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ON DISCRETE MEAN VALUES OF DIRICHLET L-FUNCTIONS

Ertan Elma, Waterloo

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Abstract. Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and let $\mathfrak{a}_\chi := \frac{1}{2}(1-\chi(-1))$. Define the mean value

$$\mathcal{M}_p(-s,\chi) := \frac{2}{p-1} \sum_{\substack{\psi \pmod{p} \\ \psi(-1)=-1}} L(1,\psi)L(-s,\chi\overline{\psi}) \quad (\sigma := \Re s > 0).$$

We give an identity for $\mathcal{M}_p(-s,\chi)$ which, in particular, shows that

$$\mathcal{M}_p(-s,\chi) = L(1-s,\chi) + \mathfrak{a}_\chi 2p^s L(1,\chi)\zeta(-s) + o(1) \quad (p \to \infty)$$

for fixed $0 < \sigma < \frac{1}{2}$ and $|t := \Im s| = o(p^{(1-2\sigma)/(3+2\sigma)})$.

Keywords: Dirichlet L-function; mean value; Dirichlet character

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1. Introduction

The Riemann zeta-function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (\sigma := \Re s > 1)$$

and Dirichlet L-functions

$$L(s,\Psi) := \sum_{n=1}^{\infty} \frac{\Psi(n)}{n^s} \quad (\sigma > 0)$$

associated with a nonprincipal Dirichlet character Ψ modulo $q \geqslant 3$ play important roles in number theory. We refer the reader to [1] and [9] for basic knowledge about

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these functions such as the functional equations

$$\zeta(s)=\pi^{s-1/2}\frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\zeta(1-s) \tag{1.1}$$

and

$$L(s,\Psi)=\frac{\tau(\Psi)}{\mathrm{i}^{\mathfrak{a}_\Psi}\sqrt{\pi}}\Big(\frac{\pi}{q}\Big)^s\frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\Psi))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}_\Psi))}L(1-s,\bar{\Psi}) \tag{1.2}$$

for primitive Dirichlet characters Ψ modulo q, where

$$\mathfrak{a}_\Psi:=\begin{cases}0 & \text{if } \Psi(-1)=1,\\ 1 & \text{if } \Psi(-1)=-1,\end{cases}$$

$$\tau(\Psi):=\sum_{1\leqslant b\leqslant q-1}\Psi(b)e\Big(\frac{b}{q}\Big)\quad (e(x):=\mathrm{e}^{2\pi\mathrm{i}x},\ x\in\mathbb{R}) \tag{1.3}$$

and $\Gamma(\cdot)$ is the Gamma function.

A part of the theory of Dirichlet L-functions is devoted to the mean values

$$\mathcal{M}(q,w,s,\varepsilon;\chi):=\frac{2}{\varphi(q)}\sum_{\substack{\psi \pmod q\\ \psi(-1)=\varepsilon}}L(w,\psi)L(s,\chi\overline{\psi}), \tag{1.4}$$

where $\varepsilon\in\{\pm1\}$, φ is the Euler totient function, χ is a Dirichlet character modulo q and $w,s\in\mathbb{C}$ except possibly the only pole of the right-hand side of (1.4) at 1, if exists. As some examples of such studies, we refer the reader to [7] for $\mathcal{M}(q,n,n,\varepsilon;\chi_0)$, to [4] and [5] for $\mathcal{M}(q,m,n,\varepsilon;\chi_0)$, where $m,n\geqslant1$ are some natural numbers and χ_0 denotes the principal Dirichlet character modulo q. For a similar mean value with complex arguments w and s but again with $\chi=\chi_0$, one may see [8] and [10]. The only related work that we were able to spot in the literature for $\chi\neq\chi_0$, is [12], in which the authors consider the mean value $\mathcal{M}(p,n,1,1;\chi_4)$, where $p\geqslant5$ is a prime number, $n\geqslant2$ is an even natural number and χ_4 is the nonprincipal Dirichlet character modulo 4.

In this work, we are interested in the mean value

$$\mathcal{M}_p(-s,\chi):=\mathcal{M}(p,1,-s,-1;\chi)=\frac{2}{p-1}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1}}L(1,\psi)L(-s,\chi\overline{\psi}), \tag{1.5}$$

where χ is a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$ and $\sigma = \Re s > 0$. The reason for considering $\mathcal{M}_p(-s,\chi)$ rather than $\mathcal{M}_p(s,\chi)$ for $\sigma > 0$ is the following. For $\mathcal{M}_p(s,\chi)$ with sufficiently large $\sigma > 0$, one can effectively use the partial sums of the Dirichlet series of the functions involved and observe that the resulting main term, for large p and bounded $|s|$, is $L(1+s,\chi)$ when $\chi(-1) = 1$. Here we are curious about whether such a behaviour occurs for $\mathcal{M}_p(-s,\chi)$ with $\sigma > 0$, that is, whether $\mathcal{M}_p(-s,\chi)$ with $\sigma > 0$ approximates to $L(1-s,\chi)$.

