Czechoslovak Mathematical Journal, 72 (147) (2022), 259–263

GENERALIZED DIVISOR PROBLEM FOR NEW FORMS OF HIGHER LEVEL

Krishnarjun Krishnamoorthy, Jhunsi

Received October 20, 2020. Published online June 21, 2021.

Abstract. Suppose that f is a primitive Hecke eigenform or a Mass cusp form for $\Gamma_0(N)$ with normalized eigenvalues $\lambda_f(n)$ and let $X > 1$ be a real number. We consider the sum

$$\mathcal{S}_k(X) := \sum_{n<X} \sum_{n=n_1,n_2,\dots,n_k} \lambda_f(n_1)\lambda_f(n_2)\dots\lambda_f(n_k)$$

and show that $\mathcal{S}_k(X) \ll_{f,\varepsilon} X^{1-3/(2(k+3))+\varepsilon}$ for every $k \geqslant 1$ and $\varepsilon > 0$. The same problem was considered for the case $N = 1$, that is for the full modular group in Lü (2012) and Kanemitsu et al. (2002). We consider the problem in a more general setting and obtain bounds which are better than those obtained by the classical result of Landau (1915) for $k \geqslant 5$. Since the result is valid for arbitrary level, we obtain, as a corollary, estimates on sums of the form $\mathcal{S}_k(X)$, where the sum involves restricted coefficients of some suitable half integral weight modular forms.

Keywords: generalized divisor problem; cusp form of higher level

MSC 2020: 11N37, 11F30

1. Introduction

The main theorem of this paper concerns the convoluted sums of Fourier coefficients of holomorphic modular forms as well as Maass forms. Since the proof follows through almost verbatim for the case of Maass forms, we shall focus only on holomorphic forms.

Let f be a primitive Hecke eigen cusp form of weight l for $\Gamma_0(N)$ and let

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(l-1)/2} e(nz) \tag{1}$$

DOI: 10.21136/CMJ.2021.0451-20

be the Fourier expansion of f at infinity. For the rest of the paper we assume that $\lambda_f(n)$s are normalized. For every $k \in \mathbb{N}$ and $X > 1$ we study sums of the form

$$\mathcal{S}_k(X) := \sum_{n<X} \lambda_{k,f}(n),$$

where

$$\lambda_{k,f}(n) = \sum_{n=n_1,n_2,\ldots,n_k} \lambda_f(n_1)\lambda_f(n_2)\ldots\lambda_f(n_k).$$

Sums of a much general type are considered in [11] for the case when f is a new form of full level. In the notation of [11], $\mathcal{S}_k(X)$ corresponds to $S_k(X,1)$. The novelty of our work is that we are introducing the generalized divisor problem for the case of new forms of higher level, which to the best of the author's knowledge has not been considered before except for the case when $k = 1$. The case of larger k was studied only for the case of modular forms of the full group, notably by Fomenko and others (see [3] for example). Latest results (for $N = 1$) include the following result of Lü, see [8],

$$\mathcal{S}_k(X) \ll \begin{cases} X^{3/5+\varepsilon}, & k = 3, \\ X^{1-3/(2k)+\varepsilon}, & k \geqslant 4. \end{cases} \tag{2}$$

Associated to f is the L function defined as the series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}. \tag{3}$$

It is known that $L(s,f)$ satisfies very nice properties including having an Euler product into local factors. From the work of Deligne, we know that $|\lambda_f(n)| \ll n^{\varepsilon}$ for every positive ε. Therefore $L(s,f)$ converges absolutely for $\Re(s) > 1$. Furthermore, it is known that $L(s,f)$ admits an analytic continuation to the whole complex plane and satisfies a functional equation. The strip $0 < \Re(s) < 1$ is known as the critical strip for $L(s,f)$.

Our proof follows from the Weyl type subconvexity bound for $L(f,s)$. Understanding the behaviour of L functions in the critical strip is an age old problem and is a source of many conjectures and open problems. Of particular interest are the so called *t-aspect bounds*. They are bounds of the form

$$|L(\tfrac{1}{2} + \mathrm{i}t, f)| \ll |t|^{\alpha}$$

for some positive number α.

Using the Phragmen-Lindelöf principle, it is possible to derive such bounds. They are called *convexity bounds*. Bounds which improve upon the convexity bounds are called *subconvexity bounds*.

