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# PAIRS OF SQUARE-FREE VALUES OF THE TYPE $n^2+1$, $n^2+2$

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*Cordially dedicated to Professor Ivan Trendafilov*
*on the occasion of his 71th birthday*

*Abstract.* We show that there exist infinitely many consecutive square-free numbers of the form $n^2+1$, $n^2+2$. We also establish an asymptotic formula for the number of such square-free pairs when $n$ does not exceed given sufficiently large positive number.

*Keywords*: square-free number; asymptotic formula; Kloosterman sum

*MSC 2020*: 11L05, 11N25, 11N37

## 1. Notations

Let $X$ be a sufficiently large positive number. By $\varepsilon$ we denote an arbitrary small positive number, not necessarily the same, in different occurrences. As usual, $\mu(n)$ is Möbius' function and $\tau(n)$ denotes the number of positive divisors of $n$. Further, $[t]$ and $\{t\}$ denote the integer part and the fractional part of $t$, respectively. We shall use the convention that a congruence $m \equiv n \pmod d$ will be written as $m \equiv n(d)$. As usual, $(m,n)$ is the greatest common divisor of $m$ and $n$. The letter $p$ will always denote prime number. We put

$$\psi(t) = \{t\} - \frac{1}{2}. \tag{1.1}$$

Moreover $e(t) = \exp(2\pi i t)$. For $x, y \in \mathbb{R}$ we write $x \equiv y(1)$ when $x - y \in \mathbb{Z}$. For any $n$ and $q$ such that $(n,q)=1$ we denote by $\bar{n}_q$ the inverse of $n$ modulo $q$. The number of distinct prime factors of a natural number $n$ we denote by $\omega(n)$. For any odd prime number $p$ we denote by $\left(\frac{\cdot}{p}\right)$ the Legendre symbol. By $K(r,h)$ we shall

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denote the incomplete Kloosterman sum

$$K(r,h)=\sum_{\substack{\alpha\leqslant x\leqslant\beta\\(x,r)=1}} e\Big(\frac{h\bar{x}_{|r|}}{r}\Big), \tag{1.2}$$

where

$$h,r\in\mathbb{Z},\quad hr\neq 0,\quad 0<\beta-\alpha\leqslant 2|r|.$$

## 2. Introduction and statement of the result

In 1931 Estermann in [6] proved that there exist infinitely many square-free numbers of the form $n^2+1$. More precisely, he proved that for $X\geqslant 2$ the asymptotic formula

$$\sum_{n\leqslant X}\mu^2(n^2+1)=c_0X+\mathcal{O}(X^{2/3}\log X)$$

holds. Here

$$c_0=\prod_{p\equiv 1(4)}\Big(1-\frac{2}{p^2}\Big).$$

Afterwards, Heath-Brown in [8] used a variant of the determinant method and improved the remainder term in the formula of Estermann with $\mathcal{O}(X^{7/12+\varepsilon})$.

On the other hand, in 1932 Carlitz in [1] showed that there exist infinitely many pairs of consecutive square-free numbers. More precisely, he proved the asymptotic formula

$$\sum_{n\leqslant X}\mu^2(n)\mu^2(n+1)=\prod_{p}\Big(1-\frac{2}{p^2}\Big)X+\mathcal{O}(X^{\theta+\varepsilon}), \tag{2.1}$$

where $\theta=2/3$. Formula (2.1) was sharpened by Heath-Brown (see [7]) to $\theta=\frac{7}{11}$ and by Reuss (see [10]) to $\theta=\frac{1}{81}(26+\sqrt{433})$.

The existence of infinitely many consecutive square-free numbers of a special form was demonstrated by the author in [2], [3], [4], [5]. In particular, in [5] he proved that there exist infinitely many consecutive square-free numbers of the form $x^2+y^2+1$, $x^2+y^2+2$. While in [5] the main role was played by the properties of Gauss sums, in this paper we use a surjective correspondence between the number of representations of numbers by binary quadratic form and the incongruent solutions of quadratic congruence.

Define

$$\Gamma(X) = \sum_{1 \leqslant n \leqslant X} \mu^2(n^2+1)\mu^2(n^2+2), \tag{2.2}$$

$$S(q_1, q_2) = \{n \in \mathbb{N}\colon\ 1 \leqslant n \leqslant q_1 q_2,\ n^2+1 \equiv 0(q_1),\ n^2+2 \equiv 0(q_2)\} \tag{2.3}$$

and

$$\lambda(q_1, q_2) = \sum_{n \in S(q_1, q_2)} 1. \tag{2.4}$$

We establish our result by combining the tasks of Estermann and Carlitz. Thus, we prove the following theorem.

**Theorem 2.1.** *For the sum $\Gamma(X)$ defined by* (2.2), *the asymptotic formula*

$$\Gamma(X) = \sigma X + \mathcal{O}(X^{8/9+\varepsilon}) \tag{2.5}$$

*holds. Here*

$$\sigma = \prod_{p>2}\Bigl(1 - \frac{(-1/p) + (-2/p) + 2}{p^2}\Bigr). \tag{2.6}$$

From Theorem 2.1 it follows that there exist infinitely many consecutive square-free numbers of the form $n^2+1$, $n^2+2$, where $n$ runs over naturals.

