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A NOTE ON CLASSES OF STRUCTURED MATRICES WITH ELLIPTICAL TYPE NUMERICAL RANGE

Natália Bebiano, Coimbra, Susana Furtado, Porto

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Abstract. We identify new classes of structured matrices whose numerical range is of the elliptical type, that is, an elliptical disk or the convex hull of elliptical disks.

Keywords: tridiagonal matrix; antitridiagonal matrix; elliptical disk; numerical range

MSC 2020: 15A21, 15A60

1. Introduction

The *numerical range* of $A \in \mathbb{C}^{n\times n}$ is the subset of the complex plane defined and denoted as

$$W(A) := \{x^*Ax\colon\ x \in \mathbb{C}^n,\ x^*x = 1\}.$$

The numerical range is *unitarily invariant*, i.e., invariant under unitary similarity transformations. The Toeplitz-Hausdorff theorem asserts that $W(A)$ is a convex set. It is also well-known that it contains $\sigma(A)$, the spectrum of A. Moreover, every extreme point (corner) of $W(A)$ is an eigenvalue of A. If A is unitarily similar to $A_1 \oplus \ldots \oplus A_p$ with $p \geqslant 2$, then $W(A)$ is the convex hull of $W(A_1) \cup \ldots \cup W(A_p)$. In particular, if A is normal, then $W(A)$ is the convex hull of $\sigma(A)$. Kippenhahn proved that $W(A)$ is the convex hull of the so called *generating curve*, see [13]. For more details and a comprehensive discussion of the basic properties of $W(A)$ see e.g. [11], [12].

The *Elliptical Range Theorem* states that the numerical range of $A \in \mathbb{C}^{2\times 2}$ is an elliptical disc with foci at λ_1 and λ_2, the eigenvalues of A, and a minor axis of length $(\operatorname{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}$. The elliptical shape of the numerical range of

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matrices of size $n > 2$ and operators has deserved the attention of several researchers, see [1], [3], [4], [5], [6], [7], [8], [9], [10], [13], [14], [15], [16]. This is the case e.g. for certain Toeplitz matrices and operators (see [1]) or for *quadratic matrices* (that is, those with minimal polynomial of degree 2), see [4]. In [1], refining an idea of Marcus and Pesce (see [15]), the numerical range of banded biperiodic Toeplitz operators is characterized by performing a reduction to the 2×2 case, and taking the convex hull of an infinite number of elliptical discs. In a recent work (see [10]), classes of 2×2 block matrices with scalar diagonal blocks whose numerical range is the convex hull of at most a finite number of ellipses have been given. The main goal of this note is to present new classes of matrices with elliptical shape numerical range.

In Section 2 we introduce some notation and auxiliary results. In Sections 3 and 4, we identify classes of antitridiagonal matrices, with at most two nonzero antidiagonals and constant main diagonal, having elliptical shape numerical range. Our approach is based on a reduction by permutation similarity of these matrices to tridiagonal ones, leaving the numerical range unchanged. We focus on the antibidiagonal case in Section 3 and on the antitridiagonal case with 0 main antidiagonal in Section 4. In Section 5 an illustrative example is presented.

We note that the antitridiagonal matrices we consider here can be partitioned into 2×2 block matrices whose principal blocks are scalar. In this perspective, we emphasize that we add new classes of block matrices to those in [10].

In our statements we will consider matrices with a 0 main diagonal. Using the elementary property

$$W(aA + bI) = W(A) + b$$

for any $a, b \in \mathbb{C}$, our results can be extended to matrices with a constant main diagonal.

