M,R))\subseteq R.$$

Also, since $\operatorname{inj\,dim}_R(R)=2$, we deduce that $\mathrm{Ext}_R^t(M,R)=0$ for all $t>2$.

Now, we directly compute $\mathrm{Ann}_R(\mathrm{Ext}_R^t(M,R))$ for all t (especially for $t=2$). It is straightforward to see that

$$\mathbf{P}\colon\ 0\longrightarrow R\xrightarrow{d_2} R^2\xrightarrow{d_1} R\xrightarrow{\varepsilon} M\longrightarrow 0$$

with $\varepsilon(f)=f+\langle X^2,XY\rangle$, $d_1(f,g)=X^2f+XYg$, $d_2(f)=(Yf,-Xf)$ for all $f,g\in R$ being a projective resolution of M. Applying the functor $\mathrm{Hom}_R(\cdot,R)$ to the delated projective resolution $\mathbf{P}_M$, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(R,R) \arrow[r, "d_1^*"] \arrow[d, "\cong"', "\alpha"] & \mathrm{Hom}_R(R^2,R) \arrow[r, "d_2^*"] \arrow[d, "\cong"', "\beta"] & \mathrm{Hom}_R(R,R) \arrow[r] \arrow[d, "\cong"', "\gamma"] & 0 \\
0 \arrow[r] & R \arrow[r, "\delta_1"] & R^2 \arrow[r, "\delta_2"] & R \arrow[r] & 0,
\end{tikzcd}$$

where α, β, γ are natural isomorphisms, $\delta_1(f) = (X^2f, XYf)$, and $\delta_2(f,g) = Yf - Xg$ for all $f, g \in R$. Hence,

$$\mathrm{Ext}^1_R(M,R) \cong \ker\delta_2/\operatorname{im}\delta_1 = \langle (X,Y)\rangle/\langle (X^2,XY)\rangle \cong R/\langle X\rangle,$$
$$\mathrm{Ext}^2_R(M,R) \cong R/\langle X,Y\rangle \quad \text{and} \quad \mathrm{Ext}^t_R(M,R) = 0 \quad \forall\, t \neq 1,2.$$

(Note that by our notation, $\ker\delta_2$ and $\operatorname{im}\delta_1$ are cyclic R-modules generated by the elements (X,Y) and (X^2,XY) of R^2, respectively.) It follows that

$$\mathrm{Ann}_R(\mathrm{Ext}^1_R(M,R)) = \langle X\rangle \quad \text{and} \quad \mathrm{Ann}_R(\mathrm{Ext}^2_R(M,R)) = \langle X,Y\rangle.$$

Thus, there is not a subset Σ of $\mathrm{Ass}_R(M)$ such that $\mathrm{Ann}_R(\mathrm{Ext}^2_R(M,R)) = \mathrm{Ann}_R\Big(M/(\bigcap_{\mathfrak{p}_i\in\Sigma} M_i)\Big)$. Moreover, for $t = 2$, this example shows that in the second inclusion of (2.1) in Remark 2.6, to obtain a better upper bound (under inclusion) of $\mathrm{Ann}_R(\mathrm{Ext}^t_R(M,R))$, we cannot replace the index set $\mathrm{MinAss}^{d-t}_R(M)$ by the larger sets $\mathrm{MinAss}^{\geqslant d-t}_R(M)$, $\mathrm{Ass}^{d-t}_R(M)$ or $\mathrm{Ass}^{\geqslant d-t}_R(M)$.

Example 2.8. Let K be a field and let $R = K[[X,Y,Z,W]]$ be the ring of formal power series over K in indeterminates X, Y, Z, W. Then R is a local ring with maximal ideal $\mathfrak{m} = \langle X,Y,Z,W\rangle$. Set $\mathfrak{p}_1 = \langle X,Y\rangle$, $\mathfrak{p}_2 = \langle Z,W\rangle$, and $M = R/(\mathfrak{p}_1\cap\mathfrak{p}_2)$. Then $\mathrm{depth}_R(R/\mathfrak{p}_1) = \mathrm{depth}_R(R/\mathfrak{p}_2) = 2$ and hence, $\mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p}_1) = \mathrm{H}^i_{\mathfrak{m}}(R/\mathfrak{p}_2) = 0$ for $i = 0,1$. Now, the exact sequence

$$0 \to M \to R/\mathfrak{p}_1 \oplus R/\mathfrak{p}_2 \to R/\mathfrak{m} \to 0$$

induces the exact sequence

$$0 \to \mathrm{H}^0_{\mathfrak{m}}(M) \to \mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}_1) \oplus \mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{p}_2) \to \mathrm{H}^0_{\mathfrak{m}}(R/\mathfrak{m}) \to \mathrm{H}^1_{\mathfrak{m}}(M)$$
$$\to \mathrm{H}^1_{\mathfrak{m}}(R/\mathfrak{p}_1) \oplus \mathrm{H}^1_{\mathfrak{m}}(R/\mathfrak{p}_2)$$

of local cohomology modules. It follows that $\mathrm{H}^0_{\mathfrak{m}}(M) = 0$ and $\mathrm{H}^1_{\mathfrak{m}}(M) \cong R/\mathfrak{m}$. Since R is a regular ring, it is Gorenstein (see [7], Proposition 3.1.20), and hence, R is the canonical module of R, see [7], Theorem 3.3.7. Therefore, by the Grothendieck duality (see [6], Theorem 11.2.8), we have $\mathrm{Hom}_R(\mathrm{Ext}^3_R(M,R),E(R/\mathfrak{m}))\cong \mathrm{H}^1_{\mathfrak{m}}(M)$. Thus, $\mathrm{Ann}_R(\mathrm{Ext}^3_R(M,R))=\mathfrak{m}$.