Our main result below gives an identity for $\mathcal{M}_p(-s,\chi)$ in a larger region, where $\sigma > -1$ and it shows that the behaviour explained above is still valid if $0 < \sigma < \frac{1}{2}$ is fixed and $|t := \Im s| = o(p^{(1-2\sigma)/(3+2\sigma)})$ as $p \to \infty$. Moreover, by differentiation, our main result gives information about the derivatives $\mathcal{M}_p^{(k)}(-s,\chi)$ in $\sigma > -1$ as well.

Theorem 1.1. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Then for $s = \sigma + \mathrm{i}t$ with $\sigma > -1$, $t \in \mathbb{R}$, we have*

$$\mathcal{M}_p(-s,\chi) = L(1-s,\chi) + \mathfrak{a}_\chi 2p^s L(1,\chi)\zeta(-s) + E_p(s,\chi), \tag{1.6}$$

where

$$E_p(s,\chi) := \frac{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\bar{\chi})}\Big(\frac{p}{\pi}\Big)^s \frac{s\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\chi))}(s+1)\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_{\bar{\chi}}(x)}{x^{s+2}}\,\mathrm{d}x$$

and

$$S_{\bar{\chi}}(x) := \sum_{1\leqslant n\leqslant x} \bar{\chi}(n).$$

For $-1 < \sigma \leqslant 1$ we have

$$E_p(s,\chi) \ll p^{\sigma-1/2}(|t|^{\sigma+3/2} + |1-(\sigma-\mathfrak{a}_\chi)^2|)\Big(\frac{1-(p^{1/2}\log p)^{-\sigma}}{\sigma(\sigma+1)}\Big).$$

In particular, if $0 < \sigma < \frac{1}{2}$ is fixed and $|t| = o(p^{(1-2\sigma)/(3+2\sigma)})$, then (1.6) *holds with $E_p(s,\chi) = o(1)$ as $p \to \infty$.*

2. A key proposition

In the proof of Theorem 1.1, we use the functional equations of the factors $L(-s,\chi\overline{\psi})$ in (1.5). Note that for general moduli, the product of two nonconjugate characters is not necessarily primitive even if both of them are primitive. However, the assumption that the modulus p is a prime number guarantees the fact that a nonprincipal Dirichlet character modulo p is primitive and thus, one can use the functional equations corresponding to such characters. This brings us to the problem of understanding the mean value of $L(1,\psi)\tau(\chi\overline{\psi})L(s+1,\bar{\chi}\psi)$ over the characters $\psi \neq \chi$ with $\psi(-1)=-1$. In Proposition 2.1 below, we relate such a mean value to the function

$$\mathbf{S}(s,\chi) := \sum_{N=1}^{\infty} \frac{S_\chi(N)}{N^s} \quad (\sigma > 1), \tag{2.1}$$

where

$$S_\chi(N) = \sum_{1\leqslant n\leqslant N} \chi(n).$$

By the Pólya-Vinogradov inequality, $|S_\chi(N)| \ll \sqrt{p}\log p$ and hence, the series in (2.1) is absolutely convergent for $\sigma > 1$.

Proposition 2.1. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$.*

(a) *For any $s \in \mathbb{C}\setminus\{1\}$ we have*

$$\mathbf{S}(s,\chi) = \frac{p}{\pi \mathrm{i}\tau(\bar{\chi})(p-1)} \sum_{\substack{\psi \ (\mathrm{mod}\ p)\\ \psi(-1)=-1\\ \psi\neq\bar{\chi}}} L(1,\psi)\tau(\overline{\chi\psi})L(s,\chi\psi) + \mathfrak{a}_\chi \frac{\tau(\chi)(p^s-1)}{\pi \mathrm{i} p^{s-1}(p-1)} L(1,\bar{\chi})\zeta(s) + \frac{L(s,\chi)}{2}. \tag{2.2}$$

(b) *For $\sigma > 0$ and $s \neq 1$ we have*

$$\mathbf{S}(s,\chi) = \frac{L(s-1,\chi)}{s-1} + \frac{L(s,\chi)}{2} + s\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}}\,\mathrm{d}x. \tag{2.3}$$

Moreover, identities (2.2) *and* (2.3) *hold for $s=1$ if $\chi(-1)=1$.*

Remark 2.1. Part (a) of the proposition above shows that the function $\mathbf{S}(s,\chi)$ is analytic everywhere on $\mathbb{C}$ if $\chi(-1)=1$; otherwise, the only pole of $\mathbf{S}(s,\chi)$ is at $s=1$, which is a simple pole with residue $\tau(\chi)/(\pi\mathrm{i})L(1,\bar{\chi})$.

