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WEAKLY COMPACT SETS IN ORLICZ SEQUENCE SPACES

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Cordially dedicated to Professor Henryk Hudzik

Abstract. We combine the techniques of sequence spaces and general Orlicz functions that are broader than the classical cases of N-functions. We give three criteria for the weakly compact sets in general Orlicz sequence spaces. One criterion is related to elements of dual spaces. Under the restriction of $\lim\limits_{u\to 0} M(u)/u = 0$, we propose two other modular types that are convenient to use because they get rid of elements of dual spaces. Subsequently, by one of these two modular criteria, we see that a set A in Riesz spaces l_p $(1 < p < \infty)$ is relatively sequential weakly compact if and only if it is normed bounded, that says $\sup\limits_{u\in A} \sum\limits_{i=1}^{\infty} |u(i)|^p < \infty$. The result again confirms the conclusion of the Banach-Alaoglu theorem.

Keywords: compact set; weak topology; Banach space; dual space; Orlicz sequence spaces

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1. Introduction and preliminaries

Since the inception of the study of Banach spaces, one of the main topics has been the study of compactness. A set A in a topological space X is called *compact* if any cover of open sets for A has a finite subcover; sequentially compact if any sequence of A has a convergent subsequence; countably compact if any countable subset of A has a cluster point in A, see [15]. The three compact types coincide if the topology is metrizable.

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These types of compactness play important roles not only in theory but also in practical applications. In 1880s, Arzelà-Ascoli's criterion was given for a compact set in a continuous function space, see [3]. Kolmogorov's criterion and Riesz's criterion for a compact set were given in Riesz function spaces and Orlicz function spaces, see [1], [17]. In 1912, Brouwer gave a fixed point theorem in compact settings (see [5]), which led to a broad and thorough development, see [6], [7]. Since then Brouwer's fixed point theorem has been a powerful tool in many theoretical and applied fields, see [8], [12], [16], [18]. A significant contribution was made by Eberlein and Smulian who proved that for the weak topology over a normed space, the three types of compactness coincide, see [14], [23]. James gave a powerful criterion for a weakly compact set in Banach spaces related to attainable functional and reflexivity, see [13].

Orlicz spaces are the extensions of Riesz spaces that have been widely adopted in recent years, especially in nonlinear problems, see [19]. In 1962, Andô gave criteria for weakly $\sigma(L_M, L_N)$-compact sets in the Orlicz function space, see [2]. In 1982, Wu studied normed compact sets and weakly compact sets in the same spaces in general sense, see [24]. In 1997, Zhang and Shi gave a criterion for normed compact sets in the Orlicz sequence space, see [26]. In 2009, Montesinos investigated the weak compactness in L_1 (using different techniques), see [10]. We refer the reader to the surveys in [4], [9], [14], [20] for an outline of the development and applications of this theory. This paper contributes to the literature on the criterion for weak compactness of Orlicz sequence spaces. We combine the techniques and ideas of sequence spaces and Orlicz functions that are much different from N-functions, see [17]. In the end, we also give some criteria which are convenient to use.

In the following part of Section 1 of the paper, we illustrate basic notions, terminology, and original results. In Section 2, some criteria of weak compactness are presented.

Let X be a real Banach space, and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. By X^* we denote the dual space of X. In the sequel, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively. $M\colon \mathbb{R} \to [0,\infty]$, where the ∞ value may be possible, is called an *Orlicz function* if M is vanishing and continuous at zero, convex, even, left continuous on $(0,\infty)$ and not identically equal to zero on $(-\infty,\infty)$. Let $p(u)$ be the right-side derivative of $M(u)$. For every Orlicz function M, we define the complementary function $N\colon \mathbb{R} \to [0,\infty]$ by the formula

$$N(v) = \sup\{u|v| - M(u)\colon\ u \geqslant 0\}.$$

The complementary function N is also an Orlicz function. Let $q(v)$ be the right-side derivative of $N(v)$. Young's inequality $uv \leqslant M(u) + N(v)$ $(u, v \in \mathbb{R})$ holds and the equality in Young's inequality applies if and only if $v = p(u)$ or $u = q(v)$, see [21]. For a given Orlicz function M and a scalar sequence $u = (u(1), u(2), \ldots)$, we define a convex function by

$$\varrho_M(u) = \varrho_M(|u|) = \sum_{i=1}^{\infty} M(|u(i)|),$$

where $|u| = (|u(1)|, |u(2)|, \ldots, |u(i)|, \ldots)$. We introduce the Orlicz sequence space l_M generated by an Orlicz function M by the formula

$$l_M = \{u\colon\ \varrho_M(\lambda u) < \infty \text{ for some } \lambda > 0 \text{ depending on } u\}.$$