On the other hand, if $M_1=\mathfrak{p}_1/(\mathfrak{p}_1\cap\mathfrak{p}_2)$ and $M_2=\mathfrak{p}_2/(\mathfrak{p}_1\cap\mathfrak{p}_2)$, then $0=M_1\cap M_2$ is a minimal primary decomposition of the zero submodule of M. Since $\mathrm{Ass}^1_R(M)=\emptyset$, we have

$$R=\mathrm{Ann}_R\Bigg(M/\bigcap_{\mathfrak{p}_i\in\mathrm{Ass}^1_R(M)} M_i\Bigg)\nsubseteq \mathrm{Ann}_R(\mathrm{Ext}^3_R(M,R)).$$

Therefore, in the first inclusion of (2.1) in Remark 2.6, to obtain a better lower bound of $\mathrm{Ann}_R(\mathrm{Ext}^t_R(M,R))$, we cannot replace the index set $\mathrm{Ass}^{\geqslant d-t}_R(M)$ by the smaller set $\mathrm{Ass}^{d-t}_R(M)$.

3. Bounds for the annihilators of local cohomology modules

In this section we investigate the annihilators of local cohomology modules. For an R-module M, we denote $\sup\{i\in\mathbb{N}_0\colon \mathrm{H}^i_{\mathfrak{a}}(M)\neq 0\}$ by $\mathrm{cd}_R(\mathfrak{a},M)$. Let $\mathfrak{a}$ be a proper ideal of R, M a nonzero finitely generated R-module of dimension d, and $0=M_1\cap\ldots\cap M_n$ a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i)=\{\mathfrak{p}_i\}$ for all $1\leqslant i\leqslant n$. If $\mathrm{cd}_R(\mathfrak{a},M)=d<\infty$, then

$$\mathrm{Ann}_R(\mathrm{H}^d_{\mathfrak{a}}(M))=\mathrm{Ann}_R\Bigg(M/\bigcap_{\mathrm{cd}_R(\mathfrak{a},R/\mathfrak{p}_i)=d} M_i\Bigg),$$

see [1] and Section 1 for more details.

For an arbitrary integer t, when $(R,\mathfrak{m})$ is a local ring, we give a bound for $\mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M))$, see Theorem 3.2. Also, whenever R is not necessarily local, in Theorem 3.4, we provide a bound for $\mathrm{Ann}_R(\mathrm{H}^{\mathrm{cd}_R(\mathfrak{a},M)}_{\mathfrak{a}}(M))$ which implies the above equality when $\mathrm{cd}_R(\mathfrak{a},M)=\dim_R(M)$. Finally, when M is Cohen-Macaulay, a bound of $\mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{a}}(M))$ is given and at $t=\mathrm{grade}(\mathfrak{a},M)$, this annihilator is computed in Theorem 3.6.

Assume $(R,\mathfrak{m})$ is a local ring. The $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a faithfully flat R-module (see [12], Theorem 8.14), and so $R\subseteq\widehat{R}$. Applying [12], Theorem 23.2, to the ring homomorphism $\varphi\colon R\to\widehat{R}$ we obtain the following lemma.

Lemma 3.1 ([12], Theorem 23.2). *Let $(R,\mathfrak{m})$ be a local ring and M an R-module. Then:*

(i) *if $\mathfrak{p}\in\mathrm{Spec}(R)$ and $\mathfrak{P}\in\mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})$, then $R\cap\mathfrak{P}=\mathfrak{p}$.*

(ii)

$$\mathrm{Ass}_{\widehat{R}}(M\otimes_R\widehat{R})=\bigcup_{\mathfrak{p}\in\mathrm{Ass}_R(M)}\mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R}).$$

This lemma is used in the proof of the following theorem which is the main theorem of this section.

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a local ring and $t \in \mathbb{N}_0$. Let M be a nonzero finitely generated R-module and $0 = M_1 \cap \ldots \cap M_n$ a minimal primary decomposition of the zero submodule of M with $\mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$. Then:*

(i) $\bigcap\limits_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geqslant t}(M)} M_i = S_M^t(0)$ *and* $\bigcap\limits_{\mathfrak{p}_i \in \mathrm{MinAss}_R^t(M)} M_i = T_M^t(0)$, *where* $S^t = R \setminus \bigcup\limits_{\mathfrak{p} \in \mathrm{Ass}_R^{\geqslant t}(M)} \mathfrak{p}$ *and* $T^t = R \setminus \bigcup\limits_{\mathfrak{p} \in \mathrm{MinAss}_R^t(M)} \mathfrak{p}$. *In particular,* $\bigcap\limits_{\mathfrak{p}_i \in \mathrm{Ass}_R^{\geqslant t}(M)} M_i$ *and* $\bigcap\limits_{\mathfrak{p}_i \in \mathrm{MinAss}_R^t(M)} M_i$ *are independent of the choice of minimal primary decomposition of the zero submodule of M.*

(ii) $S_M^t(0)$ *is the largest submodule N of M such that* $\dim_R(N) < t$.

(iii) *There is the following bound for the annihilator of* $H_m^t(M)$

$$\mathrm{Ann}_R(M/S_M^t(0)) \subseteq \mathrm{Ann}_R(\mathrm{H}_{\mathfrak{m}}^t(M)) \subseteq \mathrm{Ann}_R(M/T_M^t(0)).$$

In particular, for $t = \dim_R(M)$ there are the equalities $S_M^t(0) = T_M^t(0) = \bigcap\limits_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i$, *and*

$$\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)) = \mathrm{Ann}_R\Bigg(M/\bigcap_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i\Bigg).$$

Proof. Set $S = S_M^t(0)$ and $T = T_M^t(0)$. It is clear that $\mathrm{Ass}_R^{\geqslant t}(M)$ and $\mathrm{MinAss}_R^t(M)$ are isolated subsets of $\mathrm{Ass}_R(M)$ and hence, (i) follows from Lemma 2.2.