3. Lemmata

We start with a general result due to Louboutin in [6].

Lemma 3.1 ([6], Proposition 1)**.** *Let ψ be a Dirichlet character modulo $q \geqslant 3$ such that $\psi(-1) = -1$. Then*

$$\frac{2q}{\pi} L(1,\psi) = \sum_{1 \leqslant b \leqslant q-1} \psi(b) \cot\Big(\frac{\pi b}{q}\Big). \tag{3.1}$$

Lemma 3.2. *Let $q \geqslant 3$ and $a \in \mathbb{N}$ with $(a,q) = 1$. Then we have*

$$\cot\Big(\frac{\pi a}{q}\Big) = \frac{2q}{\pi\varphi(q)} \sum_{\substack{\psi \ (\mathrm{mod}\ q) \\ \psi(-1)=-1}} \overline{\psi}(a) L(1,\psi). \tag{3.2}$$

P r o o f. Let ψ be a Dirichlet character modulo q with $\psi(-1) = -1$ and $(a,q) = 1$. We multiply both sides of (3.1) by $\overline{\psi}(a)$ and sum over such characters. Then the left-hand side of (3.1) becomes

$$\frac{2q}{\pi} \sum_{\substack{\psi \ (\mathrm{mod}\ q) \\ \psi(-1)=-1}} \overline{\psi}(a) L(1,\psi). \tag{3.3}$$

The right-hand side of (3.1) turns into

$$\begin{aligned} \sum_{\substack{\psi \ (\mathrm{mod}\ q) \\ \psi(-1)=-1}} \sum_{1 \leqslant b \leqslant q-1} \overline{\psi}(a)\psi(b) \cot\Big(\frac{\pi b}{q}\Big) &= \sum_{1 \leqslant b \leqslant q-1} \cot\Big(\frac{\pi b}{q}\Big) \sum_{\substack{\psi \ (\mathrm{mod}\ q) \\ \psi(-1)=-1}} \overline{\psi}(a)\psi(b) \\ &= \frac{\varphi(q)}{2} \cot\Big(\frac{\pi a}{q}\Big) - \frac{\varphi(q)}{2} \cot\Big(\frac{-\pi a}{q}\Big) \\ &= \varphi(q) \cot\Big(\frac{\pi a}{q}\Big) \end{aligned} \tag{3.4}$$

by the orthogonality relation (see [6])

$$\sum_{\substack{\psi \ (\mathrm{mod}\ q) \\ \psi(-1)=-1}} \overline{\psi}(a)\psi(b) = \begin{cases} \dfrac{\varphi(q)}{2} & \text{if } b \equiv a \pmod q, \\ -\dfrac{\varphi(q)}{2} & \text{if } b \equiv -a \pmod q, \\ 0 & \text{otherwise} \end{cases}$$

for $(a,q) = 1$. Comparing (3.3) and (3.4) finishes the proof. □

Now, we give a closed formula for the partial sums

$$S_\chi(N) = \sum_{1\leqslant n\leqslant N} \chi(n)$$

of a nonprincipal Dirichlet character χ modulo a prime number $p \geqslant 3$, which is proved in [2] by the author. Here, we include its proof for the sake of completeness.

Lemma 3.3. *Let χ be a nonprincipal Dirichlet character modulo a prime number $p \geqslant 3$. Then for any natural number $N \geqslant 1$ we have*

$$\begin{aligned} S_\chi(N) = {} & \frac{p\chi(N)}{\pi \mathrm{i}\tau(\bar\chi)(p-1)} \sum_{\substack{\psi \ (\mathrm{mod}\ p)\\ \psi(-1)=-1}} L(1,\psi)\tau(\overline{\chi\psi})\psi(N) \\ & + \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi i} L(1,\bar\chi)\chi_0(N) + \frac{\chi(N)}{2}, \end{aligned} \tag{3.5}$$

where χ_0 denotes the principal Dirichlet character modulo p.