2. Notation and auxiliary results

The following notation will be used throughout. Given $\mathbf{b} = (b_1 \ldots, b_{n-1})$, $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{c} = (c_1, \ldots, c_{n-1})$, we denote by $T(n; \mathbf{b}, \mathbf{a}, \mathbf{c})$ the $n \times n$ tridiagonal complex matrix with subdiagonal $\mathbf{b}$, main diagonal $\mathbf{a}$ and super diagonal $\mathbf{c}$,

$$T = \begin{bmatrix} a_1 & c_1 & & & 0 \\ b_1 & a_2 & c_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & b_{n-1} & a_n \end{bmatrix}.$$

By $A(n;\mathbf{b},\mathbf{a},\mathbf{c})$ we denote the $n\times n$ antitridiagonal complex matrix with subantidiagonal $\mathbf{b}$, main antidiagonal $\mathbf{a}$ and super antidiagonal $\mathbf{c}$,

$$A=\begin{bmatrix} 0 & & c_1 & a_1 \\ & \iddots & \iddots & b_1 \\ c_{n-2} & \iddots & \iddots & \\ a_n & b_{n-1} & & 0 \end{bmatrix}.$$

In the case $\mathbf{b}=\mathbf{0}$ ($\mathbf{c}=\mathbf{0}$), the antibidiagonal matrix is denoted by $A(n;\mathbf{0},\mathbf{a},\mathbf{c})$ ($A(n;\mathbf{b},\mathbf{a},\mathbf{0})$).

The numerical range of a tridiagonal matrix is invariant under the switching of any two corresponding off-diagonal entries, b_j and c_j, see [4], Lemma 3.1. In [4], Theorem 3.3, a class of tridiagonal matrices with elliptical numerical range is obtained. We next state a particular case of this result that will have a crucial role in the sequel. For this purpose the following notation is needed.

By $\varrho(B)$ we denote the largest singular value of a matrix B. For complex numbers $d_1,\ldots,d_{n-1}$, we denote the $\lceil\frac{n}{2}\rceil\times\lfloor\frac{n}{2}\rfloor$ matrix

$$X(n;\mathbf{d},\mathbf{d}'):=\begin{bmatrix} d_1 & & & \\ d_2 & d_3 & & \\ & d_4 & d_5 & \\ & & \ddots & \ddots \\ & & & \end{bmatrix},$$

with $\mathbf{d}=(d_1,d_3,\ldots)$ and $\mathbf{d}'=(d_2,d_4,\ldots)$. Here $\lfloor x\rfloor$ (or by $\lceil x\rceil$) denotes the largest (smallest) integer smaller (larger) than or equal to the real number x.

Theorem 1 ([4]). *Consider the tridiagonal matrix $T(n;\mathbf{b},\mathbf{0},\mathbf{c})$. Assume that there is $J_1\subseteq\{1,\ldots,n-1\}$ and $k\in\mathbb{C}$ such that $b_j=k\overline{c_j}$ for $j\in J_1$ and $c_j=k\overline{b_j}$ for $j\in J_2=\{1,\ldots,n-1\}\setminus J_1$. Let $d_j=c_j$ for $j\in J_1$ and $d_j=b_j$ for $j\in J_2$. Let $\varrho=\varrho(X)$, where $X=X(n;\mathbf{d},\mathbf{d}')$, $\mathbf{d}=(d_1,d_3,\ldots)$ and $\mathbf{d}'=(d_2,d_4,\ldots)$. Then $W(T)$ is an elliptical disc centered at the origin, with foci, major axis length and minor axis length, respectively,*

$$\pm\varrho\sqrt{k},\quad \varrho(|k|+1),\quad \varrho||k|-1|.$$

3. Antibidiagonal matrices with elliptical shape numerical range

In this section we will be concerned with antibidiagonal matrices with zeros above the main antidiagonal. Antibidiagonal matrices with zeros below the main antidiagonal can be obtained from them by a similarity transformation via the permutation matrix $R_n = A(n; \mathbf{0}, (1, \ldots, 1), \mathbf{0})$, which preserves the numerical range.

