ON THE CONJUGATE TYPE VECTOR AND THE STRUCTURE OF A NORMAL SUBGROUP

Ruifang Chen, Lujun Guo, Xinxiang

Received September 14, 2020. Published online November 18, 2021.

Abstract. Let N be a normal subgroup of a group G. The structure of N is given when the G-conjugacy class sizes of N is a set of a special kind. In fact, we give the structure of a normal subgroup N under the assumption that the set of G-conjugacy class sizes of N is $(p_{1n_1}^{a_{1n_1}}, \ldots, p_{11}^{a_{11}}, 1) \times \ldots \times (p_{rn_r}^{a_{rn_r}}, \ldots, p_{r1}^{a_{r1}}, 1)$, where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, n_i\}$.

Keywords: index; conjugacy class size; Baer group

MSC 2020: 20E45, 20D60

1. Introduction

All groups considered in this paper are finite. Let G be a group and x an element in G. We denote by x^G the conjugacy class of G containing x, that is, $x^G = \{g^{-1}xg\colon g \in G\}$. Then the size of x^G is $|G : C_G(x)|$, which is sometimes called the *index* of x in G. Let $cs(G) = \{|x^G|\colon x \in G\}$. Suppose that $cs(G) = \{n_1, n_2, \ldots, n_r\}$, where $n_1, n_2, \ldots, n_r$ are different numbers with $n_1 > n_2 > \ldots > n_r = 1$. In 1953, Itô in [9] called the *vector* $(n_1, n_2, \ldots, n_r)$ the conjugate type vector of G, and the group G is said to be a group with type $(n_1, n_2, \ldots, n_r)$ if G has conjugate type vector $(n_1, n_2, \ldots, n_r)$. In the same paper, Itô proved that a group G is nilpotent if G has type $(n_1, 1)$. Since then, the relationship between the conjugate type of a group G and the property of G attracts interest of many authors. Camina in [8]

The research of the work is supported by the National Natural Science Foundation of China (11901169, U1504101), the Youth Science Foundation of Henan Normal University (2019QK02) and Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, College of Mathematics and Information Sciences, Henan Normal University.

DOI: 10.21136/CMJ.2021.0395-20

© Institute of Mathematics, Czech Academy of Sciences 2021.

gave the structure of a group G under the assumption that the conjugate type vector of G is the product of several conjugate type vectors, see [4], [5] for more examples.

Let N be a normal subgroup of a group G and write $cs_G(N) = \{|x^G|\colon\ x \in N\}$. Since N is a union of some G-conjugacy classes contained in N, the set $cs_G(N)$ has a strong influence on the structure of N, and many interesting results are obtained, for instance, see [1], [11].

Suppose that $cs_G(N) = \{n_1, n_2, \ldots, n_r\}$, where $n_1, n_2, \ldots, n_r$ are different numbers with $n_1 > n_2 > \ldots > n_r = 1$. In this paper, we call the vector $(n_1, n_2, \ldots, n_r)$ the *G-conjugate type vector* of N. It is obvious that $cs_G(N) = \{m_1, m_2, \ldots, m_t\}$ and for each m_i there exists n_j such that m_i is a divisor of n_j. Furthermore, if $w = (a_1, a_2, \ldots, a_s)$ and $v = (b_1, b_2, \ldots, b_t)$, we define $w \times v = \{a_i b_j\colon\ i = 1, 2, \ldots, s,$ $j = 1, 2, \ldots, t\}$.

Motivated by the results in [8], in this short paper, we consider the structure of a normal subgroup N of G under the assumption that the G-conjugate type vector of N is of a particular type, and the following theorem is obtained:

Theorem 3.1. *Let G be a group and N a normal subgroup of G. Suppose that G-conjugate type vector of N is*

$$(p_{1n_1}^{a_{1n_1}}, \ldots, p_{11}^{a_{11}}, 1) \times \ldots \times (p_{rn_r}^{a_{rn_r}}, \ldots, p_{r1}^{a_{r1}}, 1),$$

where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, n_i\}$. Then $n_i = 2$ and $N = A_1 \times \ldots \times A_r$, and the G-conjugate type vector of A_i is $(p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1)$ for each $i \in \{1, \ldots, r\}$.