Proof. Since both sides of (3.5) are zero if $p \mid N$, we assume that $p \nmid N$. We start with the expansion, see [1], Section 9

$$\chi(n) = \frac{1}{\tau(\bar\chi)} \sum_{1\leqslant a\leqslant p-1} \bar\chi(a) e\Big(\frac{an}{p}\Big) \quad (n \in \mathbb{N}), \tag{3.6}$$

where the Gauss sum $\tau(\bar\chi)$ associated with $\bar\chi$ is defined by (1.3) and it satisfies $|\tau(\bar\chi)| = \sqrt{p}$. Then, on summing both sides of (3.6) over $n \in \{1, 2, \ldots, N\}$ and interchanging the order of summations on the resulting right-hand side of (3.6), the inner sum becomes

$$\sum_{1\leqslant n\leqslant N} e\Big(\frac{an}{p}\Big) = \frac{e(a/p)}{e(a/p)-1}\Big(e\Big(\frac{aN}{p}\Big) - 1\Big).$$

Since

$$\begin{aligned} \frac{e(a/p)}{e(a/p)-1} & = \frac{e(a/p)}{e(a/2p)} \frac{1}{e(a/2p) - e(-a/2p)} = \frac{\cos(\pi a/p) + \mathrm{i}\sin(\pi a/p)}{2\mathrm{i}\sin(\pi a/p)} \\ & = \frac{\cot(\pi a/p)}{2\mathrm{i}} + \frac{1}{2}, \end{aligned}$$

we have

$$S_\chi(N) = \frac{1}{\tau(\bar\chi)} \sum_{1\leqslant a\leqslant p-1} \bar\chi(a)\Big(\frac{\cot(\pi a/p)}{2\mathrm{i}} + \frac{1}{2}\Big)\Big(e\Big(\frac{aN}{p}\Big) - 1\Big). \tag{3.7}$$

By (3.6), the contribution of the term $\frac{1}{2}$ on the right-hand side of (3.7) is

$$\frac{1}{2\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)\Big(e\Big(\frac{aN}{p}\Big)-1\Big)=\frac{1}{2\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)e\Big(\frac{aN}{p}\Big)=\frac{\chi(N)}{2}. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned} S_\chi(N) &= \frac{1}{2\mathrm{i}\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)\Big(e\Big(\frac{aN}{p}\Big)-1\Big)\cot\Big(\frac{\pi a}{p}\Big)+\frac{\chi(N)}{2} \\ &= T(\chi,N)+T(\chi)+\frac{\chi(N)}{2}, \end{aligned} \tag{3.9}$$

where

$$T(\chi,N):=\frac{1}{2\mathrm{i}\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)e\Big(\frac{aN}{p}\Big)\cot\Big(\frac{\pi a}{p}\Big) \tag{3.10}$$

and

$$\begin{aligned} T(\chi) &:= -\frac{1}{2\mathrm{i}\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)\cot\Big(\frac{\pi a}{p}\Big)=-\mathfrak{a}_\chi\frac{p}{\pi\mathrm{i}\tau(\bar{\chi})}L(1,\bar{\chi}) \\ &= \mathfrak{a}_\chi\frac{\tau(\chi)}{\pi\mathrm{i}}L(1,\bar{\chi}) \end{aligned} \tag{3.11}$$

on combining the terms a and $p-a$ if $\chi(-1)=1$, and using Lemma 3.1 and $\tau(\bar{\chi})=-\overline{\tau(\chi)}$ if $\chi(-1)=-1$.

Now, we consider $T(\chi,N)$. By Lemma 3.2, we have

$$\begin{aligned} T(\chi,N) &= \frac{1}{2\mathrm{i}\tau(\bar{\chi})}\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)e\Big(\frac{aN}{p}\Big)\frac{2p}{\pi(p-1)}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1}}\overline{\psi}(a)L(1,\psi) \\ &= \frac{p}{\pi\mathrm{i}\tau(\bar{\chi})(p-1)}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1}}L(1,\psi)\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)\overline{\psi}(a)e\Big(\frac{aN}{p}\Big). \end{aligned} \tag{3.12}$$

Note that

$$\sum_{1\leqslant a\leqslant p-1}\bar{\chi}(a)\overline{\psi}(a)e\Big(\frac{aN}{p}\Big)=\chi(N)\psi(N)\tau(\overline{\chi\psi}) \tag{3.13}$$

by (3.6) if $\overline{\chi\psi}$ is nonprincipal, and if $\overline{\chi\psi}=\chi_0$, then (3.13) holds since we assumed that $p\nmid N$ and both sides of (3.13) are -1. By (3.12) and (3.13), we have

$$T(\chi,N)=\frac{p\chi(N)}{\pi\mathrm{i}\tau(\bar{\chi})(p-1)}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1}}L(1,\psi)\tau(\overline{\chi\psi})\psi(N). \tag{3.14}$$