Let $k \in \mathbb{C}$ and denote by $\mathcal{F}_n(k)$ the class of matrices $A = A(n; \mathbf{b}, \mathbf{a}, \mathbf{0})$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_{n-1})$ are such that

$$a_j = k\overline{a_{n+1-j}} \quad \text{or} \quad a_{n+1-j} = k\overline{a_j}, \quad j = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$b_j = k\overline{b_{n-j}} \quad \text{or} \quad b_{n-j} = k\overline{b_j}, \quad j = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

For $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$d_j = \begin{cases} a_{n+1-j} & \text{if } a_j = k\overline{a_{n+1-j}}, \\ a_j & \text{if } a_{n+1-j} = k\overline{a_j}. \end{cases}$$

For $j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$, let

$$d_{\lfloor \frac{n}{2} \rfloor + j} = \begin{cases} b_{n-j} & \text{if } b_j = k\overline{b_{n-j}}, \\ b_j & \text{if } b_{n-j} = k\overline{b_j}. \end{cases}$$

Theorem 2. *Let $k \in \mathbb{C}$. Consider the antibidiagonal matrix $A = A(n; \mathbf{b}, \mathbf{a}, \mathbf{0}) \in \mathcal{F}_n(k)$, with $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_{n-1})$. Suppose that $a_{(n+1)/2} = 0$ if n is odd and $b_{n/2} = 0$ if n is even. Let $\varrho = \varrho(X)$, where $X = X(n; \mathbf{d}, \mathbf{d}')$, $\mathbf{d} = (d_1, d_2, d_3, \ldots, d_{\lfloor \frac{n}{2} \rfloor})$ and $\mathbf{d}' = (d_{\lfloor \frac{n}{2} \rfloor + 1}, d_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, d_{n-1})$. Then, $W(A)$ is the elliptical disc centered at the origin, with foci, major axis length and minor axis length, respectively,*

$$\pm \varrho\sqrt{k}, \quad \varrho(|k| + 1), \quad \varrho||k| - 1|.$$

Proof. By [2] the antibidiagonal matrix $A = A(n; \mathbf{b}, \mathbf{a}, \mathbf{0})$ in the statement is permutationally similar to $T(n; \mathbf{b}', \mathbf{0}, \mathbf{c}')$ with

$$\mathbf{b}' = (a_n, b_1, a_{n-1}, b_2, a_{n-2}, \ldots, a_{(n/2)+1}), \quad \mathbf{c}' = (a_1, b_{n-1}, a_2, b_{n-2}, a_3, \ldots, a_{n/2}),$$

if n is even, and

$$\mathbf{b}' = (a_n, b_1, a_{n-1}, b_2, a_{n-2}, \ldots, b_{(n-1)/2}), \quad \mathbf{c}' = (a_1, b_{n-1}, a_2, b_{n-2}, a_3, \ldots, b_{(n+1)/2}),$$

if n is odd.

Now the claim follows from Theorem 1. □

4. Antitridiagonal matrices with zero main antidiagonal and elliptical shape numerical range

4.1. Odd size matrices. Let $k \in \mathbb{C}$ and n be odd. Denote by $\mathcal{H}_n(k)$ the class of matrices $A = A(n; \mathbf{b}, \mathbf{0}, \mathbf{c})$, with $\mathbf{b} = (b_1, \ldots, b_{n-1})$ and $\mathbf{c} = (c_1, \ldots, c_{n-1})$ such that, for $j = 1, \ldots, \frac{1}{2}(n-1)$,

$$c_{2j-1} = k\overline{c_{n-2j+1}} \quad \text{or} \quad c_{n-2j+1} = k\overline{c_{2j-1}},$$

and

$$b_{2j} = k\overline{b_{n-2j}} \quad \text{or} \quad b_{n-2j} = k\overline{b_{2j}}.$$