To prove (ii), first note that $\mathrm{Ass}_R(S) = \mathrm{Ass}_R(M) \setminus \mathrm{Ass}_R^{\geqslant t}(M)$ by Lemma 2.1 and hence, $\dim_R(S) < t$. Now, assume that N is a submodule of M such that $\dim_R(N) < t$. Suppose, for the sake of contradiction, that $N \nsubseteq S$. Then

$$0 \neq N/(N \cap S) \cong (N + S)/S \subseteq M/S.$$

Hence,

$$\emptyset \neq \mathrm{Ass}_R(N/(N \cap S)) \subseteq \mathrm{Ass}_R(M/S) = \mathrm{Ass}_R^{\geqslant t}(M),$$

which is impossible, because $\dim_R(N/(N \cap S)) \leqslant \dim_R(N) < t$. This proves (ii).

Now, we prove (iii). In the case when $t = \dim_R(M)$, it is clear that

$$\mathrm{MinAss}_R^t(M) = \mathrm{Ass}_R^{\geqslant t}(M) = \mathrm{Assh}_R(M),$$

and so $S_M^t(0) = T_M^t(0) = \bigcap\limits_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i$. Therefore, the first part of (iii) yields the equality $\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{m}}^{\dim_R(M)}(M)) = \mathrm{Ann}_R\Big(M/\bigcap\limits_{\mathfrak{p}_i \in \mathrm{Assh}_R(M)} M_i\Big)$ whenever

$t = \dim_R(M)$. Also, we saw in (ii) that $\dim_R(S) < t$, and so we obtain $\mathrm{H}^t_{\mathfrak{m}}(M) \cong \mathrm{H}^t_{\mathfrak{m}}(M/S)$. Therefore,

$$\mathrm{Ann}_R(M/S) \subseteq \mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M/S)) = \mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M)).$$

Thus, to complete the proof of (iii), it only remains to prove that

$$\mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M)) \subseteq \mathrm{Ann}_R(M/T).$$

Set $d = \dim_R(R)$. First, assume that R is complete. By Cohen's structure theorem for complete local rings (see [12], Theorem 29.4 (ii)) there is a complete regular local ring R' such that $R = R'/I$ for some ideal I of R'. Now, let $h = \mathrm{ht}_{R'}(I)$ and $x_1, \dots, x_h$ be a maximal R'-sequence in I. Set $R'' = R'/(x_1, \dots, x_h)$ and $J = I/(x_1, \dots, x_h)$. Then R'' is a local Gorenstein ring of dimension d (see [7], Corollary 3.1.15) and $R \cong R''/J$. Now, let $\mathfrak{n}$ be the maximal ideal of R''. Then $\mathfrak{m} \cong \mathfrak{n}/J$. By the Grothendieck duality for Gorenstein rings (see [6], Theorem 11.2.5), there is the following isomorphism of R''-modules:

$$\mathrm{H}^t_{\mathfrak{n}}(M) \cong \mathrm{Hom}_{R''}(\mathrm{Ext}^{d-t}_{R''}(M, R''), E_{R''}(R''/\mathfrak{n})).$$

Also, by using the Independence Theorem under the ring homomorphism $R'' \to R''/J \cong R$, we obtain the following isomorphism of R''-modules:

$$\mathrm{H}^t_{\mathfrak{n}}(M) \cong \mathrm{H}^t_{\mathfrak{n}(R''/J)}(M) \cong \mathrm{H}^t_{\mathfrak{m}}(M)$$

(we recall that $\mathfrak{n}/J \cong \mathfrak{m}$). We refer the reader for more about the Independence Theorem to Theorem 4.2.1 of [6] or Proposition 2.11 (2) of [10]. Also, we note that any R-module N has an R''-module structure given by $r''x = (r'' + J)x = \psi(r'' + J)x$ for all $r'' \in R''$ and $x \in N$, where ψ denotes the ring isomorphism from R''/J to R. Hence, by [6], Remarks 10.2.2 (ii), we have

$$\mathrm{Ann}_{R''}(\mathrm{H}^t_{\mathfrak{m}}(M)) = \mathrm{Ann}_{R''}(\mathrm{H}^t_{\mathfrak{n}}(M)) = \mathrm{Ann}_{R''}(\mathrm{Ext}^{d-t}_{R''}(M, R'')).$$

For each $1 \leqslant i \leqslant n$ let P_i be the contraction of $\mathfrak{p}_i$ in R'' under the ring homomorphism $R'' \to R''/J \cong R$. Then $\mathrm{Ass}_{R''}(M) = \{P_1, \dots, P_n\}$ and there is the bijective correspondence between the sets $\mathrm{Ass}_{R''}(M)$ and $\mathrm{Ass}_R(M)$ given by $P_i \leftrightarrow \mathfrak{p}_i$. Also, $0 = M_1 \cap \ldots \cap M_n$ is a minimal primary decomposition of the zero submodule of M as R''-modules with $\mathrm{Ass}_{R''}(M/M_i) = \{P_i\}$ for all $1 \leqslant i \leqslant n$. Since R'' is Gorenstein, by Remark 2.6 equation (2.1) we obtain

$$\mathrm{Ann}_{R''}(\mathrm{H}^t_{\mathfrak{m}}(M)) \subseteq \mathrm{Ann}_{R''}\Bigg(M/\bigcap_{P_i \in \mathrm{MinAss}^t_{R''}(M)} M_i\Bigg).$$

For any R-module N we have $J \subseteq \mathrm{Ann}_{R''}(N)$ and so $\mathrm{Ann}_{R''/J}(N) = (\mathrm{Ann}_{R''}(N))/J$. Therefore, the above inclusion proves the claimed inclusion in the case when R is complete.