By (3.9), (3.11) and (3.14), the desired result follows. □

4. Proof of proposition 2.1

Let $\sigma > 1$. Dividing both sides of (3.5) by N^s and summing over $N \geqslant 1$ give

$$\mathbf{S}(s,\chi) = \frac{p}{\pi \mathrm{i}\tau(\bar{\chi})(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)\tau(\overline{\chi\psi})L(s,\chi\psi) + \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi \mathrm{i}} L(1,\bar{\chi})\zeta(s)\Big(1-\frac{1}{p^s}\Big) + \frac{L(s,\chi)}{2}. \tag{4.1}$$

If $\chi(-1) = -1$, then the term in the sum above with $\psi = \bar{\chi}$ contributes to

$$\frac{p}{\pi \mathrm{i}\tau(\bar{\chi})(p-1)} L(1,\bar{\chi})\tau(\chi_0)L(s,\chi_0) = \frac{\tau(\chi)}{\pi \mathrm{i}(p-1)} L(1,\bar{\chi})\zeta(s)\Big(1-\frac{1}{p^s}\Big). \tag{4.2}$$

By (4.1) and (4.2), we have

$$\mathbf{S}(s,\chi) = \frac{p}{\pi \mathrm{i}\tau(\bar{\chi})(p-1)} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1 \\ \psi \neq \bar{\chi}}} L(1,\psi)\tau(\overline{\chi\psi})L(s,\chi\psi) + \mathfrak{a}_\chi \frac{\tau(\chi)}{\pi \mathrm{i}} L(1,\bar{\chi})\zeta(s)\Big(1-\frac{1}{p^s}\Big)\Big(1+\frac{1}{p-1}\Big) + \frac{L(s,\chi)}{2},$$

which gives the first assertion of Proposition 2.1 by analytic continuation.

For the second assertion of Proposition 2.1, we start with

$$\sum_{N \leqslant pk} \frac{S_\chi(N)}{N^s} = \sum_{N \leqslant pk} \frac{1}{N^s} \sum_{n \leqslant N} \chi(n) = \sum_{n \leqslant pk} \chi(n) \sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} \tag{4.3}$$

for some $k \in \mathbb{N}$ and $\sigma > 1$. Since

$$\sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} = \frac{1}{n^s} + \sum_{N \leqslant pk} \frac{1}{N^s} - \sum_{N \leqslant n} \frac{1}{N^s},$$

we have

$$\begin{aligned} \sum_{n \leqslant pk} \chi(n) \sum_{n \leqslant N \leqslant pk} \frac{1}{N^s} &= \sum_{n \leqslant pk} \chi(n) \Big[\frac{1}{n^s} + \sum_{N \leqslant pk} \frac{1}{N^s} - \sum_{N \leqslant n} \frac{1}{N^s}\Big] \\ &= \sum_{n \leqslant pk} \frac{\chi(n)}{n^s} - \sum_{n \leqslant pk} \chi(n) \sum_{N \leqslant n} \frac{1}{N^s} = S_1 - S_2, \end{aligned} \tag{4.4}$$

where

$$S_1 := \sum_{n \leqslant pk} \frac{\chi(n)}{n^s}, \quad S_2 := \sum_{n \leqslant pk} \chi(n) \sum_{N \leqslant n} \frac{1}{N^s}.$$

It is known, [11], Equation 3.5.3, that

$$\zeta(s) = \sum_{N\leqslant n} \frac{1}{N^s} + s\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x + \frac{n^{1-s}}{s-1} - \frac{1}{2n^s} \quad (\sigma > 0).$$

Thus,

$$\begin{aligned} S_2 &= \sum_{n\leqslant pk} \chi(n)\Big[\zeta(s) - s\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x - \frac{n^{1-s}}{s-1} + \frac{1}{2n^s}\Big] \\ &= -\frac{1}{s-1}\sum_{n\leqslant pk}\frac{\chi(n)}{n^{s-1}} - s\sum_{n\leqslant pk}\chi(n)\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x + \frac{1}{2}\sum_{n\leqslant pk}\frac{\chi(n)}{n^s} \end{aligned}$$