This family is linear and is usually equipped with one of the two following equivalent norms:

▹ the Luxemburg norm defined by:

$$\|u\|_{(M)} = \inf\Big\{\lambda > 0\colon\ \varrho_\Phi\Big(\frac{u}{\lambda}\Big) \leqslant 1\Big\}$$

▹ or the Orlicz norm defined by:

$$\|u\|_M = \sup_{\varrho_N(v) \leqslant 1} \sum_{i=1}^{\infty} u(i)v(i).$$

It forms a Banach space which is called an *Orlicz sequence space*, denoted by

$$l_{(M)} = (l_M, \|\cdot\|_{(M)}), \quad l_M = (l_M, \|\cdot\|_M).$$

Let $h_0 = \{u = (u(1), \ldots, u(i), 0, \ldots)\colon\ i = 1, 2, \ldots\}$. The closure of h_0 in $l_{(M)}$ or l_M is denoted by $h_{(M)}$ or h_M, respectively. Further details about Orlicz spaces can be found in [11], [22].

Below we recall the basic facts about Orlicz sequence spaces that will be used in this paper. The proofs can be found in [22].

Lemma 1.1 ([22]). *For all* $u \in l_{(M)}$,

(1) $\|u\|_{(M)} \leqslant 1$ *if and only if* $\varrho_M(u) \leqslant 1$;

(2) $\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}$;

(3) $\|u\|_{(M)} \leqslant 1$ *implies* $\varrho_M(u) \leqslant \|u\|_{(M)}$.

Lemma 1.2 ([22]). *Hölder's inequality:*

$$\sum_{i=1}^{\infty} |u(i)||v(i)| \leqslant \|u\|_{(M)}\|v\|_N \quad (u \in l_{(M)}, v \in l_N),$$

$$\sum_{i=1}^{\infty} |u(i)||v(i)| \leqslant \|u\|_M\|v\|_{(N)} \quad (u \in l_M, v \in l_{(N)}).$$

Lemma 1.3 ([22]). *Representation of Riesz type. In the Orlicz sequence space,*

$$h^*_{(M)} \cong l_N, \quad h^*_M \cong l_{(N)},$$

where $\langle u, v\rangle = \sum\limits_{i=1}^{\infty} u(i)v(i)$ *for all* $u \in h_{(M)}$, $v \in l_N$ *or* $u \in h_M$, $v \in l_{(N)}$.

2. Main results

We recall that a set A is said relatively (sequentially, countably) compact in a topological space if the closure of A is (sequentially, countably) compact. In the sequel, we assume $A \neq \{\theta\}$ because it is trivial to discuss the compactness of a singleton set A.

Theorem 2.1 ([25]). *For a set A in an Orlicz sequence space $l_{(M)}$, A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if and only if*

(1) *A is normed bounded,*

(2) *for each $v \in l_N$*

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0.$$

P r o o f. Sufficiency: Since $l_{(M)}$ is the dual space of h_N (see [22]), we have that the normed bounded A is w^*-compact according to the Banach-Alaoglu theorem. That says, A is relatively weakly $\sigma(l_{(M)}, h_N)$-compact. Since h_N is normed seperable (see [17], [22]), the w^*-topology on A can be metrizable, thus the bounded set A is relatively sequentially weakly $\sigma(l_{(M)}, h_N)$-compact. We obtain that for any sequence $u_n \in A$, there exist $u \in l_M$ and a subsequence u_{n_k} such that $u_{n_k} \to u$ $\sigma(l_{(M)}, h_N)$-weakly. Moreover, $u_{n_k}(i) \to u(i)$ for all natural numbers i.