## 3. Lemmas

The first lemma we need gives us important expansions.

**Lemma 3.1.** *For any $M \geqslant 2$ we have*

$$\psi(t) = -\sum_{1 \leqslant |m| \leqslant M} \frac{e(mt)}{2\pi \mathrm{i} m} + \mathcal{O}(f_M(t)),$$

*where $f_M(t)$ is a positive function of $t$ which is infinitely many times differentiable and periodic with period* 1. *It can be expanded into the Fourier series*

$$f_M(t) = \sum_{m=-\infty}^{\infty} b_M(m) e(mt)$$

*with coefficients* $b_M(m)$ *such that*

$$b_M(m) \ll \frac{\log M}{M} \quad \forall\, m$$

*and*

$$\sum_{|m| > M^{1+\varepsilon}} |b_M(m)| \ll M^{-A}.$$

*Here* $A > 0$ *is arbitrarily large and the constant in the* $\ll$ *symbol depends on* $A$ *and* $\varepsilon$*.*

P r o o f. See [11], Theorem 1. □

The next lemma we need is well-known.

**Lemma 3.2.** *Let* $A, B \in \mathbb{Z} \setminus \{0\}$ *and* $(A, B) = 1$*. Then*

$$\frac{\bar{A}_{|B|}}{B} + \frac{\overline{B}_{|A|}}{A} \equiv \frac{1}{AB}(1).$$

P r o o f. See [12], Lemma 17.5.1. □

**Lemma 3.3.** *For the sum denoted by* (1.2) *the estimate*

$$K(r, h) \ll |r|^{1/2+\varepsilon}(r, h)^{1/2}$$

*holds.*

P r o o f. Follows easily from Weil's estimate for the Kloosterman sum. See [9], Chapter 11, Corollary 11.12. □

**Lemma 3.4.** *Let* $n \geqslant 5$*. There exists a surjective function from the solution set of the equation*

$$x^2 + 2y^2 = n, \quad (x, y) = 1, \quad x \in \mathbb{N}, \quad y \in \mathbb{Z} \setminus \{0\} \tag{3.1}$$

*to the incongruent solutions modulo* $n$ *of the congruence*

$$z^2 + 2 \equiv 0(n). \tag{3.2}$$

P r o o f. Let $F$ denote the set of ordered pairs $(x, y)$ satisfying (3.1) and $E$ denote the set of solutions of the congruence (3.2). We consider each residue class modulo $n$ with representatives satisfying (3.2) as one solution of (3.2).

Let $(x, y) \in F$. From (3.1) it follows that $(n, y) = 1$. Therefore, there exists a unique residue class $z$ modulo $n$ such that

$$zy \equiv x(n). \tag{3.3}$$

For this class we have

$$(z^2 + 2)y^2 \equiv (zy)^2 + 2y^2 \equiv x^2 + 2y^2 \equiv 0(n).$$

From the last congruence and $(n, y) = 1$ we deduce $z^2 + 2 \equiv 0(n)$ which means that $z \in E$. We define the map

$$\beta\colon\ F \to E \tag{3.4}$$

that associates to each pair $(x, y) \in F$ the residue class $z = x\overline{y}_n$ satisfying (3.3).

We shall prove that the map (3.4) is a surjection. Let $z \in E$. From Dirichlet's approximation theorem it follows that there exist integers $a$ and $q$ such that

$$\left|\frac{z}{n} - \frac{a}{q}\right| < \frac{1}{q\sqrt{n}}, \quad 1 \leqslant q \leqslant \sqrt{n}, \quad (a, q) = 1. \tag{3.5}$$

Replace

$$r = zq - an. \tag{3.6}$$

Hence

$$r^2 + 2q^2 = z^2q^2 - 2zqan + a^2n^2 + 2q^2 \equiv (z^2 + 2)q^2(n). \tag{3.7}$$

From (3.2) and (3.7) it follows

$$r^2 + 2q^2 \equiv 0(n). \tag{3.8}$$

By (3.5) and (3.6) we deduce

$$|r| < \sqrt{n}. \tag{3.9}$$

Using (3.5) and (3.9) we obtain

$$0 < r^2 + 2q^2 < 3n. \tag{3.10}$$

Bearing in mind (3.8) and (3.10) we conclude that $r^2 + 2q^2 = n$ or $r^2 + 2q^2 = 2n$. Consider two cases.

*Case 1*:

$$r^2 + 2q^2 = n. \tag{3.11}$$

From (3.6) and (3.11) we get

$$n = (zq - an)^2 + 2q^2 = (zq - an)zq - (zq - an)an + 2q^2 = (zq - an)zq - ran + 2q^2$$

and therefore

$$ra + 1 = kq, \tag{3.12}$$

where

$$k = \frac{z^2 + 2}{n} q - az. \tag{3.13}$$

By (3.2) and (3.13) it follows that $k \in \mathbb{Z}$ and taking into account (3.12) we deduce

$$(r, q) = 1. \tag{3.14}$$

Using (3.11), (3.14) and $n \geqslant 5$ we establish that $r \neq 0$.