and

$$\begin{aligned} S_1 - S_2 &= \sum_{n\leqslant pk}\frac{\chi(n)}{n^s} + \frac{1}{s-1}\sum_{n\leqslant pk}\frac{\chi(n)}{n^{s-1}} \\ &\quad + s\sum_{n\leqslant pk}\chi(n)\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x - \frac{1}{2}\sum_{n\leqslant pk}\frac{\chi(n)}{n^s} \\ &= \frac{1}{s-1}\sum_{n\leqslant pk}\frac{\chi(n)}{n^{s-1}} + \frac{1}{2}\sum_{n\leqslant pk}\frac{\chi(n)}{n^s} + s\sum_{n\leqslant pk}\chi(n)\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x. \end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned} \sum_{n\leqslant pk}\chi(n)\int_n^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\,\mathrm{d}x &= \int_1^\infty \frac{\lfloor x\rfloor - x + \frac{1}{2}}{x^{s+1}}\Big(\sum_{\substack{n\leqslant pk\\ n\leqslant x}}\chi(n)\Big)\,\mathrm{d}x \\ &= \int_1^{pk} \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}}\,\mathrm{d}x. \end{aligned} \tag{4.6}$$

By (4.3)–(4.6) and letting $k\to\infty$ for $\sigma > 1$, we obtain

$$\mathbf{S}(s,\chi) = \frac{1}{s-1}L(s-1,\chi) + \frac{1}{2}L(s,\chi) + s\int_1^\infty \frac{(\lfloor x\rfloor - x + \frac{1}{2})S_\chi(x)}{x^{s+1}}\,\mathrm{d}x.$$

Since $S_\chi(x) \ll_p 1$, the integral above is convergent for $\sigma > 0$ and hence the desired result follows.

5. Proof of Theorem 1.1

Replacing s by $s+1$ in Proposition 2.1 and equating the expressions in (2.2) and (2.3), we have

$$T_1+T_2+T_3=(s+1)\int_1^\infty \frac{(\lfloor x\rfloor - x+\frac{1}{2})S_\chi(x)}{x^{s+2}}\,\mathrm{d}x \tag{5.1}$$

for $\sigma>-1$, where

$$T_1 := \frac{p}{\pi \mathrm{i}\tau(\bar\chi)(p-1)}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1\\ \psi\neq\bar\chi}} L(1,\psi)\tau(\overline{\chi\psi})L(s+1,\chi\psi),$$
$$T_2 := \mathfrak{a}_\chi\frac{\tau(\chi)(p^{s+1}-1)}{\pi\mathrm{i}p^s(p-1)}L(1,\bar\chi)\zeta(s+1),$$
$$T_3 := -\frac{L(s,\chi)}{s}.$$

Now, we consider T_1. Note that if $\psi(-1)=-1$ and $\psi\neq\bar\chi$, we have

$$\mathfrak{a}_{\chi\psi}=1-\mathfrak{a}_\chi$$

and

$$\tau(\overline{\chi\psi})\tau(\chi\psi)=\chi\psi(-1)\overline{\tau(\chi\psi)}\tau(\chi\psi)=-\chi(-1)p.$$

Thus, for such characters χ and ψ we have

$$\begin{aligned}\tau(\overline{\chi\psi})L(s+1,\chi\psi) &= \tau(\overline{\chi\psi})\frac{\tau(\chi\psi)}{\mathrm{i}^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\Big(\frac{\pi}{p}\Big)^{s+1}\frac{\Gamma(\frac{1}{2}(-s+1-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}L(-s,\overline{\chi\psi})\\ &= -\frac{\chi(-1)p}{\mathrm{i}^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\Big(\frac{\pi}{p}\Big)^{s+1}\frac{\Gamma(\frac{1}{2}(-s+1-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}L(-s,\overline{\chi\psi})\end{aligned} \tag{5.2}$$

by the functional equation (1.2). By (5.2), we have

$$\begin{aligned}T_1 &= \frac{p}{\pi\mathrm{i}\tau(\bar\chi)(p-1)}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1\\ \psi\neq\bar\chi}} L(1,\psi)\Big[-\frac{\chi(-1)p}{\mathrm{i}^{1-\mathfrak{a}_\chi}\sqrt{\pi}}\Big(\frac{\pi}{p}\Big)^{s+1}\frac{\Gamma(\frac{1}{2}(-s+1-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}L(-s,\overline{\chi\psi})\Big]\\ &= \frac{\mathrm{i}^{\mathfrak{a}_\chi}\tau(\chi)}{\sqrt{\pi}}\Big(\frac{\pi}{p}\Big)^s\frac{\Gamma(\frac{1}{2}(1-s-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}\frac{1}{p-1}\sum_{\substack{\psi \pmod p\\ \psi(-1)=-1\\ \psi\neq\bar\chi}} L(1,\psi)L(-s,\overline{\chi\psi}).\end{aligned}$$

Recall that

$$\mathcal{M}_p(-s,\chi) := \frac{2}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1}} L(1,\psi)L(-s,\chi\overline{\psi}).$$