For each $v \in l_N$ and for any positive number ε, by the condition (2), we have a natural number I_0 such that

$$\sup_{u\in A} \sum_{i=I_0}^{\infty} |u(i)v(i)| < \frac{\varepsilon}{4}.$$

By Hölder's inequality, $\sum\limits_{i=1}^{\infty} |u(i)||v(i)| < \infty$, there exists a natural number $I_1 \geqslant I_0$ such that

$$\sum_{i=I_1}^{\infty} |u(i)v(i)| < \frac{\varepsilon}{4}.$$

From $u_{n_k}(i) \to u(i)$ for all natural numbers i, we have a natural number k_0 such that for all $k \geqslant k_0$

$$\sum_{i=1}^{I_1} |(u_{n_k} - u)(i)||v(i)| < \frac{\varepsilon}{4}.$$

Therefore, for all $k \geqslant k_0$

$$
\begin{aligned}
|\langle v, u_{n_k} - u\rangle| &= \left|\sum_{i=1}^{\infty}(u_{n_k}(i) - u(i))v(i)\right| \\
&= \left|\sum_{i=1}^{I_1}(u_{n_k} - u)(i)v(i) + \sum_{i=I_1+1}^{\infty}(u_{n_k} - u)(i)v(i)\right| \\
&\leqslant \left|\sum_{i=1}^{I_1}(u_{n_k} - u)(i)v(i)\right| + \left|\sum_{i=I_1+1}^{\infty}(u_{n_k} - u)(i)v(i)\right| \\
&\leqslant \sum_{i=1}^{I_1}|(u_{n_k} - u)(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|(u_{n_k} - u)(i)||v(i)| \\
&\leqslant \sum_{i=1}^{I_1}|(u_{n_k} - u)(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|u_{n_k}(i)||v(i)| + \sum_{i=I_1+1}^{\infty}|u(i)||v(i)| \\
&\leqslant \sum_{i=1}^{I_1}|(u_{n_k} - u)(i)||v(i)| + \sup_{u\in A}\sum_{i=I_0}^{\infty}|u(i)v(i)| + \sum_{i=I_1+1}^{\infty}|u(i)||v(i)| \\
&\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}
$$

It follows that for all $v \in l_N$, $\langle v, u_{n_k} - u\rangle \to 0$, that says, A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact.

Necessity: At first, A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact implies that A is $\sigma(l_{(M)}, l_N)$-bounded and by the Banach all bounded principle, we get (1), i.e., A is normed bounded.

Next we prove (2), i.e., for all $v \in l_N$

$$\lim_{I\to\infty}\sup_{u\in A}\sum_{i=I}^{\infty}|u(i)||v(i)| = 0.$$

Otherwise, for some $v \in l_N$ and positive ε_0, there must exist a strictly increasing sequence of natural numbers I_n such that

$$\sup_{u\in A}\sum_{i=I_n}^{\infty}|u(i)v(i)| > \varepsilon_0.$$

We take $u_n \in A$ such that

$$\sum_{i=I_n}^{\infty}|u_n(i)v(i)| > \varepsilon_0.$$

Since A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact, we deduce that the sequence $\{u_n\}$ has a subsequence and we still write $\{u_n\}$ for simplicity. For this subsequence, there is $u \in l_M$ such that $u_n \to u$ $\sigma(l_{(M)}, l_N)$-weakly. Then $u_n(i) \to u(i)$

for all natural numbers i. By Hölder's inequality, we see that $\sum\limits_{i=1}^{\infty}|u(i)||v(i)| \leqslant \|u\|_{(M)}\|v\|_N < \infty$. So there is a natural number I' such that

$$\sum_{i=I'}^{\infty}|u(i)||v(i)| \leqslant \frac{\varepsilon_0}{2}.$$

Thus for all $I_n \geqslant I'$,

$$\begin{aligned}\sum_{i=I_n}^{\infty}|(u_n(i)-u(i))v(i)| &\geqslant \sum_{i=I_n}^{\infty}|u_n(i)v(i)|-|u(i))v(i)| \\ &= \sum_{i=I_n}^{\infty}|u_n(i)v(i)|-\sum_{i=I_n}^{\infty}|u(i)v(i)| > \varepsilon_0-\frac{\varepsilon_0}{2}=\frac{\varepsilon_0}{2}.\end{aligned}$$

For simplicity, we write u_n-u as w_n. Then for all natural numbers n,

$$\sum_{i=I_n}^{\infty}|w_n(i)v(i)| \geqslant \frac{\varepsilon_0}{2}.$$

Since w_n is $\sigma(l_{(M)}, l_N)$-weakly convergent to θ, we see that $w_n(i) \to 0$ for all natural numbers i.