Now, suppose that R is not necessarily complete. Assume $0 = \bigcap\limits_{k \in K} \mathcal{M}_k$ is a minimal $\widehat{R}$-primary decomposition of the zero submodule of $\widehat{M}$ with $\mathrm{Ass}_{\widehat{R}}(\widehat{M}/\mathcal{M}_k) = \{\mathfrak{P}_k\}$. Since $\mathrm{Ass}_{\widehat{R}}(\widehat{M}) = \bigcup\limits_{i=1}^{n} \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}_i\widehat{R})$, there exist subsets $K_1, \ldots, K_n$ of K such that $K = \bigcup\limits_{i=1}^{n} K_i$, and for each i, $\mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}_i\widehat{R}) = \{\mathfrak{P}_k \colon k \in K_i\}$. Also, the subsets $K_1, \ldots, K_n$ of K are disjoint by Lemma 3.1 (i).

Assume $x \in \mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{m}}(M))$ and $\mathfrak{p}_i \in \mathrm{MinAss}^t_R(M)$. By the complete case,

$$x\widehat{R} \subseteq \mathrm{Ann}_{\widehat{R}}(\mathrm{H}^t_{\mathfrak{m}\widehat{R}}(\widehat{M})) \subseteq \mathrm{Ann}_{\widehat{R}}\Bigg(\widehat{M}/ \bigcap_{\mathfrak{P}_k \in \mathrm{MinAss}^t_{\widehat{R}}(\widehat{M})} \mathcal{M}_k\Bigg). \tag{3.1}$$

Now, suppose that $k \in K_i$ and $\mathfrak{P}_k \in \mathrm{Assh}_{\widehat{R}}(\widehat{M}/\widehat{M_i})$ (note that, by Lemma 3.1, $\mathrm{Ass}_{\widehat{R}}(\widehat{M}/\widehat{M_i}) = \mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}_i\widehat{R})$). We have

$$\dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}_k) = \dim_{\widehat{R}}(\widehat{M}/\widehat{M_i}) = \dim_R(M/M_i) = \dim_R(R/\mathfrak{p}_i) = t.$$

We show that $\mathfrak{P}_k$ is a minimal element of $\mathrm{Ass}_{\widehat{R}}(\widehat{M})$. Assume that $1 \leqslant i' \leqslant n$, $k' \in K_{i'}$ and $\mathfrak{P}_{k'} \subseteq \mathfrak{P}_k$. Then $\mathfrak{p}_{i'} = \mathfrak{P}_{k'} \cap R \subseteq \mathfrak{P}_k \cap R = \mathfrak{p}_i$. Since $\mathfrak{p}_i$ is a minimal element of $\mathrm{Ass}_R(M)$ and $K_1, \ldots, K_n$ are disjoint sets, we deduce that $i = i'$. It follows that both $\mathfrak{P}_k$ and $\mathfrak{P}_{k'}$ are elements of $\mathrm{Ass}_{\widehat{R}}(\widehat{M}/\widehat{M_i})$. Therefore,

$$\dim_{\widehat{R}}(\widehat{M}/\widehat{M_i}) = \dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}_k) \leqslant \dim_{\widehat{R}}(\widehat{R}/\mathfrak{P}_{k'}) \leqslant \dim_{\widehat{R}}(\widehat{M}/\widehat{M_i}),$$

and hence, $\mathfrak{P}_k = \mathfrak{P}_{k'}$. Thus, $\mathfrak{P}_k \in \mathrm{MinAss}^t_{\widehat{R}}(\widehat{M})$ and inclusion (3.1) yields $x\widehat{M} \subseteq \mathcal{M}_k$. Since $\mathfrak{P}_k$ is a minimal element of $\mathrm{Ass}_{\widehat{R}}(\widehat{M}/\widehat{M_i})$, it follows that the contraction of $(\widehat{M_i})_{\mathfrak{P}_k}$ under the canonical map $\widehat{M} \to \widehat{M}_{\mathfrak{P}_k}$, denoted by $\mathcal{N}_k$, is the $\mathfrak{P}_k$-primary component of each minimal primary decomposition of $\widehat{M_i}$ in $\widehat{M}$ (see Lemma 2.2 or [12], Theorem 6.8.3 (iii)). Hence, $\mathcal{N}_k/\widehat{M_i}$ is the $\mathfrak{P}_k$-primary component of each minimal primary decomposition of 0 in $\widehat{M}/\widehat{M_i}$. Also, we have $\mathcal{M}_k \subseteq \mathcal{N}_k$ because $\mathcal{M}_k$ is the contraction of the zero submodule under the map $\widehat{M} \to \widehat{M}_{\mathfrak{P}_k}$. Therefore, $x(\widehat{M}/\widehat{M_i}) \subseteq \mathcal{N}_k/\widehat{M_i}$. Since $\mathfrak{P}_k$ is an arbitrary element of $\mathrm{Assh}_{\widehat{R}}(\widehat{M}/\widehat{M_i})$, we have $x(\widehat{M}/\widehat{M_i}) \subseteq \bigcap\limits_{\mathfrak{P}_k \in \mathrm{Assh}_{\widehat{R}}(\widehat{M}/\widehat{M_i})} \mathcal{N}_k/\widehat{M_i}$. Hence, by Lemma 2.1, $\mathrm{Ass}_{\widehat{R}}(x(\widehat{M}/\widehat{M_i})) \subseteq \mathrm{Ass}_{\widehat{R}}(\widehat{M}/\widehat{M_i}) \setminus \mathrm{Assh}_{\widehat{R}}(\widehat{M}/\widehat{M_i})$. This yields

$$\dim_R(x(M/M_i)) = \dim_{\widehat{R}}(x(\widehat{M}/\widehat{M_i})) < \dim_{\widehat{R}}(\widehat{M}/\widehat{M_i}) = t.$$