Consider first $r > 0$. Replace

$$x = r, \quad y = q. \tag{3.15}$$

From (3.11), (3.14) and (3.15) it follows that $(x, y) \in F$. Also (3.6) and (3.15) give us (3.3). Consequently $\beta(x, y) = z$.

Next we consider $r < 0$. Put

$$x = -r, \quad y = -q. \tag{3.16}$$

Again (3.11), (3.14) and (3.16) lead to $(x, y) \in F$. As well from (3.6) and (3.16) follows (3.3). Therefore $\beta(x, y) = z$.

*Case 2*:

$$r^2 + 2q^2 = 2n. \tag{3.17}$$

From (3.6) and (3.17) we find

$$2n = (zq - an)^2 + 2q^2 = (zq - an)zq - (zq - an)an + 2q^2 = (zq - an)zq - ran + 2q^2$$

and thus

$$ra + 2 = kq, \tag{3.18}$$

where $k$ is defined by (3.13). From (3.18) we conclude

$$(r, q) \leqslant 2. \tag{3.19}$$

By (3.17), (3.19) and $n \geqslant 5$ we deduce that $r \neq 0$.

On the other hand, from (3.17) it follows that $r$ is even. We replace $r = 2r_0$ in (3.17) and obtain

$$q^2 + 2r_0^2 = n. \tag{3.20}$$

We shall verify that

$$(r_0, q) = 1. \tag{3.21}$$

If we assume that $(r_0, q) > 1$, then (3.19) gives us

$$(r_0, q) = 2. \tag{3.22}$$

From (3.20) and (3.22) it follows

$$n \equiv 0(4). \tag{3.23}$$

Finally (3.2) and (3.23) imply

$$z^2 + 2 \equiv 0(4),$$

which is impossible. This proves (3.21).

No matter whether $r$ is positive or negative we replace

$$x = q, \quad y = -r_0. \tag{3.24}$$

Using (3.20), (3.21) and (3.24) we deduce that $(x, y) \in F$.

By (3.6) and (3.24) we get

$$2(zy - x) = -2(zr_0 + q) = -zr - 2q = -(z^2 + 2)q + zan. \tag{3.25}$$

From (3.2) and (3.25) we conclude

$$2(zy - x) \equiv 0(n). \tag{3.26}$$

If $n$ is odd, then (3.26) gives us (3.3). Consequently $\beta(x, y) = z$.

Let $n$ be even. Since (3.23) is impossible,

$$n = 2n_0, \quad n_0 \text{ is odd.} \tag{3.27}$$

By (3.20) and (3.27) it follows

$$q \equiv 0(2), \tag{3.28}$$

i.e., $q$ is even.

On the other hand, (3.2) and (3.27) imply that

$$z \equiv 0(2), \tag{3.29}$$

i.e., $z$ is even.

Now (3.24), (3.28) and (3.29) give us

$$zy - x \equiv 0(2), \tag{3.30}$$

i.e., $zy - x$ is even.

Finally from (3.26), (3.27) and (3.30) we obtain (3.3). Therefore $\beta(x, y) = z$.

The lemma is proved. □

## 4. Proof of the theorem

Using (2.2) and the well-known identity $\mu^2(n) = \sum_{d^2|n} \mu(d)$ we get

$$\Gamma(X) = \sum_{\substack{d_1,d_2 \\ (d_1,d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{1\leqslant n\leqslant X \\ n^2+1\equiv 0(d_1^2) \\ n^2+2\equiv 0(d_2^2)}} 1 = \Gamma_1(X) + \Gamma_2(X), \tag{4.1}$$

where

$$\Gamma_1(X) = \sum_{\substack{d_1 d_2\leqslant z \\ (d_1,d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \tag{4.2}$$

$$\Gamma_2(X) = \sum_{\substack{d_1 d_2 > z \\ (d_1,d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(X, d_1^2, d_2^2), \tag{4.3}$$

$$\Sigma(X, d_1^2, d_2^2) = \sum_{\substack{1\leqslant n\leqslant X \\ n^2+1\equiv 0(d_1^2) \\ n^2+2\equiv 0(d_2^2)}} 1, \tag{4.4}$$

$$\sqrt{X} \leqslant z < X, \tag{4.5}$$

where $z$ is to be chosen later.

**4.1. Estimation of $\Gamma_1(X)$.** Suppose that $q_1 = d_1^2$, $q_2 = d_2^2$, where $d_1$ and $d_2$ are square-free, $(q_1, q_2) = 1$ and $d_1 d_2 \leqslant z$.

Denote

$$\Omega(X, q_1, q_2, n) = \sum_{\substack{m \leqslant X \\ m \equiv n(q_1 q_2)}} 1. \tag{4.6}$$

Using (2.3), (4.4) and (4.6) we obtain upon partitioning sum (4.4) into residue classes modulo $q_1 q_2$

$$\Sigma(X, q_1, q_2) = \sum_{n \in S(q_1, q_2)} \Omega(X, q_1, q_2, n). \tag{4.7}$$

It is easy to see that

$$\Omega(X, q_1, q_2, n) = \frac{X}{q_1 q_2} + \mathcal{O}(1). \tag{4.8}$$

From (2.4), (4.7) and (4.8) we find

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(\lambda(q_1, q_2)). \tag{4.9}$$