By Hölder's inequality, $\sum\limits_{i=1}^{\infty}|w_1(i)||v(i)| < \infty$. We write $I_1=0$ and we take a natural number $I_1' > I_1$ such that $\sum\limits_{i=I_1'+1}^{\infty}|w_1(i)||v(i)| < \frac{1}{8}\varepsilon_0$. We write $w_{n_1}=w_1$ and we have

$$\begin{aligned}\sum_{i=I_1+1}^{I_1'}|w_{n_1}(i)||v(i)| &= \sum_{i=I_1+1}^{\infty}|w_1(i)||v(i)|-\sum_{i=I_1'+1}^{\infty}|w_1(i)||v(i)| \\ &\geqslant \frac{\varepsilon_0}{2}-\frac{\varepsilon_0}{8}=\frac{3\varepsilon_0}{8}.\end{aligned}$$

Since $w_n(i) \to 0$ for all natural numbers i, we take a natural number n_2 such that $I_{n_2} > I_1'$, $\sum\limits_{i=1}^{I_1'}|w_{n_2}(i)||v(i)| < \frac{1}{8}\varepsilon_0$.

By Hölder's inequality, $\sum\limits_{i=1}^{\infty}|w_{n_2}(i)||v(i)| < \infty$. We take a natural number $I_{n_2}' > I_{n_2}$ such that $\sum\limits_{i=I_{n_2}'+1}^{\infty}|w_{n_2}(i)||v(i)| < \frac{1}{8}\varepsilon_0$, then

$$\begin{aligned}\sum_{i=I_{n_1}'+1}^{I_{n_2}'}|w_{n_2}(i)v(i)| &= \sum_{i=I_{n_1}'+1}^{I_{n_2}}|w_{n_2}(i)v(i)|+\sum_{i=I_{n_2}+1}^{I_{n_2}'}|w_{n_2}(i)v(i)| \geqslant \sum_{i=I_{n_2}+1}^{I_{n_2}'}|w_{n_2}(i)v(i)| \\ &= \sum_{i=I_{n_2}+1}^{\infty}|w_{n_2}(i)v(i)|-\sum_{i=I_{n_2}'+1}^{\infty}|w_{n_2}(i)v(i)| \geqslant \frac{\varepsilon_0}{2}-\frac{\varepsilon_0}{8} \geqslant \frac{3\varepsilon_0}{8}.\end{aligned}$$

In the induction procedure, since $w_n(i) \to 0$ for all natural numbers i, we take a natural number n_k such that $I_{n_k} > I'_{n_{k-1}}$ and

$$\sum_{i=1}^{I'_{n_{k-1}}} |w_{n_k}(i)||v(i)| < \frac{\varepsilon_0}{8}.$$

By Hölder's inequality, $\sum\limits_{i=1}^{\infty} |w_{n_k}(i)||v(i)| < \infty$. We take a natural number $I'_{n_k} > I_{n_k}$ such that

$$\sum_{i=I'_{n_k}+1}^{\infty} |w_{n_k}(i)||v(i)| < \frac{\varepsilon_0}{8}.$$

Then we have

$$\begin{aligned}\sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |w_{n_k}(i)v(i)| &= \sum_{i=I'_{n_{k-1}}+1}^{I_{n_k}} |w_{n_k}(i)v(i)| + \sum_{i=I_{n_k}+1}^{I'_{n_k}} |w_{n_k}(i)v(i)| \geqslant \sum_{i=I_{n_k}+1}^{I'_{n_k}} |w_{n_k}(i)v(i)| \\ &= \sum_{i=I_{n_k}+1}^{\infty} |w_{n_k}(i)v(i)| - \sum_{i=I'_{n_k}+1}^{\infty} |w_{n_k}(i)v(i)| \geqslant \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{8} = \frac{3\varepsilon_0}{8}.\end{aligned}$$