Furthermore, one of the following holds for A_i (up to multiplication by central Sylow subgroups):

(1) *A_i is abelian;*

(2) *A_i is a non-abelian p_{i1} or p_{i2}-group;*

(3) *A_i is a non-nilpotent $\{p_{i1}, p_{i2}\}$-group with abelian Sylow subgroups.*

Recall that a group G is called a *p-Baer group* if every p-element in G has prime power index in G, and G is called a *Baer group* if every element of the group with prime power order has prime power index in G. The structure of p-Baer groups and Baer groups are characterized in [2]. If S is a nonempty subset of G, following [4], we set $K_S = \{x \in G\colon\ xS = S\}$. Then $|K_S|$ divides $|S|$. Other notation and terminology are standard, see [10] for instance.

In this section, we give some lemmas which are useful in the proofs of our main results.

The following lemma is a famous result as Thompson's Lemma, and the proof can be found in many books of group theory, see Theorem 8.2.8 of [10] for example.

Lemma 2.1. *Let $P \times Q$ be the direct product of a p-group P and a p'-group Q and suppose that $P \times Q$ acts on a p-group G. If $C_G(P) \subseteq C_G(Q)$, then Q acts trivially on G.*

Lemma 2.2 (Wielandt)**.** *Let G be a group and x an element of G. If both $|x|$ and $|x^G|$ are powers of a prime p, then $x \in O_p(G)$.*

Lemma 2.3. *Let G be a group and N a normal subgroup of G. Suppose that p^a is the highest power of the prime p which divides the G-conjugacy class sizes of elements in N. If there is a p-element x in N such that $|x^G| = p^a$, then N has a normal p-complement.*

Proof. Since x is a p-element and $|x^G| = p^a$, we have that $\langle x^G \rangle \leqslant O_p(G)$ by Lemma 2.2. Therefore, $\langle x^G \rangle \leqslant O_p(G) \cap N \leqslant O_p(N)$. Write $H = \langle x^G \rangle$ and $Z = C_G(H) \cap N = C_N(H)$. For every p'-element $y \in C_N(x)$, the hypothesis of this lemma implies that $(p, |C_G(x) : C_G(xy)|) = 1$. Since N is normal in G, we conclude that $(p, |C_N(x) : C_N(xy)|) = 1$. That is, $(p, |C_N(x) : C_N(x) \cap C_N(y)|) = 1$. Therefore, $H \cap C_N(x) \leqslant H \cap C_N(y)$, that is, $C_H(x) \leqslant C_H(y)$. Now by Lemma 2.1, we have that $y \in C_N(H) = Z$.

Since $|x^N|$ divides $|x^G|$, we have that $|x^N|$ is a power of p. From the above paragraph, we see that Z contains all the p'-elements in $C_N(x)$, and thus $|N : Z|$ is a power of p. Now let w be an arbitrary p'-element in Z. By the previous argument, p does not divide $|C_N(x) : C_N(w) \cap C_N(x)|$. As Z is a normal subgroup of $C_N(x)$, we have that p does not divide $|Z : C_Z(w)|$. Therefore, every p'-element in Z has index in Z prime to p, so by [6], Lemma 1, we have that $Z = K \times P$, where K is a p-complement of Z and P is a Sylow p-subgroup of Z. Therefore, K is a normal p-complement of N since $|N : Z|$ is a power of p. □

Lemma 2.4. *Let G be a group and N a normal subgroup of G such that p^a is the highest power of the prime p which divides the G-conjugacy class size of an element in N. Assume that there exists a p-element x in N such that $|x^G| = p^a$. If m is a G-conjugacy class size in N such that $(m, p) = 1$, then there exists a p'-element in N, say y, such that $|(xy)^G| = p^a m$.*

Proof. By Lemma 2.3, we see that N has a normal p-complement K. As $|x^N|$ divides $|x^G|$, $|x^N|$ is a power of p, and thus $K \leqslant C_N(x)$. Let u be a p'-element in $C_N(x)$. Then p does not divide $|C_G(x) : C_G(ux)|$. Since N is normal in G, we have that p does not divide $|C_N(x) : C_N(ux)|$. That is to say, p does not divide the index of u in $C_N(x)$. Therefore, by [6], Lemma 1, $C_N(x) = P_x \times K$ with P_x a Sylow p-subgroup of $C_N(x)$ and K a normal p-complement of $C_N(x)$. Let y be an element in N such that $|y^G| = m$. Then p does not divide $|y^N|$ since $|y^N|$ divides $|y^G|$, whence y centralizes a Sylow p-subgroup of N, and thus y centralizes $O_p(N)$. Since $x \in O_p(G)$ by Lemma 2.2, we have that $x \in O_p(G) \cap N \leqslant O_p(N)$. Therefore, y centralizes x. We may assume that $y \in K$, and thus $|(xy)^G| = p^a m$, as required. □