When f is a holomorphic cusp form of full level, the convexity bound on $|L(s,f)|$ was broken by Good, see [4]. It is a long standing record and is yet to be improved upon[1]. For the case of a primitive holomorphic new form of higher level, similar bounds have been obtained recently by Booker et al. (see [2]) when $f \in S_k(\Gamma_1(N))$ and by Aggarwal (see [1]) for holomorphic modular forms as well as Maass forms of higher level. In particular we have the following theorem, see Theorem 1.3 of [1].

Theorem 1 (Aggarwal)**.** *Let f be a holomorphic or Hecke-Maass cusp form for $\Gamma_0(N)$. Then for every $\varepsilon > 0$, we have*

$$L(\tfrac{1}{2} + \mathrm{i}t, f) \ll_{f,\varepsilon} |t|^{1/3+\varepsilon}.$$

Using this subconvexity bound we prove the following theorem.

Theorem 2. *Let f be a holomorphic modular form or a Maass cusp form for $\Gamma_0(N)$ and $\mathcal{S}_k(X)$ be as above. Then for every $\varepsilon > 0$ we have*

$$\mathcal{S}_k(X) \ll_{f,\varepsilon} X^{1-3/(2(k+3))+\varepsilon}.$$

2. Main results

2.1. Preliminaries. We recall a few basic facts that will be used in the rest of the paper.

We start with the following equality, valid whenever $\Re(s) > 1$,

$$L(s,f)^k = \sum_{n=1}^{\infty} \frac{\lambda_{k,f}(n)}{n^s}. \tag{4}$$

Our second ingredient in the proof of Theorem 2 is Perron's formula, see Proposition 5.54 of [5]. Therefore from equation (4) and Perron's formula we have

$$\mathcal{S}_k(X) = \frac{1}{2\pi \mathrm{i}} \int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} L(s,f)^k \frac{X^s}{s}\,\mathrm{d}s + O\Big(\frac{X^{1+\varepsilon}}{T}\Big). \tag{5}$$

For the remainder of the article, let ε denote an arbitrary small positive number which may be different at each occurrence.

[1] Recently Munshi has announced an improvement on Good's subconvexity bound, see [9].

2.2. Proof of Theorem 2. Let $\varepsilon > 0$ and $X > 1$. Let $1 < T < X$ be a positive quantity which will be optimally chosen later. We remark here that all the implied constants are dependent on f, ε which are hence omitted from the notation for the sake of brevity. We start with equation (5) and shift the line of integration to $\Re(s) = \frac{1}{2} + \varepsilon$. Since $L(s, f)$ is entire, it has no poles, and therefore no residues. Therefore from the Cauchy residue theorem we have

$$\begin{aligned}(6)\quad & \mathcal{S}_k(X) \\ &= \frac{1}{2\pi \mathrm{i}}\left\{\int_{1/2+\varepsilon-\mathrm{i}T}^{1/2+\varepsilon+\mathrm{i}T} + \int_{1/2+\varepsilon-\mathrm{i}T}^{1+\varepsilon-\mathrm{i}T} + \int_{1+\varepsilon+\mathrm{i}T}^{1/2+\varepsilon+\mathrm{i}T}\right\} L(s, f)^k \frac{X^s}{s}\,\mathrm{d}s + O\Big(\frac{X^{1+\varepsilon}}{T}\Big).\end{aligned}$$

Suppose we call the integrals in equation (6) as J_1, J_2, J_3, respectively. Then for J_1 we have

$$|J_1| \leqslant \int_{1/2+\varepsilon-\mathrm{i}T}^{1/2+\varepsilon+\mathrm{i}T} \Big|L(s, f)^k \frac{X^s}{s}\Big|\,\mathrm{d}s \ll X^{1/2+\varepsilon} + X^{1/2+\varepsilon}\int_1^T \Big|L\Big(\frac{1}{2} + \varepsilon + \mathrm{i}t, f\Big)\Big|^k t^{-1}\,\mathrm{d}t.$$

From Theorem 1 we have

$$|J_1| \ll X^{1/2+\varepsilon} + X^{1/2+\varepsilon}\int_1^T t^{k/3-1}\,\mathrm{d}t \ll X^{1/2+\varepsilon} + X^{1/2+\varepsilon}T^{k/3}. \tag{7}$$