We set $\tilde{v}(i) = |v(i)| \operatorname{sign} w_{n_k}(i)$ as $I'_{n_{k-1}} < i \leqslant I'_{n_k}$, where $I'_{n_0} = 0$. Obviously, $|\tilde{v}(i)| = |v(i)|$ for all i. Since l_N is symmetric, we get $\tilde{v} \in l_N$. But for all k,

$$\begin{aligned}\langle \tilde{v}, w_{n_k} \rangle &= \sum_{i=1}^{\infty} w_{n_k}(i)\tilde{v}(i) \\ &= \sum_{i=1}^{I'_{n_{k-1}}} w_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} w_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_k}+1}^{\infty} w_{n_k}(i)\tilde{v}(i) \\ &= \sum_{i=1}^{I'_{n_{k-1}}} w_{n_k}(i)\tilde{v}(i) + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |w_{n_k}(i)||v(i)| + \sum_{i=I'_{n_k}+1}^{\infty} w_{n_k}(i)\tilde{v}(i) \\ &\geqslant -\sum_{i=1}^{I'_{n_{k-1}}} |w_{n_k}(i)\tilde{v}(i)| + \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |w_{n_k}(i)||v(i)| - \sum_{i=I'_{n_k}+1}^{\infty} |w_{n_k}(i)\tilde{v}(i)| \\ &= \sum_{i=I'_{n_{k-1}}+1}^{I'_{n_k}} |w_{n_k}(i)||v(i)| - \sum_{i=1}^{I'_{n_{k-1}}} |w_{n_k}(i)\tilde{v}(i)| - \sum_{i=I'_{n_k}+1}^{\infty} |w_{n_k}(i)v(i)| \\ &\geqslant \frac{3\varepsilon_0}{8} - \frac{\varepsilon_0}{8} - \frac{\varepsilon_0}{8} = \frac{\varepsilon_0}{8}.\end{aligned}$$

This is a contradiction with that w_n is $\sigma(l_{(M)}, l_N)$-weakly convergent to θ. It ends the proof. □

If a set A is sequentially weakly $\sigma(l_{(M)}, l_N)$-compact, A is $\sigma(l_{(M)}, l_N)$-closed. We have immediately:

Corollary 2.2 ([25]). *Given a set A in an Orlicz sequence space $l_{(M)}$, A is sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if and only if*

(1) *A is $\sigma(l_{(M)}, l_N)$-closed,*

(2) *A is normed bounded,*

(3) *for each $v \in l_N$*

$$\lim_{I\to\infty} \sup_{u\in A} \sum_{i=I}^{\infty} |u(i)||v(i)| = 0.$$

Next, we give a modular criterion, which gets rid of the elements of l_N of Theorem 2.1 and is easier to use.

Theorem 2.3 ([2], [25]). *Given $\lim\limits_{u\to 0} M(u)/u = 0$, a set A in an Orlicz sequence space $l_{(M)}$ is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if and only if*

$$\lim_{\xi\to 0} \sup_{u\in A} \frac{\varrho_M(\xi u)}{\xi} = 0.$$

P r o o f. Sufficiency: It is enough to show that the conditions (1) and (2) of Theorem 2.1 hold. As $\lim\limits_{\xi\to 0} \sup\limits_{u\in A} \varrho_M(\xi u)/\xi = 0$, we take $0 < \xi_1 \leqslant 1$ such that $\sup\limits_{u\in A} \varrho_M(\xi_1 u)/\xi_1 \leqslant 1$. Then $\sup\limits_{u\in A} \varrho_M(\xi_1 u) \leqslant \xi_1 \leqslant 1$. By Lemma 1.1, $\sup\limits_{u\in A} \|\xi_1 u\|_{(M)} \leqslant 1$, $\sup\limits_{u\in A} \|u\|_{(M)} \leqslant 1/\xi_1$. We get that (1) holds, i.e., A is normed bounded.