Lemma 2.5 ([7], Proposition 1). *Let G be a group and p a prime. Suppose that $x \in G$ such that $|x^G|$ is a power of p. Then $[x^G, x^G] \subseteq O_p(G)$.*

Lemma 2.6. *Let G be a group and p and r two primes. Suppose that there is an r-element $x \in G$ such that $|x^G|$ is a power of p. If we set $B = x^G$, then $\langle BB^{-1}\rangle \subseteq O_{p,r}(G)$.*

Proof. First suppose that $O_p(G) = 1$. Then by Lemma 2.5, $[B, B] = 1$. It follows that $\langle B\rangle$ is an abelian normal subgroup of G, and thus $\langle B\rangle \leqslant F(G)$. As x is an r-element, we have that $\langle B\rangle \leqslant O_r(G)$. Since $\langle BB^{-1}\rangle \leqslant \langle B\rangle$, we have that $\langle BB^{-1}\rangle \leqslant O_r(G)$.

Now suppose that $O_p(G) \neq 1$, we can set $\overline{G} = G/O_p(G)$. Then $O_p(\overline{G}) = 1$. Since $|\bar{x}^{\overline{G}}|$ divides $|x^G|$, we have that $|\bar{x}^{\overline{G}}|$ is a power of p. By the above paragraph, we have that $\langle \overline{B}\overline{B}^{-1}\rangle \leqslant O_r(\overline{G})$. Therefore, $\langle BB^{-1}\rangle \subseteq O_{p,r}(G)$. □

Lemma 2.7. *Let G be a group and N a normal subgroup of G. Suppose that $x, y \in N$ such that $|x^G| = p^a$ and $|y^G| = q^b$, where p and q are distinct primes with $p^a < q^b$. If there is no element in N with G-conjugacy class size divisible by pq, then x is a q-element (up to multiplication by central elements).*

Proof. Write $x = x_1x_2\dots x_s$ such that each x_i is an element of a prime power order, $x_ix_j = x_jx_i$ for all i and j and $(|x_i|, |x_j|) = 1$ for $i \neq j$. Since x is not central in G, we may assume that $x_1 \notin Z(G)$ and that x_1 is an r-element for a prime r.

Write $B = x_1^G$, $C = y^G$ and $D = CB$. Since $|B|$ divides $|x^G|$, we have that $(|B|, |y^G|) = 1$. Therefore, similarly as in [3], Lemma 1(b), we see that D is a G-conjugacy class contained in N and $|D|$ divides $|C||B|$. In fact, since $(|y^G|, |x_1^G|) = 1$, we have that $G = C_G(y)C_G(x_1)$. For every $y^g x_1^h \in CB$, we have that $gh^{-1} \in G = C_G(y)C_G(x_1)$. Then there exist $a \in C_G(y)$ and $b \in C_G(x_1)$ such that $gh^{-1} = a^{-1}b$. So $ag = bh$. Furthermore, $y^g x_1^h = y^{ag} x_1^{bh} = (yx_1)^{ag} \in (yx_1)^G$. Therefore, $CB \subseteq (yx_1)^G$. Conversely, it is obvious that $(yx_1)^G \subseteq y^G x_1^G = CB$. Therefore, $CB = y^G x_1^G = (yx_1)^G$ is a conjugacy class. Now it is clear that $|D| \geqslant |C|$. So by the hypothesis of this lemma, $|D| = |C|$. Repeating the argument we see that DB^{-1} is a G-conjugacy class contained in N, and thus $C = CBB^{-1}$ since $C \subseteq CBB^{-1}$. Therefore, $H = \langle BB^{-1} \rangle \leqslant K_C$. Since $|K_c|$ divides $|C|$, we have that $|H|$ divides $|C|$, whence $|H|$ is a power of q. According to Lemma 2.6, we have that $\langle BB^{-1} \rangle \subseteq O_{p,r}(G)$, which forces $r = q$. Therefore, $x_2 x_3 \dots x_s$ is central in G and by replacing x with x_1 we can assume that x is a q-element. □

Lemma 2.8 ([11], Lemma 2.2). *Let G be a group. A prime p does not divide any conjugacy class size of G if and only if G has a central Sylow p-subgroup.*

3. Proof of the Main Result

In this section, we give the proof of the main result.