For the horizontal segments we proceed similarly. From the Phragmen-Lindelöf principle, we have

$$|J_2| + |J_3| \ll \int_{1/2+\varepsilon}^{1+\varepsilon} X^{\sigma} T^{2k(1-\sigma)/3-1}\,\mathrm{d}\sigma,$$

where $\sigma = \Re(s)$. Therefore we have

$$|J_2| + |J_3| \ll T^{2k/3-1}\int_{1/2+\varepsilon}^{1+\varepsilon} X^{\sigma} T^{-2k\sigma/3}\,\mathrm{d}\sigma \ll T^{2k/3-1}\max_{1/2+\varepsilon\leqslant\sigma\leqslant 1+\varepsilon} X^{\sigma}T^{-2k\sigma/3}. \tag{8}$$

The maximum in equation (8) will occur either at $\sigma = \frac{1}{2} + \varepsilon$ or $\sigma = 1 + \varepsilon$, respectively, when $T^{2k/3} > X$ and $T^{2k/3} < X$. From equation (7), however, it is clear that the better bound for $\mathcal{S}_k(X)$ is obtained when we assume the latter case, that is $T^{2k/3} < X$. Therefore we have

$$|J_2| + |J_3| \ll T^{2k/3-1}X^{1+\varepsilon}T^{-2k/3} = X^{1+\varepsilon}T^{-1}. \tag{9}$$

Then, if we suppose $T = X^{\alpha}$ we see from equations (6), (7) and (9) that the optimal choice for α is $\alpha = 3/(2(k+3))$. Therefore we have that

$$\mathcal{S}_k(X) \ll_{f,\varepsilon} X^{1-3/(2(k+3))+\varepsilon} \tag{10}$$

thus completing the proof. □

3. Concluding remarks

Remark 1. We remark here that for $k \geqslant 5$, equation (10) provides a sharper bound than the one obtained by employing the classical result of Landau which gives the exponent as $1 - 2/(2k+1) + \varepsilon$, but is weaker than the estimates of Lü for the case of modular forms of full level, see equation (2).

Remark 2. We also remark here that, since Theorem 2 holds for an arbitrary level, using the Shimura lift (or the Shimura correspondence), it is possible to obtain an estimate for sums of the type $\mathcal{S}_k(X)$ with the coefficients being the non-square free Hecke eigenvalues of certain suitably chosen half integral weight modular forms, see [10] for more details. The previous results did not supply this corollary as they were not applicable for integral weight modular forms of higher level.

Acknowledgements. The author is grateful to Dr. Kalyan Chakraborty for introducing the author to this problem and to the Kerala School of Mathematics for their generous hospitality. The author also thanks the referee for carefully reading the manuscript and suggesting corrections.

References

[1] *K. Aggarwal*: Weyl bound for GL(2) in t-aspect via a simple delta method. J. Number Theory *208* (2020), 72–100. zbl MR doi

[2] *A. R. Booker, M. B. Milinovich, N. Ng*: Subconvexity for modular form L-functions in the t aspect. Adv. Math. *341* (2019), 299–335. zbl MR doi

[3] *O. M. Fomenko*: On summatory functions for automorphic L-functions. J. Math. Sci., New York *184* (2012), 776–785. zbl MR doi

[4] *A. Good*: The square mean of Dirichlet series associated to cusp forms. Mathematika *29* (1982), 278–295. zbl MR doi

[5] *H. Iwaniec, E. Kowalski*: Analytic Number Theory. Colloquium Publications 53. American Mathematical Society, Providence, 2004. zbl MR doi

[6] *S. Kanemitsu, A. Sankaranarayanan, Y. Tanigawa*: A mean value theorem for Dirichlet series and a general divisor problem. Monatsh. Math. *136* (2002), 17–34. zbl MR doi

[7] *E. Landau*: Über die Anzahl der Gitterpunkte in gewissen Bereichen. Gött. Nachr. *1915* (1915), 209–243. (In German.) zbl

[8] *G. Lü*: On general divisor problems involving Hecke eigenvalues. Acta. Math. Hung. *135* (2012), 148–159. zbl MR doi

[9] *R. Munshi*: Sub-Weyl bounds for $GL(2)$ L-functions. Available at `https://arxiv.org/abs/1806.07352` (2018), 30 pages.

[10] *G. Shimura*: On modular forms of half integral weight. Ann. Math. (2) *97* (1973), 440–481. zbl MR doi

[11] *W. Zhang*: Some results on divisor problems related to cusp forms. Ramanujan J. *53* (2020), 75–83. zbl MR doi

Author's address: Krishnarjun Krishnamoorthy, Harish-Chandra Research Institute, Homi Bhabha National Institute, Chhatnag Road, Jhunsi, Allahabad, 211 019, India, e-mail: `krishnarjunk@hri.res.in`.