Therefore, $\mathfrak{p}_i \notin \mathrm{Ass}_R(x(M/M_i))$ and hence, $\mathrm{Ass}_R(x(M/M_i)) = \emptyset$ or equivalently, $xM \subseteq M_i$. This proves the claimed inclusion and completes the proof. $\square$

Now, in the following theorem, we give a bound for the annihilator of top local cohomology module without the local assumption on R. But before that, we need the following lemma.

Lemma 3.3 ([9], Theorem 2.2). *Let $\mathfrak{a}$ be an ideal of R and M, N two finitely generated R-modules such that $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(N)$. Then $\operatorname{cd}_R(\mathfrak{a}, M) \leqslant \operatorname{cd}_R(\mathfrak{a}, N)$.*

Assume $\mathfrak{a}$ is an ideal of R and M is a finitely generated R module. Since $\operatorname{Supp}_R(M) = \operatorname{Supp}_R\Bigl(\bigoplus_{\mathfrak{p}\in\operatorname{Ass}_R(M)} R/\mathfrak{p}\Bigr)$, the above lemma implies that

$$\operatorname{cd}_R(\mathfrak{a}, M) = \operatorname{cd}_R\left(\mathfrak{a}, \bigoplus_{\mathfrak{p}\in\operatorname{Ass}_R(M)} R/\mathfrak{p}\right) = \sup\{\operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p})\colon\ \mathfrak{p} \in \operatorname{Ass}_R(M)\}.$$

By [6], Exercise 6.2.6 and Theorem 6.2.7, $\mathrm{H}^i_{\mathfrak{a}}(M)$ is zero for all i if and only if $M = \mathfrak{a}M$, and so in this case, we have $\operatorname{cd}_R(\mathfrak{a}, M) = \sup\emptyset = -\infty$. On the other hand, if $\mathfrak{a}$ is generated by $t \in \mathbb{N}_0$ elements, then $\operatorname{cd}_R(\mathfrak{a}, M) \leqslant t < \infty$, see [6], Theorem 3.3.1. Hence, $\operatorname{cd}_R(\mathfrak{a}, M)$ is a nonnegative integer if and only if $M \neq \mathfrak{a}M$.

Theorem 3.4. *Let M be a nonzero finitely generated R-module and $\mathfrak{a}$ an ideal of R such that $M \neq \mathfrak{a}M$. Let $c = \operatorname{cd}_R(\mathfrak{a}, M)$ and $0 = M_1 \cap \ldots \cap M_n$ be a minimal primary decomposition of the zero submodule of M with $\operatorname{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$ for all $1 \leqslant i \leqslant n$. Set $\Delta = \{\mathfrak{p} \in \operatorname{Ass}_R(M)\colon\ \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = c\}$ and $\Sigma = \{\mathfrak{p} \in \operatorname{Ass}_R(M)\colon \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = \dim_R(R/\mathfrak{p}) = c\}$. Then*

(i) *$\bigcap_{\mathfrak{p}_i\in\Delta} M_i = S_M(0)$, where $S = R \setminus \bigcup_{\mathfrak{p}_i\in\Delta} \mathfrak{p}$. In particular, $\bigcap_{\mathfrak{p}_i\in\Delta} M_i$ is independent of the choice of minimal primary decomposition of the zero submodule of M.*

(ii) *$S_M(0)$ is the largest submodule N of M such that $\operatorname{cd}_R(\mathfrak{a}, N) < c$,*

(iii)
$$\operatorname{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i\in\Delta} M_i\right) \subseteq \operatorname{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M)) \subseteq \operatorname{Ann}_R\left(M/\bigcap_{\mathfrak{p}_i\in\Sigma} M_i\right).$$

In particular, when $c = \dim_R(M)$, there are the equalities $\Delta = \Sigma$ and

$$\operatorname{Ann}_R(\mathrm{H}^{\dim_R(M)}_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M/S_M(0)).$$

Proof. Set $S = \bigcap_{\mathfrak{p}_i\in\Delta} M_i$ and $T = \bigcap_{\mathfrak{p}_i\in\Sigma} M_i$.

(i) If $\mathfrak{q} \in \operatorname{Ass}_R(M)$ and $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Delta$, then by Lemma 3.3,

$$c = \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{p}) \leqslant \operatorname{cd}_R(\mathfrak{a}, R/\mathfrak{q}) \leqslant \operatorname{cd}_R(\mathfrak{a}, M) = c.$$

It follows that $\mathfrak{q} \in \Delta$, and hence Δ is an isolated subset of $\operatorname{Ass}_R(M)$. Therefore (i) follows from Lemma 2.2.