For each $v \in l_N$, by the definition of l_N, we take a positive number λ with $\varrho_N(\lambda v) < \infty$. For any $\varepsilon > 0$, by the given condition, there exists a positive number ξ such that

$$\sup_{u\in A} \frac{\varrho_M(\xi u)}{\xi} < \frac{\lambda\varepsilon}{2}.$$

We take a natural number I_0 such that $\sum\limits_{i=I_0+1}^{\infty} N(\lambda v(i)) < \frac{1}{2}\lambda\xi\varepsilon$. Then for all $u \in A$ and all natural numbers $I > I_0$

$$\begin{aligned}\sum_{i=I}^{\infty} |u(i)v(i)| &\leqslant \sum_{i=I_0}^{\infty} |u(i)v(i)| = \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} \xi|u(i)|\lambda|v(i)| \leqslant \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} M(\xi|u(i)|) + N(\lambda|v(i)|) \\ &= \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} M(\xi u(i)) + \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} N(\lambda v(i)) \\ &\leqslant \frac{1}{\xi\lambda} \varrho_M(\xi u(i)) + \frac{1}{\xi\lambda} \sum_{i=I_0}^{\infty} N(\lambda v(i)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, (2) of Theorem 2.1 holds:

$$\lim_{I\to\infty}\sup_{u\in A}\sum_{i=I}^{\infty}|u(i)v(i)|=0.$$

Combing (1) and (2), by Theorem 2.1 we get that A is relatively sequentially weakly $\sigma(l_{(M)},l_N)$-compact.

Necessity: We will prove that

$$\lim_{\xi\to 0}\sup_{u\in A}\frac{\varrho_M(\xi u)}{\xi}=0.$$

Otherwise, there exists a positive number ε_0 such that

$$\inf_{\xi>0}\sup_{u\in A}\frac{\varrho_M(\xi u)}{\xi}=\lim_{\xi\to 0}\sup_{u\in A}\frac{\varrho_M(\xi u)}{\xi}>\varepsilon_0,$$

where the identity holds due to $M(u)/u$ being nondecreasing. We take $u_n\in l_M$ such that for each natural number n

$$\varrho_M\Big(\frac{u_n}{2^{n+1}}\Big)2^{n+1}>\varepsilon_0.$$

From Young's inequality [17], we see that

$$N\Big(p\Big(\frac{u_n(t)}{2^{n+1}}\Big)\Big)\leqslant N\Big(p\Big(\frac{u_n(t)}{2^{n+1}}\Big)\Big)+M\Big(\frac{u_n(t)}{2^{n+1}}\Big)\leqslant M\Big(2\frac{u_n(t)}{2^{n+1}}\Big).$$

By Lemma 1.1, we have that for each n

$$\varrho_N\Big(p\Big(\frac{u_n}{2^{n+1}}\Big)\Big)\leqslant\varrho_M\Big(2\frac{u_n}{2^{n+1}}\Big)\leqslant 2\frac{1}{2^{n+1}}\varrho_M(u_n)\leqslant 2\frac{1}{2^{n+1}}=\frac{1}{2^n}.$$

We set

$$v(i)=\sup_n p\Big(\frac{u_n(i)}{2^{n+1}}\Big),\quad i=1,2,\dots$$

By the left continuity of an Orlicz function N, we have

$$\begin{aligned}\varrho_N(v)&=\sum_{i=1}^{\infty}N(v(i))=\sum_{i=1}^{\infty}N\Big(\sup_n p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\Big)=\sum_{i=1}^{\infty}\sup_n N\Big(p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\Big)\\&\leqslant\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}N\Big(p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\Big)=\sum_{n=1}^{\infty}\sum_{i=1}^{\infty}N(p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\\&\leqslant\sum_{n=1}^{\infty}\varrho_N\Big(p\Big(\frac{u_n}{2^{n+1}}\Big)\Big)\leqslant\sum_{n=1}^{\infty}\frac{1}{2^n}=1.\end{aligned}$$

Now v is well defined and $v \in l_N$. Since A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact, by Theorem 2.1, there exists a natural number I such that

$$\sup_{u\in A} \sum_{i=I}^{\infty} |u(i)v(i)| \leqslant \frac{\varepsilon_0}{4}.$$

We note that $|u(i)| \leqslant c$, where $c := \inf\{t > 0\colon\ M(t) > 1\} < \infty$ for all $u \in A$ and $\lim\limits_{u\to 0} M(u)/u = 0$. Then we deduce that for n large enough

$$\sum_{i=1}^{I} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1} \leqslant IM\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} \leqslant \frac{\varepsilon_0}{4}.$$