For $j = 1, \ldots, \frac{1}{2}(n-1)$, let

$$d_{2j-1} = \begin{cases} c_{n-2j+1} & \text{if } c_{2j-1} = k\overline{c_{n-2j+1}}, \\ c_{2j-1} & \text{if } c_{n-2j+1} = k\overline{c_{2j-1}}, \end{cases} \quad \text{and} \quad d_{2j} = \begin{cases} b_{n-2j} & \text{if } b_{2j} = k\overline{b_{n-2j}}, \\ b_{2j} & \text{if } b_{n-2j} = k\overline{b_{2j}}. \end{cases}$$

Theorem 3. *Let $k \in \mathbb{C}$ and n be odd. Consider the antitridiagonal matrix $A = A(n; \mathbf{b}, \mathbf{0}, \mathbf{c}) \in \mathcal{H}_n(k)$, with $\mathbf{b} = (b_1, \ldots, b_{n-1})$ and $\mathbf{c} = (c_1, \ldots, c_{n-1})$. Let $\varrho = \varrho(X)$, where $X = X(n; \mathbf{d}, \mathbf{d}')$ for*

$$\mathbf{d} = (d_1, d_3, \ldots, d_{n-2}) \quad \text{and} \quad \mathbf{d}' = (d_2, d_4, \ldots, d_{n-1}).$$

Then, $W(A)$ is the elliptical disc centered at the origin and with foci, major axis length and minor axis length, respectively

$$\pm\varrho\sqrt{k}, \quad \varrho(|k|+1), \quad \varrho||k|-1|.$$

P r o o f. From [2], the antitridiagonal matrix $A = A(n; \mathbf{b}, \mathbf{0}, \mathbf{c})$ is permutationally similar to $T(n; \mathbf{b}', \mathbf{0}, \mathbf{c}')$ with

$$\mathbf{b}' = (c_{n-1}, b_2, c_{n-3}, b_4, \ldots, c_2, b_{n-1}) \quad \text{and} \quad \mathbf{c}' = (c_1, b_{n-2}, c_3, b_{n-4}, \ldots, c_{n-2}, b_1).$$

The result is now a consequence of Theorem 1. □

We notice that the class of matrices considered in the previous theorem includes the real antitridiagonal Hankel matrices of odd size with zero main antidiagonal. Recall that a Hankel matrix has constant entries on each antidiagonal.

4.2. Even size matrices. Let $k_1, k_2 \in \mathbb{C}$ and n be even. Denote by $\mathcal{G}_n(k_1, k_2)$ the class of matrices $A = A(n; \mathbf{b}, \mathbf{0}, \mathbf{c})$, with $\mathbf{b} = (b_1, \ldots, b_{n-1})$ and $\mathbf{c} = (c_1, \ldots, c_{n-1})$ such that

$$c_{2j-1} = k_1 \overline{c_{n-2j+1}} \quad \text{or} \quad c_{n-2j+1} = k_1 \overline{c_{2j-1}},$$

and

$$b_{2j-1} = k_2 \overline{b_{n-2j+1}} \quad \text{or} \quad b_{n-2j+1} = k_2 \overline{b_{2j-1}},$$

where $j = 1, \ldots, \lfloor \frac{n}{4} \rfloor$, and

$$b_{2j} = k_1 \overline{b_{n-2j}} \quad \text{or} \quad b_{n-2j} = k_1 \overline{b_{2j}},$$

and

$$c_{2j} = k_2 \overline{c_{n-2j}} \qquad \text{or} \qquad c_{n-2j} = k_2 \overline{c_{2j}},$$

where $j = 1, \ldots, \left\lfloor \frac{n-2}{4} \right\rfloor$.