Taking into account (2.3), (2.4), Chinese remainder theorem and that the number of solutions of the congruence $n^2 \equiv a(q_1 q_2)$ is less than or equal to $\tau(q_1 q_2)$, we get

$$\lambda(q_1, q_2) \ll \tau(q_1 q_2). \tag{4.10}$$

From (4.9), (4.10) and the inequalities

$$\tau(q_1 q_2) \ll (q_1 q_2)^{\varepsilon} \ll X^{\varepsilon}$$

it follows

$$\Sigma(X, q_1, q_2) = X \frac{\lambda(q_1, q_2)}{q_1 q_2} + \mathcal{O}(X^{\varepsilon}). \tag{4.11}$$

Bearing in mind (4.2), (4.5) and (4.11) we obtain

$$\begin{aligned} \Gamma_1(X) &= X \sum_{\substack{d_1 d_2 \leqslant z \\ (d_1, d_2) = 1}} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(zX^{\varepsilon}) \\ &= \sigma X - X \sum_{\substack{d_1 d_2 > z \\ (d_1, d_2) = 1}} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2, d_2^2)}{d_1^2 d_2^2} + \mathcal{O}(zX^{\varepsilon}), \end{aligned} \tag{4.12}$$

where

$$\sigma = \sum_{\substack{d_1,d_2=1\\(d_1,d_2)=1}}^{\infty} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^2 d_2^2}. \tag{4.13}$$

Using (4.10) we find

$$\sum_{\substack{d_1d_2>z\\(d_1,d_2)=1}} \frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^2 d_2^2} \ll \sum_{\substack{d_1d_2>z\\(d_1,d_2)=1}} \frac{(d_1d_2)^\varepsilon}{(d_1d_2)^2} \ll \sum_{n>z} \frac{\tau(n)}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}. \tag{4.14}$$

It remains to see that product (2.6) and sum (4.13) coincide. From definition (2.4) it follows that the function $\lambda(q_1,q_2)$ is multiplicative, i.e. if

$$(q_1q_2,q_3q_4) = (q_1,q_2) = (q_3,q_4) = 1,$$

then

$$\lambda(q_1q_2,q_3q_4) = \lambda(q_1,q_3)\lambda(q_2,q_4). \tag{4.15}$$

The proof is elementary and we leave it to the reader.

From property (4.15) and $(d_1,d_2)=1$ it follows

$$\lambda(d_1^2,d_2^2) = \lambda(d_1^2,1)\lambda(1,d_2^2). \tag{4.16}$$

Bearing in mind (4.13) and (4.16) we get

$$\sigma = \sum_{d_1=1}^{\infty} \frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^2} \sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^2} f_{d_1}(d_2), \tag{4.17}$$

where

$$f_{d_1}(d_2) = \begin{cases} 1 & \text{if } (d_1,d_2)=1, \\ 0 & \text{if } (d_1,d_2)>1. \end{cases}$$

Clearly the function

$$\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is multiplicative with respect to $d_2$ and the series

$$\sum_{d_2=1}^{\infty} \frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^2} f_{d_1}(d_2)$$

is absolutely convergent.

Applying the Euler product we obtain

$$\sum_{d_2=1}^{\infty}\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^2}f_{d_1}(d_2)=\prod_{p\nmid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)$$
$$=\prod_{p}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)\prod_{p\mid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}. \tag{4.18}$$

From (4.17) and (4.18) it follows

$$\sigma=\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^2}\prod_{p}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)\prod_{p\mid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}$$
$$=\prod_{p}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^2}\prod_{p\mid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}. \tag{4.19}$$

Obviously the function

$$\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^2}\prod_{p\mid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}$$

is multiplicative with respect to $d_1$ and the series

$$\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^2}\prod_{p\mid d_1}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}$$

is absolutely convergent.

Applying again the Euler product from (2.4) and (4.19) we find

$$\sigma=\prod_{p}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)\prod_{p}\Bigl(1-\frac{\lambda(p^2,1)}{p^2}\Bigl(1-\frac{\lambda(1,p^2)}{p^2}\Bigr)^{-1}\Bigr)$$
$$=\prod_{p}\Bigl(1-\frac{\lambda(p^2,1)+\lambda(1,p^2)}{p^2}\Bigr)=\prod_{p>2}\Bigl(1-\frac{(-1/p)+(-2/p)+2}{p^2}\Bigr). \tag{4.20}$$

Bearing in mind (4.5), (4.12), (4.14) and (4.20) we get

$$\Gamma_1(X)=\sigma X+\mathcal{O}(zX^{\varepsilon}), \tag{4.21}$$

where $\sigma$ is given by product (2.6).