We reache a contradiction:

$$\begin{aligned}
\varepsilon_0 < \varrho_{M}\Big(\frac{u_n}{2^{n+1}}\Big)2^{n+1} &= \sum_{i=1}^{I} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1}\\
&\leqslant \sum_{i=1}^{I} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} N\Big(p\Big(\frac{u(i)}{2^{n+1}}\Big)\Big)\\
&\leqslant \sum_{i=1}^{I} M\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} M\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} N\Big(p\Big(\frac{u(i)}{2^{n+1}}\Big)\Big)\\
&= \sum_{i=1}^{I} M\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} \frac{|u_n(i)|}{2^{n+1}} p\Big(\frac{u_n(i)}{2^{n+1}}\Big)2^{n+1}\\
&\leqslant \sum_{i=1}^{I} M\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} |u_n(i)| p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\\
&\leqslant \sum_{i=1}^{I} M\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} |u_n(i)| \sup_n p\Big(\frac{u_n(i)}{2^{n+1}}\Big)\\
&\leqslant \sum_{i=1}^{I} M\Big(\frac{c}{2^{n+1}}\Big)2^{n+1} + \sum_{i=I+1}^{\infty} |u_n(i)||v(t)| \leqslant \frac{\varepsilon_0}{4} + \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{2}.
\end{aligned}$$

This ends the proof. □

Due to the same reason, from Corollary 2.2, stating that a sequentially weakly $\sigma(l_{(M)}, l_N)$-compact set is $\sigma(l_{(M)}, l_N)$-closed, we have:

Corollary 2.4 ([2], [25]). *Given $\lim\limits_{u\to 0} M(u)/u = 0$, a set A in an Orlicz sequence space $l_{(M)}$ is sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if and only if*

(1) *A is $\sigma(l_{(M)}, l_N)$-closed,*

(2) $\lim\limits_{\xi\to 0} \sup\limits_{u\in A} \varrho_M(\xi u)/\xi = 0.$

Finally, we give a criterion of a modular type which gets rid of the computation of limit.

Definition 2.5 ([2]). For Orlicz functions Φ and M over the real field $\mathbb{R}$ we call Φ *strictly rapider* than M for small u (write $\Phi \succ M$) provided that for any positive number κ, there are positive numbers D and d such that for all $u \geqslant 0$ with $M(u) \leqslant d$, we have $\Phi(Du) \geqslant D\kappa M(u)$.

Theorem 2.6 ([2], [25])). *Given* $\lim\limits_{u\to 0} M(u)/u = 0$, *a set* A *in an Orlicz sequence space* $l_{(M)}$ *is relatively sequentially weakly* $\sigma(l_{(M)}, l_N)$*-compact if and only if there exists an Orlicz function* Φ *strictly rapider than* M *(write* $\Phi \succ M$*) such that*

$$\sup_{u\in A} \varrho_\Phi(u) \leqslant 1.$$

P r o o f. Sufficiency: By $\Phi \succ M$, for any positive number κ, there are positive numbers D and d such that for all $u \geqslant 0$ with $M(u) \leqslant d$, we have $\Phi(Du) \geqslant D\kappa M(u)$. Put $I_d = \text{Cardinal}\{i\colon\ M(u(i)) > d,\ \sum\limits_{i=1}^{\infty} \Phi(u(i)) = \varrho_\Phi(u) \leqslant 1\}$, then I_d is a finite number.

For any positive number $\varepsilon < 1$ and positive number $\xi \leqslant \min\{(1/c)c_d, 1/D\}$, where $c_d = \inf\{t > 0\colon\ M((t)) > d\}$, we have

$$\begin{aligned}
\frac{\varrho_M(\xi u)}{\xi} &= \frac{1}{\xi}\Bigg[\sum_{M(\xi|u(i)|)\leqslant d} M(\xi u(i)) + \sum_{M(\xi|u(i)|)>d} M(\xi u(i))\Bigg]\\
&= \frac{1}{\xi}\sum_{M(\xi|u(i)|)\leqslant d} M(\xi u(i)) + \frac{1}{\xi}\sum_{M(\xi|u(i)|)>d} M(\xi u(i))\\
&= \sum_{M(\xi|u(i)|)\leqslant d} \frac{M(\xi u(i))}{\xi} + \sum_{M(\xi|u(i)|)>d} \frac{M(\xi u(i))}{\xi}\\
&\leqslant \sum_{M(\xi|u(i)|)\leqslant d} \Phi(D\xi u(i))\frac{\varepsilon}{2D\xi} + \sum_{M(\xi|u(i)|)>d} \frac{M(\xi u(i))}{\xi}\\
&\leqslant \sum_{M(\xi|u(i)|)\leqslant d} \Phi(u(i))\frac{D\xi\varepsilon}{2D\xi} + I_d\frac{M(\xi c)}{\xi} \leqslant \frac{\varepsilon}{2}\varrho_\Phi(u) + \frac{\varepsilon}{2} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where $c = \inf\{t > 0\colon\ M(t) > 1\}$. That says, $\lim\limits_{\xi\to 0}\sup\limits_{u\in A} \varrho_M(\xi u)/\xi = 0$. By Theorem 2.3, it follows that A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact.