For $j = 1, \ldots, \lfloor \frac{n}{4} \rfloor$, let

$$d^{(1)}_{2j-1} = \begin{cases} c_{n-2j+1} & \text{if } c_{2j-1} = k_1 \overline{c_{n-2j+1}}, \\ c_{2j-1} & \text{if } c_{n-2j+1} = k_1 \overline{c_{2j-1}}, \end{cases}$$

and

$$d^{(2)}_{2j-1} = \begin{cases} b_{n-2j+1} & \text{if } b_{2j-1} = k_2 \overline{b_{n-2j+1}}, \\ b_{2j-1} & \text{if } b_{n-2j+1} = k_2 \overline{b_{2j-1}}. \end{cases}$$

For $j = 1, \ldots, \lfloor \frac{n-2}{4} \rfloor$, let

$$d^{(1)}_{2j} = \begin{cases} b_{2j} & \text{if } b_{n-2j} = k_1 \overline{b_{2j}}, \\ b_{n-2j} & \text{if } \overline{b_{2j}} = k_1 b_{n-2j}, \end{cases}$$

$$d^{(2)}_{2j} = \begin{cases} c_{2j} & \text{if } c_{n-2j} = k_2 \overline{c_{2j}}, \\ c_{n-2j} & \text{if } c_{2j} = k_2 \overline{c_{n-2j}}. \end{cases}$$

Theorem 4. *Let $k_1, k_2 \in \mathbb{C}$ and n be even. Consider the antitridiagonal matrix $A = A(n; \mathbf{b}, \mathbf{0}, \mathbf{c}) \in \mathcal{G}_n(k_1, k_2)$ with $\mathbf{b} = (b_1, \ldots, b_{n-1})$, $\mathbf{c} = (c_1, \ldots, c_{n-1})$ and $b_{n/2} = c_{n/2} = 0$. Let*

$$\mathbf{d}_1 = (d^{(1)}_1, d^{(1)}_3, \ldots, d^{(1)}_{2\lfloor \frac{n}{4} \rfloor - 1}), \quad \mathbf{d}'_1 = (d^{(1)}_2, d^{(1)}_4, \ldots, d^{(1)}_{2\lfloor \frac{n-2}{4} \rfloor}),$$

and

$$\mathbf{d}_2 = (d^{(2)}_{(n/2)-1}, \ldots, d^{(2)}_3, d^{(2)}_1), \quad \mathbf{d}'_2 = (d^{(2)}_{(n/2)-2}, \ldots, d^{(2)}_4, d^{(2)}_2),$$

if $\frac{1}{2}n$ is even, and

$$\mathbf{d}_2 = (d^{(2)}_{(n/2)-1}, \ldots, d^{(2)}_4, d^{(2)}_2), \quad \mathbf{d}'_2 = (d^{(2)}_{(n/2)-2}, \ldots, d^{(2)}_3, d^{(2)}_1),$$

if $\frac{1}{2}n$ is odd. Let $\varrho_1 = \varrho(X_1)$ and $\varrho_2 = \varrho(X_2)$, where $X_1 = X(\frac{1}{2}n, \mathbf{d}_1, \mathbf{d}'_1)$ and $X_2 = X(\frac{1}{2}n, \mathbf{d}_2, \mathbf{d}'_2)$. Then, $W(A)$ is the convex hull of two elliptical discs $\mathcal{C}_1$ and $\mathcal{C}_2$ centered at the origin, where $\mathcal{C}_i$, $i = 1, 2$, has foci, major axis length and minor axis length, respectively,

$$\pm\varrho_i\sqrt{k_i}, \quad \varrho_i(|k_i| + 1), \quad \varrho_i||k_i| - 1|.$$

P r o o f. By [2], matrix A is permutationally similar to $T_1 \oplus T_2$, where $T_1 = T(\frac{1}{2}n; \mathbf{b}'_1, \mathbf{0}, \mathbf{c}'_1)$ and $T_2 = T(\frac{1}{2}n; \mathbf{b}'_2, \mathbf{0}, \mathbf{c}'_2)$ with