**4.2. Estimation of $\Gamma_2(X)$.** Using (4.3), (4.4) and splitting the range of $d_1$ and $d_2$ into dyadic subintervals of the form $D_1 \leqslant d_1 < 2D_1$, $D_2 \leqslant d_2 < 2D_2$ we write

$$\Gamma_2(X) \ll (\log X)^2 \sum_{n\leqslant X} \sum_{\substack{D_1\leqslant d_1<2D_1\\ n^2+1\equiv 0(d_1^2)}} \sum_{\substack{D_2\leqslant d_2<2D_2\\ n^2+2\equiv 0(d_2^2)}} 1, \tag{4.22}$$

where

$$\frac{1}{2} \leqslant D_1, \quad D_2 \leqslant \sqrt{X^2+2}, \quad D_1D_2 > \frac{z}{4}. \tag{4.23}$$

On the one hand, (4.22) gives us

$$\Gamma_2(X) \ll X^\varepsilon \Sigma_1, \tag{4.24}$$

where

$$\Sigma_1 = \sum_{n\leqslant X} \sum_{\substack{D_1\leqslant d_1<2D_1\\ n^2+1\equiv 0(d_1^2)}} 1. \tag{4.25}$$

On the other hand, (4.22) implies

$$\Gamma_2(X) \ll X^\varepsilon \Sigma_2, \tag{4.26}$$

where

$$\Sigma_2 = \sum_{n\leqslant X} \sum_{\substack{D_2\leqslant d_2<2D_2\\ n^2+2\equiv 0(d_2^2)}} 1. \tag{4.27}$$

**Estimation of $\Sigma_1$.** Define

$$\mathcal{N}_1(d) = \{n \in \mathbb{N}\colon\ 1 \leqslant n \leqslant d,\ n^2+1 \equiv 0(d)\}, \tag{4.28}$$

$$\mathcal{N}_1'(d) = \{n \in \mathbb{N}\colon\ 1 \leqslant n \leqslant d^2,\ n^2+1 \equiv 0(d^2)\}. \tag{4.29}$$

By (4.25) and (4.29) we obtain

$$\begin{aligned}\Sigma_1 &= \sum_{D_1\leqslant d_1<2D_1} \sum_{n\in\mathcal{N}_1'(d_1)} \sum_{\substack{m\leqslant X\\ m\equiv n(d_1^2)}} 1 \\ &= \sum_{D_1\leqslant d_1<2D_1} \sum_{n\in\mathcal{N}_1'(d_1)} \left(\left[\frac{X-n}{d_1^2}\right] - \left[\frac{-n}{d_1^2}\right]\right) \\ &= \sum_{D_1\leqslant d_1<2D_1} \sum_{n\in\mathcal{N}_1'(d_1)} \left(\frac{X}{d_1^2} + \psi\left(\frac{-n}{d_1^2}\right) - \psi\left(\frac{X-n}{d_1^2}\right)\right) \\ &\ll X^{1+\varepsilon}D_1^{-1} + |\Sigma_1'| + |\Sigma_1''|,\end{aligned} \tag{4.30}$$

where

$$\Sigma_1' = \sum_{D_1 \leqslant d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \psi\Big(\frac{-n}{d_1^2}\Big), \tag{4.31}$$

$$\Sigma_1'' = \sum_{D_1 \leqslant d_1 < 2D_1} \sum_{n \in \mathcal{N}_1'(d_1)} \psi\Big(\frac{X-n}{d_1^2}\Big) \tag{4.32}$$

and $\psi(t)$ is defined by (1.1).

Firstly, we consider the sum $\Sigma'$. We note that the sum over $n$ in (4.31) does not contain terms with $n = \frac{1}{2}d_1^2$ and $n = d_1^2$. Moreover, for any $n$ satisfying the congruences $n^2 + 1 \equiv 0(d_1^2)$ and such that $1 \leqslant n < \frac{1}{2}d_1^2$, the number $d_1^2 - n$ satisfies the same congruence and we have,

$$\psi\Big(\frac{-n}{d_1^2}\Big) + \psi\Big(\frac{-(d_1^2 - n)}{d_1^2}\Big) = 0.$$

Bearing in mind these arguments for the sum $\Sigma'$ denoted by (4.31) we have that

$$\Sigma' = 0. \tag{4.33}$$

Next, we consider the sum $\Sigma''$ denoted by (4.32). Let $D_1 \leqslant X^{1/2}$. The trivial estimation gives us

$$\Sigma_1'' \ll \sum_{D_1 \leqslant d_1 < 2D_1} d_1^{\varepsilon} \ll X^{1/2+\varepsilon}. \tag{4.34}$$

Let

$$D_1 > X^{1/2}. \tag{4.35}$$

From the theory of the quadratic congruences we know that when $\#\mathcal{N}_1'(d) \neq 0$, then $d$ is odd and

$$\#\mathcal{N}_1(d) = \#\mathcal{N}_1'(d) = 2^{\omega(d)}. \tag{4.36}$$

Denote

$$k = 2^{\omega(d)}, \tag{4.37}$$

$$n_1, \ldots, n_k \in \mathcal{N}_1(d), \quad n_1', \ldots, n_k' \in \mathcal{N}_1'(d). \tag{4.38}$$