Theorem 3.1. *Let G be a group and N a normal subgroup of G. Suppose that the G-conjugate type vector of N is*

$$(p_{1n_1}^{a_{1n_1}}, \dots, p_{11}^{a_{11}}, 1) \times \dots \times (p_{rn_r}^{a_{rn_r}}, \dots, p_{r1}^{a_{r1}}, 1),$$

where $r > 1$, $n_i > 1$ and p_{ij} are distinct primes for $i \in \{1, 2, \dots, r\}$, $j \in \{1, 2, \dots, n_i\}$. Then $n_i = 2$ and $N = A_1 \times \dots \times A_r$, and the G-conjugate type vector of A_i is $(p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1)$ for each $i \in \{1, \dots, r\}$.

Furthermore, one of the following holds for A_i (up to multiplication by central Sylow subgroups):

(1) *A_i is abelian;*
(2) *A_i is a non-abelian p_{i1} or p_{i2}-group;*
(3) *A_i is a non-nilpotent $\{p_{i1}, p_{i2}\}$-group with abelian Sylow subgroups.*

P r o o f. We first consider the case $r = 2$. Let $x, y_i \in N$ such that $|x^G| = p_{11}^{a_{11}}$ and $|y_i^G| = p_{1i}^{a_{1i}}$ for $2 \leqslant i \leqslant s$. Then by Lemma 2.7, x is a p_{1i}-element for each i. Thus, $n_1 = 2$ and x is a p_2-element. For every p'_{12}-element $y \in C_N(x)$, the hypothesis implies that p_{12} does not divide $|C_N(x) : C_N(xy)|$, whence $C_N(x) = P_2 \times L$ by [6], Lemma 1, where P_2 is a Sylow p_{12}-subgroup of N. Therefore, p_{12} does not divide the index of any p_{2j}-element in N for $j = 1, \dots, t$. Similarly, we have $n_2 = 2$, and if z is an element in N such that $|z^G| = p_{21}^{a_{21}}$, then z is a p_{22}-element. Furthermore, we have that $C_N(z) = Q_2 \times K$, where Q_2 is a Sylow p_{22}-subgroup of N, and p_{22} does not divide the index of any p_{1i}-element in N for $i = 1, 2$.

Now assume that w is an element in N such that $|w^G| = p_{12}^{a_{12}}$. By the above paragraph we see that w is neither a p_{21}-element nor a p_{22}-element. If w is a p_{12}-element, we may assume that $w \in P_2$. It follows that $L \leqslant C_G(w)$. Then $p_{12}^{a_{12}} = |w^G|$ divides $|G : L| = |G : C_G(x)||C_G(x) : C_N(x)||C_N(x) : K|$, which is a contradiction. Therefore, w must be a p_{11}-element. Let v be an arbitrary p'_{11}-element in $C_N(w)$. Since $p_{11}p_{12}$ does not divide any G-conjugacy class size of element in N, p_{11} does not divide $|C_G(w) : C_G(wv)|$ and thus, p_{11} does not divide $|C_N(w) : C_N(wv)|$ since N is normal in G. Therefore, $C_N(w) = P_1 \times M$, where P_1 is a Sylow p_{11}-subgroup of N. Recall that $|N : C_N(w)|$ is a p_{12}-number. If u is a p_{21}- or p_{22}-element in N, then u is contained in a conjugation of M and thus, p_{11} does not divide $|u^N|$. Combining this with the above paragraph, we see that $|u^N|$ is a power of p_{21} or p_{22}. Similarly, if h is a p_{11}- or p_{12}-element in N, then $|h^N|$ is a power of p_{11} or p_{12}.