$$\begin{aligned} \mathbf{b}'_1 &= (c_{n-1}, b_2, c_{n-3}, b_4, \ldots, c_{(n/2)+1}), \quad & \mathbf{b}'_2 &= (b_{(n/2)+1}, \ldots, c_4, b_{n-3}, c_2, b_{n-1}), \\ \mathbf{c}'_1 &= (c_1, b_{n-2}, c_3, b_{n-4}, \ldots, c_{(n/2)-1}), \quad & \mathbf{c}'_2 &= (b_{(n/2)-1}, \ldots, c_{n-4}, b_3, c_{n-2}, b_1), \end{aligned}$$

if $\frac{1}{2}n$ is even, and

$$\begin{aligned} \mathbf{b}'_1 &= (c_{n-1}, b_2, c_{n-3}, b_4, \ldots, b_{(n/2)-1}), \quad & \mathbf{b}'_2 &= (c_{(n/2)-1}, \ldots, c_4, b_{n-3}, c_2, b_{n-1}), \\ \mathbf{c}'_1 &= (c_1, b_{n-2}, c_3, b_{n-4}, \ldots, b_{(n/2)+1}), \quad & \mathbf{c}'_2 &= (c_{(n/2)+1}, \ldots, c_{n-4}, b_3, c_{n-2}, b_1), \end{aligned}$$

if $\frac{1}{2}n$ is odd. The claim follows from Theorem 1, being in mind that the numerical range of a direct sum of matrices is the convex hull of the numerical ranges of the direct summands. □

5. An example

Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the notation in Section 4.2, we have $A \in \mathcal{G}_8(\frac{3}{2}, 5)$. Moreover,

$$\begin{aligned} \mathbf{d}_1 &= (c_7, c_5), \quad & \mathbf{d}'_1 &= (b_2), \\ \mathbf{d}_2 &= (b_7, b_5), \quad & \mathbf{d}'_2 &= (c_2). \end{aligned}$$

Then, ϱ_1 and ϱ_2 are the largest singular values of

$$X_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad X_2 = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

respectively, that is, $\varrho_1 = 1.618$ and $\varrho_2 = 0.040451$. According to Theorem 5, $W(A)$ is the convex hull of the elliptical discs $\mathcal{C}_1$ and $\mathcal{C}_2$, centered at the origin, with focus, major axis and minor axis, respectively,

$$\pm\varrho_1\sqrt{k_1} = \pm 1.9816, \quad \varrho_1(|k_1|+1) = 4.045, \quad \varrho_1||k_1|-1| = 0.809$$

for $\mathcal{C}_1$ and

$$\pm\varrho_2\sqrt{k_2} = \pm 0.9045, \quad \varrho_2(|k_2|+1) = 2.4271, \quad \varrho_2||k_2|-1| = 1.6180$$

for $\mathcal{C}_2$. See Figure 1.

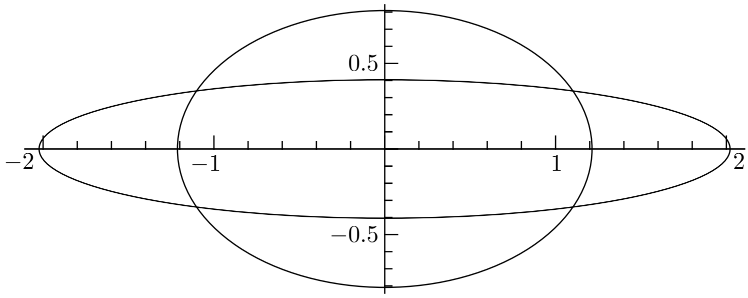

Figure 1. Boundary generating curves of $W(A)$, for A in the example.

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Authors' addresses: Natália Bebiano, Mathematics Center of the University of Coimbra, Mathematics Department, University of Coimbra, 3001-454 Coimbra, Portugal, e-mail: `bebiano@mat.uc.pt`; Susana Furtado (corresponding author), Center for Functional Analysis, Linear Structures and Applications, Faculty of Economics, University of Porto, Rua Dr. Roberto Frias, 4200-464 Porto Portugal, e-mail: `sbf@fep.up.pt`.