(ii) Lemma 2.1 implies that $\mathrm{Ass}_R(S) = \{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) < c\}$. Hence, by Lemma 3.3, $\mathrm{cd}_R(\mathfrak{a}, S) < c$. Also, if N is a submodule of M such that $\mathrm{cd}_R(\mathfrak{a}, N) < c$, then

$$\mathrm{Ass}_R(N/(N \cap S)) = \mathrm{Ass}_R((N+S)/S) \subseteq \mathrm{Ass}_R(M/S) = \Delta.$$

Thus, if $\mathrm{Ass}_R(N/(N \cap S)) \neq \emptyset$, then $c = \mathrm{cd}_R(\mathfrak{a}, N/(N \cap S)) \leqslant \mathrm{cd}_R(\mathfrak{a}, N)$, which is impossible. Therefore, $N \subseteq S$ and the proof of (ii) is completed.

(iii) We proved in (ii) that $\mathrm{cd}_R(\mathfrak{a}, S) < c$. Therefore, $\mathrm{H}^c_{\mathfrak{a}}(M) \cong \mathrm{H}^c_{\mathfrak{a}}(M/S)$ and hence,

$$\mathrm{Ann}_R(M/S) \subseteq \mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M/S)) = \mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M)).$$

This proves the first inclusion. Now, we prove the second inclusion claimed in (iii).

Case 1: Assume that $c = \dim_R(M)$ and $(R, \mathfrak{m})$ is a complete local ring. For each prime ideal $\mathfrak{p}$, in view of Grothendieck's Vanishing Theorem (see [6], Theorem 6.1.2), we have $\mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) \leqslant \dim_R(R/\mathfrak{p})$. It follows that $\Delta = \Sigma$, and so $S = T$. Also, we have $\Delta = \{\mathfrak{p} \in \mathrm{Assh}_R(M)\colon \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}\}$ by the Lichtenbaum-Hartshorne Theorem. Therefore,

$$\sqrt{\mathfrak{a} + \mathrm{Ann}_R(M/S)} = \sqrt{\mathfrak{a} + \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R(M/S)} \mathfrak{p}} = \sqrt{\mathfrak{a} + \bigcap_{\mathfrak{p} \in \Delta} \mathfrak{p}}.$$

Since M is a finitely generated R-module, the set Δ is finite and so

$$\sqrt{\mathfrak{a} + \bigcap_{\mathfrak{p} \in \Delta} \mathfrak{p}} = \sqrt{\bigcap_{\mathfrak{p} \in \Delta} (\mathfrak{a} + \mathfrak{p})} = \bigcap_{\mathfrak{p} \in \Delta} \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}.$$

(Note that for ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and prime ideal $\mathfrak{q}$ we have $(\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c}) \subseteq \mathfrak{q}$ if and only if $\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c}) \subseteq \mathfrak{q}$. Therefore, $\sqrt{(\mathfrak{a} + \mathfrak{b}) \cap (\mathfrak{a} + \mathfrak{c})} = \sqrt{\mathfrak{a} + (\mathfrak{b} \cap \mathfrak{c})}$). Hence, $\sqrt{\mathfrak{a} + \mathrm{Ann}_R(M/S)} = \mathfrak{m}$ and we deduce from the Independence Theorem that

$$\mathrm{H}^c_{\mathfrak{a}}(M) \cong \mathrm{H}^c_{\mathfrak{a}}(M/S) \cong \mathrm{H}^c_{\mathfrak{m}}(M/S).$$

Also, since $\mathrm{Ass}_R(M/S) = \Delta = \Sigma \subseteq \mathrm{Assh}_R(M)$ and Δ is not empty, we have $\dim_R(M/S) = \dim_R(M) = c$ and $\mathrm{Assh}_R(M/S) = \mathrm{Ass}_R(M/S)$. Therefore, the previous theorem yields

$$\mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{m}}(M/S)) = \mathrm{Ann}_R(M/S).$$

Case 2: Assume that $c = \dim_R(M)$ and R is not necessarily local. As in the previous case, we have $\Delta = \Sigma$ and $S = T$. To prove $\mathrm{Ann}_R(\mathrm{H}^c_{\mathfrak{a}}(M)) \subseteq \mathrm{Ann}_R(M/S)$, assume that $x \in R$ and $xM \nsubseteq S$ and we show $x\mathrm{H}^c_{\mathfrak{a}}(M) \neq 0$. By (ii), $\mathrm{H}^c_{\mathfrak{a}}(xM) \neq 0$.

Thus, there exists a prime ideal $\mathfrak{m}$ such that $\mathrm{H}^c_{\mathfrak{a}R_\mathfrak{m}}(xM_\mathfrak{m}) \neq 0$ and consequently, $\mathrm{H}^c_{\mathfrak{a}\widehat{R_\mathfrak{m}}}(x\widehat{M_\mathfrak{m}}) \neq 0$. Therefore, $c \leqslant \mathrm{cd}_{\widehat{R_\mathfrak{m}}}(\mathfrak{a}\widehat{R_\mathfrak{m}}, x\widehat{M_\mathfrak{m}})$. It follows from Lemma 3.3 and Grothendieck's Vanishing Theorem that

$$c \leqslant \mathrm{cd}_{\widehat{R_\mathfrak{m}}}(\mathfrak{a}\widehat{R_\mathfrak{m}}, x\widehat{M_\mathfrak{m}}) \leqslant \mathrm{cd}_{\widehat{R_\mathfrak{m}}}(\mathfrak{a}\widehat{R_\mathfrak{m}}, \widehat{M_\mathfrak{m}}) \leqslant \dim_{\widehat{R_\mathfrak{m}}}(\widehat{M_\mathfrak{m}}) \leqslant \dim_R(M) = c.$$