In the following, we suppose that $r > 2$. Let x be an element of N such that $|x^G| = p_{11}^{a_{11}}$. Then as in the first paragraph of the proof we have that $n_1 = 2$ and that x is a p_{12}-element. For every p'_{12}-element $y \in C_N(x)$ we have that

$$|C_G(x) : C_G(x) \cap C_G(y)| = |C_G(x) : C_G(xy)|$$

is prime to p_{12}. Since $C_N(x) \trianglelefteq C_G(x)$, we have that $|y^{C_N(x)}|$ is a p'_{12}-number. Therefore, we have that $C_N(x) = P_{12} \times K$, where P_{12} is a Sylow p_{12}-subgroup of N. It is easy to see that N is a p_{12}-Baer group. Furthermore, all $\{p_{11}, p_{12}\}'$-elements have index coprime to p_{12}. On the other hand, it follows from Lemma 2.4 that all p_{12}-elements have index $p_{11}^{a_{11}}$ or are central. So an element of index $p_{12}^{a_{12}}$ must be a p_{11}-element, we can assume that w is such an element. Then by arguing similarly as for the element x, we have that N is a p_{11}-Baer group. Thus, by [7], Theorem A, we see that $P_{11}P_{12}$ is a normal subgroup of N. Notice that every element of order prime to p_{11} and p_{12} have index prime to p_{11} and p_{12}. Therefore, $P_{11}P_{12}$ is centralized by all $\{p_{11}, p_{12}\}'$-elements of N. If we set $A_1 = P_{11}P_{12}$, then A_1 satisfies the theorem. Similarly, we can find all A_i for $2 \leqslant i \leqslant r$.

Let $i \in \{1, \dots, r\}$. Suppose that A_i is not abelian. For every element $x \in A_i \setminus Z(A_i)$, since A_i is normal in G, we have that $|x^{A_i}|$ divides $|x^G|$. Since $cs_G(A_i) = \{p_{i2}^{a_{i2}}, p_{i1}^{a_{i1}}, 1\}$, we have that $|x^{A_i}|$ is a power of p_{i1} or p_{i2}. Then according to Lemma 2.8, the $\{p_{i1}, p_{i2}\}$-complement of A_i is central in A_i. Up to multiplication by central Sylow subgroups, we can assume that A_i is a $\{p_{i1}, p_{i2}\}$-group. Recall that $p_{i1}p_{i2}$ does not divide $|x^{A_i}|$ for any $x \in A_i$. If A_i is nilpotent, then A_i is a p_{i1} or a p_{i2}-group. If A_i is not nilpotent, since A_i is a Baer group, we have that every Sylow subgroup of A_i is abelian by Theorem of [2]. □

Acknowledgments. The authors wish to thank the reviewers for their helpful suggestion.

References

[1] *Z. Akhlaghi, A. Beltrán, M. J. Felipe, M. Khatami*: Normal subgroups and p-regular G-class sizes. J. Algebra *336* (2011), 236–241. zbl MR doi

[2] *R. Baer*: Group elements of prime power index. Trans. Am. Math. Soc. *75* (1953), 20–47. zbl MR doi

[3] *A. Beltrán, M. J. Felipe*: Finite groups with a disconnected p-regular conjugacy class graph. Commun. Algebra *32* (2004), 3503–3516. zbl MR doi

[4] *E. A. Bertram, M. Herzog, A. Mann*: On a graph related to conjugacy classes of groups. Bull. London Math. Soc. *22* (1990), 569–575. zbl MR doi

[5] *A. R. Camina*: Arithmetical conditions on the conjugacy class numbers of a finite group. J. Lond. Math. Soc., II. Ser. *5* (1972), 127–132. zbl MR doi

[6] *A. R. Camina*: Finite groups of conjugate rank 2. Nagoya Math. J. *53* (1974), 47–57. zbl MR doi

[7] *A. R. Camina, R. D. Camina*: Implications of conjugacy class size. J. Group Theory *1* (1998), 257–269. zbl MR doi

[8] *A. R. Camina, R. D. Camina*: Recognizing direct products from their conjugate type vectors. J. Algebra *234* (2000), 604–608. zbl MR doi

[9] *N. Itô*: On finite groups with given conjugate types. I. Nagoya Math. J. *6* (1953), 17–28. zbl MR doi

[10] *H. Kurzweil, B. Stellmacher*: The Theory of Finite Groups: An Introduction. Universitext. Springer, New York, 2004. zbl MR doi

[11] *X. Zhao, X. Guo*: On the normal subgroup with exactly two G-conjugacy class sizes. Chin. Ann. Math., Ser. B *30* (2009), 427–432. zbl MR doi

Authors' address: Ruifang Chen (corresponding author), Lujun Guo, College of Mathematics and Information Science, Henan Normal University, Changsha City Peach Lake Road 15, Xinxiang, Henan, P. R. China, 453007, e-mail: `fang119128@126.com`, `lujunguo0301@163.com`.