Since

$$\frac{1}{p-1} \sum_{\substack{\psi \pmod p \\ \psi(-1)=-1 \\ \psi\neq\bar{\chi}}} L(1,\psi)L(-s,\overline{\chi\psi}) = \frac{\mathcal{M}_p(-s,\bar{\chi})}{2} - \mathfrak{a}_\chi L(1,\bar{\chi})\zeta(-s)\frac{1-p^s}{p-1},$$

T_1 can be written as

$$\begin{aligned} T_1 &= \frac{\mathrm{i}^{\mathfrak{a}_\chi}\tau(\chi)}{2\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(\frac{1}{2}(1-s-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}\mathcal{M}_p(-s,\bar{\chi}) \\ &\quad + \mathfrak{a}_\chi \frac{\mathrm{i}\tau(\chi)}{\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(-\frac{1}{2}s)}{\Gamma(\frac{1}{2}(s+1))}\frac{p^s-1}{p-1}L(1,\bar{\chi})\zeta(-s). \end{aligned} \tag{5.3}$$

For T_2, we use the functional equation (1.1) of $\zeta(s)$ and write

$$\begin{aligned} T_2 &= \mathfrak{a}_\chi \frac{\tau(\chi)(p^{s+1}-1)}{\pi \mathrm{i} p^s(p-1)}L(1,\bar{\chi})\pi^{s+1/2}\frac{\Gamma(-\frac{1}{2}s)}{\Gamma(\frac{1}{2}(s+1))}\zeta(-s) \\ &= \mathfrak{a}_\chi \frac{\mathrm{i}\tau(\chi)}{\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(-\frac{1}{2}s)}{\Gamma(\frac{1}{2}(s+1))}\frac{1-p^{s+1}}{p-1}L(1,\bar{\chi})\zeta(-s). \end{aligned} \tag{5.4}$$

For T_3 we have

$$T_3 = -\frac{1}{s}\frac{\tau(\chi)}{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))}L(1-s,\bar{\chi}) \tag{5.5}$$

by the functional equation (1.2). Thus, by (5.3)–(5.5), we have

$$\begin{aligned} T_1+T_2+T_3 &= \frac{\mathrm{i}^{\mathfrak{a}_\chi}\tau(\chi)}{2\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(\frac{1}{2}(1-s-\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+2-\mathfrak{a}_\chi))}\mathcal{M}_p(-s,\bar{\chi}) \\ &\quad - \mathfrak{a}_\chi \frac{\mathrm{i}\tau(\chi)}{\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(-\frac{1}{2}s)}{\Gamma(\frac{1}{2}(s+1))}p^s L(1,\bar{\chi})\zeta(-s) \\ &\quad - \frac{1}{s}\frac{\tau(\chi)}{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}\left(\frac{\pi}{p}\right)^s \frac{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))}L(1-s,\bar{\chi}), \end{aligned}$$

which is equivalent to

$$(5.6)\quad T_1+T_2+T_3=\frac{1}{s}\frac{\tau(\chi)}{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}\Big(\frac{\pi}{p}\Big)^s\frac{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}{\Gamma(\frac12(s+\mathfrak{a}_\chi))}$$
$$\times\Big[\frac{\mathrm{i}^{2\mathfrak{a}_\chi}}{2}\frac{\Gamma(\frac12(1-s-\mathfrak{a}_\chi))}{\Gamma(\frac12(s+2-\mathfrak{a}_\chi))}\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}\mathcal{M}_p(-s,\bar\chi)$$
$$-\mathfrak{a}_\chi\mathrm{i}^{1+\mathfrak{a}_\chi}\frac{\Gamma(-\frac12 s)}{\Gamma(\frac12(s+1))}\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}p^sL(1,\bar\chi)\zeta(-s)-L(1-s,\bar\chi)\Big].$$

Since $s\Gamma(s)=\Gamma(s+1)$, we have

$$(5.7)\qquad \frac{\mathrm{i}^{2\mathfrak{a}_\chi}}{2}\frac{\Gamma(\frac12(1-s-\mathfrak{a}_\chi))}{\Gamma(\frac12(s+2-\mathfrak{a}_\chi))}\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}=1$$

and

$$(5.8)\qquad \mathfrak{a}_\chi\mathrm{i}^{1+\mathfrak{a}_\chi}\frac{\Gamma(-\frac12 s)}{\Gamma(\frac12(s+1))}\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}=2\mathfrak{a}_\chi\frac{-\frac12 s\Gamma(-\frac12 s)}{\Gamma(\frac12(2-s))}=2\mathfrak{a}_\chi.$$