Hence, $\dim_{\widehat{R_\mathfrak{m}}}(\widehat{M_\mathfrak{m}}) = \mathrm{cd}_{\widehat{R_\mathfrak{m}}}(\mathfrak{a}\widehat{R_\mathfrak{m}}, \widehat{M_\mathfrak{m}}) = c$. Since $\mathrm{H}^c_{\mathfrak{a}\widehat{R_\mathfrak{m}}}(x\widehat{M_\mathfrak{m}}) \neq 0$, we obtain $x\widehat{M_\mathfrak{m}} \nsubseteq S'$, where S' is the largest submodule of $\widehat{M_\mathfrak{m}}$ such that $\mathrm{cd}_{\widehat{R_\mathfrak{m}}}(\mathfrak{a}\widehat{R_\mathfrak{m}}, S') < c$. So, by the complete case, we have $x\mathrm{H}^c_{\mathfrak{a}\widehat{R_\mathfrak{m}}}(\widehat{M_\mathfrak{m}}) \neq 0$ and therefore, $x\mathrm{H}^c_\mathfrak{a}(M) \neq 0$. This proves the claimed inclusion (in fact equality) in the case when $c = \dim_R(M)$.

Case 3: Assume $c < \dim_R(M)$. If $\Sigma = \emptyset$, then $T = M$ and there is nothing to prove. Assume $\Sigma \neq \emptyset$. Since $\mathrm{cd}_R(\mathfrak{a}, T) \leqslant c$, the short exact sequence

$$0 \to T \to M \to M/T \to 0$$

induces the epimorphism $\mathrm{H}^c_\mathfrak{a}(M) \to \mathrm{H}^c_\mathfrak{a}(M/T)$. It follows that $\mathrm{Ann}_R(\mathrm{H}^c_\mathfrak{a}(M)) \subseteq \mathrm{Ann}_R(\mathrm{H}^c_\mathfrak{a}(M/T))$. Since $\mathrm{Ass}_R(M/T) = \Sigma \neq \emptyset$, we have

$$\mathrm{cd}_R(\mathfrak{a}, M/T) = \max_{\mathfrak{p} \in \mathrm{Ass}_R(M/T)} \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = \max_{\mathfrak{p} \in \Sigma} \mathrm{cd}_R(\mathfrak{a}, R/\mathfrak{p}) = c$$

and

$$\dim_R(M/T) = \max_{\mathfrak{p} \in \mathrm{Ass}_R(M/T)} \dim_R(R/\mathfrak{p}) = \max_{\mathfrak{p} \in \Sigma} \dim_R(R/\mathfrak{p}) = c.$$

Thus, $\dim_R(M/T) = \mathrm{cd}_R(\mathfrak{a}, M/T) = c$, and so $\mathrm{Ann}_R(\mathrm{H}^c_\mathfrak{a}(M/T)) = \mathrm{Ann}_R(M/T)$ by the previous case. This completes the proof. $\square$

When $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, $\mathfrak{a}$ is a nonzero proper ideal of R and $t = \mathrm{grade}(\mathfrak{a}, R)$, Bahmanpour calculated the annihilator of $\mathrm{H}^t_\mathfrak{a}(R)$ in [4], Theorem 2.2. The following theorem generalizes his result for Cohen-Macaulay modules whenever R is not necessarily local.

Lemma 3.5 ([8], Theorem 2.1)**.** *Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module such that $\mathfrak{a}M \neq M$. Then*

$$\mathrm{Ass}_R(\mathrm{H}^{\mathrm{grade}(\mathfrak{a},M)}_\mathfrak{a}(M)) = \{\mathfrak{p} \in V(\mathfrak{a})\colon\ \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \mathrm{grade}(\mathfrak{a}, M)\}.$$

Let M be an R-module. For $\mathfrak{p} \in \mathrm{Supp}_R(M)$, the M-height of $\mathfrak{p}$, denoted $\mathrm{ht}_M(\mathfrak{p})$, is the supremum of the lengths t of strictly descending chains

$$\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \ldots \supset \mathfrak{p}_t$$

of prime ideals in $\mathrm{Supp}_R(M)$. For an arbitrary ideal $\mathfrak{a}$ we define the M-height of $\mathfrak{a}$, denoted $\mathrm{ht}_M(\mathfrak{a})$, by

$$\mathrm{ht}_M(\mathfrak{a}) = \inf\{\mathrm{ht}_M(\mathfrak{p})\colon\ \mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a})\}.$$

In particular, if $\mathrm{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a}) = \emptyset$, then $\mathrm{ht}_M(\mathfrak{a}) = \inf \emptyset = \infty$.

Theorem 3.6. *Let $\mathfrak{a}$ be an ideal of R, M a nonzero finitely generated Cohen-Macaulay R-module, and $0 = M_1 \cap \ldots \cap M_n$ with $\mathrm{Ass}_R(M/M_i) = \mathfrak{p}_i$ for all $1 \leqslant i \leqslant n$ a minimal primary decomposition of the zero submodule of M. Then for each $t \in \mathbb{N}_0$,*

$$\mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) \subseteq \mathrm{Ann}_R\Bigl(M/\bigcap_{\mathrm{ht}_M(\mathfrak{a}+\mathfrak{p}_i)=t} M_i\Bigr).$$

Moreover, if $M \neq \mathfrak{a}M$ and $t = \mathrm{grade}(\mathfrak{a}, M)$, then the equality holds.