From (4.28), (4.29), (4.35)–(4.38) and $d \geqslant D_1 > X^{1/2}$ it follows

$$(4.39)\qquad \sum_{n\in\mathcal{N}_1'(d_1)} \psi\Big(\frac{X-n}{d_1^2}\Big) = \sum_{n\in\mathcal{N}_1'(d_1)} \Big(\frac{X-n}{d_1^2} - \frac{1}{2}\Big)$$
$$= \sum_{n\in\mathcal{N}_1'(d_1)} \Big(\frac{X}{d_1^2} - \frac{1}{2}\Big) - \frac{n_1' + \ldots + n_{k/2}' + (d_1^2 - n_1') + \ldots + (d_1^2 - n_{k/2}')}{d_1^2}$$
$$= \sum_{n\in\mathcal{N}_1(d_1)} \Big(\frac{X}{d_1^2} - \frac{1}{2}\Big) - \frac{n_1 + \ldots + n_{k/2} + (d_1 - n_1) + \ldots + (d_1 - n_{k/2})}{d_1}$$
$$= \sum_{n\in\mathcal{N}_1(d_1)} \Big(\frac{X}{d_1^2} - \frac{1}{2}\Big) - \sum_{n\in\mathcal{N}_1(d_1)} \frac{n}{d_1}$$
$$= \sum_{n\in\mathcal{N}_1(d_1)} \Big(\frac{X}{d_1^2} - \frac{\sqrt{X}}{d_1}\Big) + \sum_{n\in\mathcal{N}_1(d_1)} \Big(\frac{\sqrt{X}-n}{d_1} - \frac{1}{2}\Big)$$
$$= \sum_{n\in\mathcal{N}_1(d_1)} \Big(\frac{X}{d_1^2} - \frac{\sqrt{X}}{d_1}\Big) + \sum_{n\in\mathcal{N}_1(d_1)} \psi\Big(\frac{\sqrt{X}-n}{d_1}\Big).$$

By (4.32), (4.35) and (4.39) we obtain

$$(4.40)\qquad \Sigma_1'' \ll X^{1/2+\varepsilon} + |\Sigma_3|,$$

where

$$(4.41)\qquad \Sigma_3 = \sum_{D_1\leqslant d_1<2D_1} \sum_{n\in\mathcal{N}_1(d_1)} \psi\Big(\frac{\sqrt{X}-n}{d_1}\Big).$$

Using (4.41) and Lemma 3.1 with

$$(4.42)\qquad M_1 = X^{1/2}$$

we find

$$\Sigma_3 = \sum_{D_1\leqslant d_1<2D_1} \sum_{n\in\mathcal{N}_1(d_1)} \Bigg(-\sum_{1\leqslant|m|\leqslant M_1} \frac{e(m(\sqrt{X}-n)/d_1)}{2\pi i m} + \mathcal{O}\Big(f_{M_1}\Big(\frac{\sqrt{X}-n}{d_1}\Big)\Big)\Bigg).$$

Arguing as in [12], Theorem 17.1.1 we deduce

$$(4.43)\qquad \Sigma_3 \ll X^{\varepsilon}(D_1 M_1^{-1} + D_1^{3/4} + X^{1/2} M_1 D_1^{-1/4}).$$

Bearing in mind (4.30), (4.33), (4.34), (4.40), (4.42) and (4.43) we get

$$(4.44)\qquad \Sigma_1 \ll X^{1+\varepsilon} D_1^{-1/4}.$$

**Estimation of $\Sigma_2$.** Our argument is a modification of Tolev (see [12], Theorem 17.1.1) argument.

Define

$$\mathcal{N}_2(d) = \{n \in \mathbb{N} \colon\ 1 \leqslant n \leqslant d,\, n^2 + 2 \equiv 0(d)\}. \tag{4.45}$$

Working as in $\Sigma_1$, from (4.27) and (4.45) we find

$$\Sigma_2 \ll X^{1+\varepsilon} D_2^{-1} \tag{4.46}$$

for $D_2 \leqslant X^{1/2}$ and

$$\Sigma_2 \ll X^{1/2+\varepsilon} + |\Sigma_4| \tag{4.47}$$

for

$$D_2 > X^{1/2}, \tag{4.48}$$

where

$$\Sigma_4 = \sum_{D_2 \leqslant d_2 < 2D_2} \sum_{n \in \mathcal{N}_2(d_2)} \psi\Big(\frac{\sqrt{X} - n}{d_2}\Big). \tag{4.49}$$

From (4.49) and Lemma 3.1 with

$$M_2 = X^{1/2} \tag{4.50}$$

we obtain

$$\begin{aligned}\Sigma_4 &= \sum_{D_2 \leqslant d_2 < 2D_2} \sum_{n \in \mathcal{N}_2(d_2)} \Bigg(- \sum_{1 \leqslant |m| \leqslant M_2} \frac{e(m((\sqrt{X} - n)/d_2))}{2\pi i m} + \mathcal{O}\Big(f_{M_2}\Big(\frac{\sqrt{X} - n}{d_2}\Big)\Big)\Bigg) \\ &= \Sigma_5 + \Sigma_6,\end{aligned} \tag{4.51}$$

where

$$\Sigma_5 = \sum_{1 \leqslant |m| \leqslant M_2} \frac{\Theta_m}{2\pi i m}, \tag{4.52}$$

$$\Theta_m = \sum_{D_2 \leqslant d_2 < 2D_2} e\Big(\frac{\sqrt{X} m}{d_2}\Big) \sum_{n \in \mathcal{N}_2(d_2)} e\Big(-\frac{nm}{d_2}\Big), \tag{4.53}$$

$$\Sigma_6 = \sum_{D_2 \leqslant d_2 < 2D_2} \sum_{n \in \mathcal{N}_2(d_2)} f_{M_2}\Big(\frac{\sqrt{X} - n}{d_2}\Big). \tag{4.54}$$