By (5.6)–(5.8) and (5.1), we have

$$(5.9)\quad \mathcal{M}_p(-s,\bar\chi)-\mathfrak{a}_\chi 2p^sL(1,\bar\chi)\zeta(-s)-L(1-s,\bar\chi)$$
$$=\frac{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\chi)}\Big(\frac{p}{\pi}\Big)^s\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}(s+1)\int_1^\infty\frac{(\lfloor x\rfloor-x+\frac12)S_\chi(x)}{x^{s+2}}\,\mathrm{d}x$$

for $\sigma>-1$. This finishes the proof of the first statement in Theorem 1.1 by replacing χ by $\bar\chi$ and reorganizing the terms in (5.9).

Let

$$E_p(s,\chi):=\frac{\mathrm{i}^{\mathfrak{a}_\chi}\sqrt{\pi}}{\tau(\bar\chi)}\Big(\frac{p}{\pi}\Big)^s\frac{s\Gamma(\frac12(s+\mathfrak{a}_\chi))}{\Gamma(\frac12(1-s+\mathfrak{a}_\chi))}(s+1)\int_1^\infty\frac{(\lfloor x\rfloor-x+\frac12)S_{\bar\chi}(x)}{x^{s+2}}\,\mathrm{d}x$$

for $-1<\sigma\leqslant 1$. By the Pólya-Vinogradov inequality, we have

$$\int_1^\infty\frac{(\lfloor x\rfloor-x+\frac12)S_{\bar\chi}(x)}{x^{s+2}}\,\mathrm{d}x\ll\int_1^A x^{-\sigma-1}\,\mathrm{d}x+p^{1/2}\log p\int_A^\infty x^{-\sigma-2}\,\mathrm{d}x$$
$$=\begin{cases}\log A+p^{1/2}(\log p)A^{-1} & \text{if } \sigma=0,\\ -\dfrac{1}{\sigma}(A^{-\sigma}-1)+p^{1/2}(\log p)\dfrac{A^{-\sigma-1}}{\sigma+1} & \text{if } \sigma\neq 0.\end{cases}$$

Taking $A=p^{1/2}\log p$ and noting that $\lim_{\sigma\to0}(1-A^{-\sigma})/\sigma=\log A$, we see that

$$\int_1^\infty\frac{(\lfloor x\rfloor-x+\frac12)S_{\bar\chi}(x)}{x^{s+2}}\,\mathrm{d}x\ll\frac{1-(p^{1/2}\log p)^{-\sigma}}{(\sigma+1)\sigma}\qquad(-1<\sigma\leqslant 1),$$

where the right-hand side above is to be interpreted as the limit $\sigma \to 0$ if $\sigma = 0$. By Stirling's formula (see [3], Equation A.34), we know that

$$|\Gamma(s)| = (2\pi)^{1/2}|t|^{\sigma-1/2}e^{-\pi|t|/2}\Big(1 + O\Big(\frac{1}{|t|}\Big)\Big) \quad (-1 < \sigma \leqslant 1,\ |t| \geqslant 1),$$

where the implied constant is absolute. Thus,

$$\frac{s(s+1)\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\chi))} \ll |t|^{\sigma+3/2} \quad (-1 < \sigma \leqslant 1,\ |t| \geqslant 1).$$

Now we consider the remaining case where $|t| < 1$. Since $\Gamma(s)$ is never zero and it has simple poles at nonpositive integers, we have

$$\frac{s(s+1)\Gamma(\frac{1}{2}(s+\mathfrak{a}_\chi))}{\Gamma(\frac{1}{2}(1-s+\mathfrak{a}_\chi))} \ll \frac{|s(s+1)(1-s+\mathfrak{a}_\chi)|}{|s+\mathfrak{a}_\chi|} \quad (-1 < \sigma \leqslant 1,\ |t| < 1).$$

Thus,

$$E_p(s,\chi) \ll p^{\sigma-1/2}(|t|^{\sigma+3/2} + |(\sigma+1-\mathfrak{a}_\chi)(1-\sigma+\mathfrak{a}_\chi)|)\Big(\frac{1-(p^{1/2}\log p)^{-\sigma}}{(\sigma+1)\sigma}\Big)$$

for $-1 < \sigma \leqslant 1$ and $t \in \mathbb{R}$, which finishes the proof of Theorem 1.1.

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Author's address: Ertan Elma, Department of Pure Mathematics, University of Waterloo, 200 University Ave. West, N2L 3G1, Waterloo, ON, Canada, e-mail: `eelma@uwaterloo.ca`.