Proof. Set $\Sigma(t) = \{\mathfrak{p} \in \mathrm{Ass}_R(M)\colon\ \mathrm{ht}_M(\mathfrak{a}+\mathfrak{p}) = t\}$. To prove the claimed inclusion, assume that $x \in R$ and $x \notin \mathrm{Ann}_R(M/\bigcap_{\mathfrak{p}_i \in \Sigma(t)} M_i)$ and we show $x \notin \mathrm{Ann}_R(\mathrm{H}^t_{\mathfrak{a}}(M))$. Hence, $xM \nsubseteq M_i$ for some $\mathfrak{p}_i \in \Sigma(t)$. Therefore, $\mathrm{Ass}_R(x(M/M_i)) = \mathrm{Ass}_R(M/M_i) = \{\mathfrak{p}_i\}$. Suppose that $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{a}+\mathfrak{p}_i$ such that $\mathrm{ht}_M(\mathfrak{q}) = \mathrm{ht}_M(\mathfrak{a}+\mathfrak{p}_i) = t$. Then

$$\mathrm{Ass}_{R_{\mathfrak{q}}}(x(M/M_i)_{\mathfrak{q}}) = \mathrm{Ass}_{R_{\mathfrak{q}}}(M/M_i)_{\mathfrak{q}} = \{\mathfrak{p}_i R_{\mathfrak{q}}\}.$$

Therefore,

$$\sqrt{\mathfrak{a}R_{\mathfrak{q}} + \mathrm{Ann}_{R_{\mathfrak{q}}}(M/M_i)_{\mathfrak{q}}} = \sqrt{\mathfrak{a}R_{\mathfrak{q}} + \mathfrak{p}_i R_{\mathfrak{q}}} = \mathfrak{q}R_{\mathfrak{q}}.$$

Also, since $M_{\mathfrak{q}}$ is Cohen-Macaulay and $\mathfrak{p}_i R_{\mathfrak{q}} \in \mathrm{Ass}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$, we have

$$\dim_{R_{\mathfrak{q}}}(R_{\mathfrak{q}}/\mathfrak{p}_i R_{\mathfrak{q}}) = \dim_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = t.$$

Hence, $\dim_{R_{\mathfrak{q}}}((M/M_i)_{\mathfrak{q}}) = t$ and by Theorem 3.2, in view of the Independence Theorem we have

$$\mathrm{Ann}_{R_{\mathfrak{q}}}(\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}((M/M_i)_{\mathfrak{q}})) = \mathrm{Ann}_{R_{\mathfrak{q}}}(\mathrm{H}^t_{\mathfrak{q}R_{\mathfrak{q}}}((M/M_i)_{\mathfrak{q}})) = \mathrm{Ann}_{R_{\mathfrak{q}}}(M/M_i)_{\mathfrak{q}}.$$

Thus, $x\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}((M/M_i)_{\mathfrak{q}}) \neq 0$ because $x(M/M_i)_{\mathfrak{q}} \neq 0$. On the other hand, the exact sequence

$$0 \to (M_i)_{\mathfrak{q}} \to M_{\mathfrak{q}} \to (M/M_i)_{\mathfrak{q}} \to 0$$

induces the epimorphism $\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \to \mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}((M/M_i)_{\mathfrak{q}})$. Thus, $x\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \neq 0$ and consequently, $x\mathrm{H}^t_{\mathfrak{a}}(M) \neq 0$. This proves the claimed inclusion.

Finally, assume $t=\operatorname{grade}(\mathfrak{a},M)$ and we prove the reverse inclusion. Let $x\in R$ be such that $x\mathrm{H}^t_{\mathfrak{a}}(M)\neq 0$. Hence, there exists $\mathfrak{q}\in\operatorname{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(M))\subseteq\operatorname{Supp}_R(M/\mathfrak{a}M)$ such that $x\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}})\neq 0$. By the above lemma, $\operatorname{ht}_M(\mathfrak{q})=\operatorname{ht}_M(\mathfrak{a})$, and hence $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{a}+\operatorname{Ann}_R(M)$. Since $M_{\mathfrak{q}}$ is a Cohen-Macaulay module of dimension t, Theorem 3.2 and Independence Theorem yield

$$\operatorname{Ann}_{R_{\mathfrak{q}}}(\mathrm{H}^t_{\mathfrak{a}R_{\mathfrak{q}}}(M_{\mathfrak{q}}))=\operatorname{Ann}_{R_{\mathfrak{q}}}(\mathrm{H}^t_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}))=\operatorname{Ann}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}),$$

and so we have $xM_{\mathfrak{q}}\neq 0$. If $\mathfrak{q}\in\operatorname{Supp}_R\Big(\bigcap_{\mathfrak{p}_i\in\Sigma(t)}M_i\Big)$, then there is a $\mathfrak{p}\in\operatorname{Ass}_R\Big(\bigcap_{\mathfrak{p}_i\in\Sigma(t)}M_i\Big)=\operatorname{Ass}_R(M)\setminus\Sigma(t)$ such that $\mathfrak{p}\subseteq\mathfrak{q}$. Therefore,

$$t=\operatorname{ht}_M(\mathfrak{a})\leqslant\operatorname{ht}_M(\mathfrak{a}+\mathfrak{p})\leqslant\operatorname{ht}_M(\mathfrak{q})=t.$$

Hence, $\operatorname{ht}_M(\mathfrak{a}+\mathfrak{p})=t$, and so $\mathfrak{p}\in\Sigma(t)$, a contradiction. Thus, $\Big(\bigcap_{\mathfrak{p}_i\in\Sigma(t)}M_i\Big)_{\mathfrak{q}}=0$. It follows that $xM_{\mathfrak{q}}\nsubseteq\Big(\bigcap_{\mathfrak{p}_i\in\Sigma(t)}M_i\Big)_{\mathfrak{q}}$ and consequently, $xM\nsubseteq\bigcap_{\mathfrak{p}_i\in\Sigma(t)}M_i$. This proves the claimed equality in the case when $t=\operatorname{grade}(\mathfrak{a},M)$ and completes the proof. $\square$

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