By (4.53), (4.54) and Lemma 3.1 it follows

$$
\begin{aligned}
(4.55)\qquad \Sigma_6 &= \sum_{D_2\leqslant d_2<2D_2}\ \sum_{n\in\mathcal{N}_2(d_2)}\ \sum_{m=-\infty}^{\infty} b_{M_2}(m)e\Big(\frac{\sqrt{X}-n}{d_2}m\Big)\\
&= \sum_{m=-\infty}^{\infty} b_{M_2}(m)\Theta_m\\
&\ll \frac{\log M_2}{M_2}|\Theta_0| + \frac{\log M_2}{M_2}\sum_{1\leqslant |m|\leqslant M_2^{1+\varepsilon}} |\Theta_m| + \sum_{|m|>M_2^{1+\varepsilon}} |b_{M_2}(m)||\Theta_m|\\
&\ll \frac{\log M_2}{M_2}D_2^{1+\varepsilon} + \frac{\log M_2}{M_2}\sum_{1\leqslant m\leqslant M_2^{1+\varepsilon}} |\Theta_m| + D_2^{1+\varepsilon}\sum_{|m|>M_2^{1+\varepsilon}} |b_{M_2}(m)|\\
&\ll \frac{\log M_2}{M_2}D_2^{1+\varepsilon} + \frac{\log M_2}{M_2}\sum_{1\leqslant m\leqslant M_2^{1+\varepsilon}} |\Theta_m|.
\end{aligned}
$$

Using (4.51), (4.52) and (4.55) we get

$$
(4.56)\qquad \Sigma_4 \ll X^{\varepsilon}\bigg(\frac{D_2}{M_2} + \sum_{1\leqslant m\leqslant M_2^{1+\varepsilon}} \frac{|\Theta_m|}{m}\bigg).
$$

Define

$$
(4.57)\qquad \mathcal{F}(d) = \{(u,v)\colon\ u^2+2v^2=d,\ (u,v)=1,\ u\in\mathbb{N},\ v\in\mathbb{Z}\setminus\{0\}\}.
$$

According to Lemma 3.4 there exists a surjection

$$
\beta\colon\ \mathcal{F}(d)\to\mathcal{N}_2(d)
$$

from $\mathcal{F}(d)$ to $\mathcal{N}_2(d)$ defined by (4.45) that associates to each couple $(u,v)\in\mathcal{F}(d)$ the element $n\in\mathcal{N}_2(d)$ satisfying

$$
(4.58)\qquad nv\equiv u(d).
$$

Consequently, there exists a subset $\mathcal{F}_0(d)\subset\mathcal{F}(d)$ such that the restriction

$$
\beta|_{\mathcal{F}_0(d)}\colon\ \mathcal{F}_0(d)\to\mathcal{N}_2(d)
$$

of $\beta$ to $\mathcal{F}_0(d)$ is bijection.

Let $\beta|_{\mathcal{F}_0(d)}(u,v)=n_{u,v}$. Now (4.58) gives us

$$
n_{u,v}\equiv u\overline{v}_d(d)
$$

and therefore

$$\frac{n_{u,v}}{d} \equiv u\frac{\overline{v}_{u^2+2v^2}}{u^2+2v^2}(1). \tag{4.59}$$

Bearing in mind (4.59) and Lemma 3.2 we deduce

$$\frac{n_{u,v}}{d} \equiv \frac{u}{v(u^2+2v^2)} - \frac{\bar{u}_{|v|}}{v}(1), \tag{4.60}$$

$$\frac{n_{u,v}}{d} \equiv -\frac{2v}{u(u^2+2v^2)} + \frac{\overline{v}_u}{u}(1). \tag{4.61}$$

From (4.53), (4.60) and (4.61) we find

$$\begin{aligned}
\Theta_m &= \sum_{D_2\leqslant d_2<2D_2} e\Big(\frac{m\sqrt{X}}{d_2}\Big) \sum_{(u,v)\in\mathcal{F}_0(d_2)} e\Big(-\frac{n_{u,v}}{d_2}m\Big) \\
&= \sum_{D_2\leqslant d_2<2D_2} e\Big(\frac{m\sqrt{X}}{d_2}\Big) \sum_{\substack{(u,v)\in\mathcal{F}_0(d_2)\\ 0<u<|v|}} e\Big(-\frac{mu}{v(u^2+2v^2)} + \frac{m\bar{u}_{|v|}}{v}\Big) \\
&\quad + \sum_{D_2\leqslant d_2<2D_2} e\Big(\frac{m\sqrt{X}}{d_2}\Big) \sum_{\substack{(u,v)\in\mathcal{F}_0(d_2)\\ 0<|v|<u}} e\Big(\frac{2mv}{u(u^2+2v^2)} - \frac{m\overline{v}_u}{u}\Big) \\
&= \sum_{\substack{D_2\leqslant u^2+2v^2<2D_2\\ 0<u<|v|\\ (u,v)=1}} e\Big(\frac{m\sqrt{X}}{u^2+2v^2} - \frac{mu}{v(u^2+2v^2)} + \frac{m\bar{u}_{|v|}}{v}\Big) \\
&\quad + \sum_{\substack{D_2\leqslant u^2+2v^2<2D_2\\ 0<|v|<u\\ (u,v)=1}} e\Big(\frac{m\sqrt{X}}{u^2+2v^2} + \frac{2mv}{u(u^2+2v^2)} - \frac{m\overline{v}_u}{u}\Big) \\
&= \Theta'_m + \Theta''_m.
\end{aligned} \tag{4.62}$$

Let us consider $\Theta'_m$. Denote

$$f(u) = e\Big(\frac{m\sqrt{X}}{u^2+2v^2} - \frac{mu}{v(u^2+2v^2)}\Big), \tag{4.63}$$

$$\eta_1(v) = \sqrt{\max(0, D_2-2v^2)}, \quad \eta_2(v) = \sqrt{\min(v^2, 2D_2-2v^2)}, \tag{4.64}$$

$$K_{v,m}(t) = \sum_{\substack{\eta_1(v)\leqslant u\leqslant t\\ (u,v)=1}} e\Big(\frac{m\bar{u}_{|v|}}{v}\Big). \tag{4.65}$$

Using (4.62)–(4.65) and Abel's summation formula we obtain

$$\begin{aligned}(4.66)\quad \Theta'_m &= \sum_{\sqrt{D_2/3}\leqslant|v|<\sqrt{D_2}} \sum_{\substack{\eta_1(v)\leqslant u\leqslant\eta_2(v)\\(u,v)=1}} f(u)e\Big(\frac{m\bar{u}_{|v|}}{v}\Big)\\ &= \sum_{\sqrt{D_2/3}\leqslant|v|<\sqrt{D_2}} \Big(f(\eta_2(v))K_{v,m}(\eta_2(v)) - \int_{\eta_1(v)}^{\eta_2(v)} K_{v,m}(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t}f(t)\Big)\,\mathrm{d}t\Big)\\ &\ll \sum_{\sqrt{D_2/3}\leqslant|v|<\sqrt{D_2}} \Big(1+\frac{m\sqrt{X}}{v^2}\Big) \max_{\eta_1(v)\leqslant t\leqslant\eta_2(v)} |K_{v,m}(t)|.\end{aligned}$$

We are now in a good position to apply Lemma 3.3 because the sum defined by (4.65) is incomplete Kloosterman sum. Thus,

$$(4.67)\qquad K_{v,m}(t) \ll |v|^{1/2+\varepsilon}(v,m)^{1/2}.$$

By (4.66) and (4.67) we get

$$\begin{aligned}(4.68)\qquad \Theta'_m &\ll \sum_{\sqrt{D_2/3}\leqslant|v|<\sqrt{D_2}} \Big(1+\frac{m\sqrt{X}}{v^2}\Big)|v|^{1/2+\varepsilon}(v,m)^{1/2}\\ &\ll X^{\varepsilon}(D_2^{1/4}+mX^{1/2}D_2^{-3/4}) \sum_{0<v<\sqrt{D_2}} (v,m)^{1/2}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(4.69)\qquad \sum_{0<v<\sqrt{D_2}} (v,m)^{1/2} &\leqslant \sum_{l\mid m} l^{1/2} \sum_{\substack{v\leqslant\sqrt{D_2}\\ v\equiv 0(l)}} 1\\ &\ll D_2^{1/2}\sum_{l\mid m} l^{-1/2} \ll D_2^{1/2}\tau(m) \ll X^{\varepsilon}D_2^{1/2}.\end{aligned}$$

Estimations (4.68) and (4.69) imply

$$(4.70)\qquad \Theta'_m \ll X^{\varepsilon}(D_2^{3/4}+mX^{1/2}D_2^{-1/4}).$$

Proceeding in a similar way for $\Theta''_m$ from (4.62) we deduce

$$(4.71)\qquad \Theta''_m \ll X^{\varepsilon}(D_2^{3/4}+mX^{1/2}D_2^{-1/4}).$$

Now (4.62), (4.70) and (4.71) give us

$$(4.72)\qquad \Theta_m \ll X^{\varepsilon}(D_2^{3/4}+mX^{1/2}D_2^{-1/4}).$$

From (4.56) and (4.72) it follows

$$\Sigma_4 \ll X^{\varepsilon}(D_2 M_2^{-1} + D_2^{3/4} + X^{1/2} M_2 D_2^{-1/4}). \tag{4.73}$$

Taking into account (4.50) and (4.73) we find

$$\Sigma_4 \ll X^{1+\varepsilon} D_2^{-1/4}. \tag{4.74}$$

Using (4.46), (4.47) and (4.74) we obtain

$$\Sigma_2 \ll X^{1+\varepsilon} D_2^{-1/4}. \tag{4.75}$$

**Estimation of $\Gamma_2(X)$.** Summarizing (4.23), (4.24), (4.26), (4.44) and (4.75) we get

$$\Gamma_2(X) \ll X^{1+\varepsilon} z^{-1/8}. \tag{4.76}$$

**4.3. The end of the proof.** Bearing in mind (4.1), (4.21), (4.76) and choosing $z = X^{8/9}$ we establish the asymptotic formula (2.5).

The theorem is proved. □

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