Necessity: By Theorem 2.3, A being relatively weakly $\sigma(l_{(M)}, l_N)$-compact implies

$$\lim_{\xi\to 0}\sup_{u\in A} \frac{\varrho_M(\xi u)}{\xi} = 0.$$

We take ξ_k, $1 > \xi_1 > \ldots > \xi_k > \ldots \to 0$, such that

$$\sup_{u\in A} \frac{\varrho_M(\xi_k u)}{\xi_k} < \frac{1}{2^{2k}}.$$

We recall an Orlicz function given by Andô, see [2]. For any $u \in \mathbb{R}$, we set

$$\Phi(u) = \sum_{k=1}^{\infty} 2^k \frac{M(\xi_k u)}{\xi_k}.$$

Then Φ is an Orlicz function. Further, $\Phi \succ M$ and for any positive κ we take a natural number k' with $2^{2k'} \geqslant \kappa$. We set $D = 1/\xi_{k'}$. Thus for all $u \in \mathbb{R}$,

$$\begin{aligned}\Phi(Dv)_{v=\xi_{k'}u} &= \Phi\Big(\frac{\xi_{k'}u}{\xi_{k'}}\Big) = \Phi(u) = \sum_{k=1}^{\infty} 2^{2k} \frac{M(\xi_k u)}{\xi_k} \geqslant 2^{2k'} \frac{M(\xi_{k'} u)}{\xi_{k'}} \\ &= 2^{2k'} \frac{M(v)}{\xi_{k'}} \geqslant \kappa D M(v).\end{aligned}$$

It holds for all $v > 0$ due to the arbitrary choice of u. Furthermore, for all $u \in A$,

$$\begin{aligned}\varrho_\Phi(u) &= \sum_{i=1}^{\infty} \Phi(u(i)) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} 2^k \frac{M(\xi_k u(i))}{\xi_k} = \sum_{k=1}^{\infty} 2^k \sum_{i=1}^{\infty} \frac{M(\xi_k u(i))}{\xi_k} \\ &\leqslant \sum_{k=1}^{\infty} 2^k \frac{1}{2^{2k}} = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.\end{aligned}$$

It ends the proof. □

Analogously, we have:

Corollary 2.7 ([2], [25]). *Given* $\lim_{u\to 0} M(u)/u = 0$, *a set* A *in an Orlicz sequence space* $l_{(M)}$ *is sequentially weakly* $\sigma(l_{(M)}, l_N)$*-compact if and only if*

(1) A *is* $\sigma(l_{(M)}, l_N)$*-closed,*

(2) *there exists an Orlicz function* Φ *strictly rapider than* M *(write* $\Phi \succ M$*) such that*

$$\sup_{u\in A} \varrho_\Phi(u) \leqslant 1.$$

From Theorems 2.1, 2.3, 2.6, we see:

Remark 2.8. In an Orlicz sequence space $l_{(M)}$, a set A is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact if and only if $|A|$ is relatively sequentially weakly $\sigma(l_{(M)}, l_N)$-compact, where $|A| = \{|u|\colon\ u \in A\}$.

Since $\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}$, $l_{(M)}$ is isomorphic to l_M. Since sequentially weak compactness is invariant under an isomorphism in an Orlicz sequence space l_M with Orlicz norm, we have:

Remark 2.9. All the main results obtained in Section 2 for the Luxemburg norm of $l_{(M)}$ hold for the Orlicz norm of l_M. That means that replacing $l_{(M)}$ by l_M, in Theorems 2.1, 2.3, 2.6, Corollaries 2.2, 2.4, 2.7, and Remark 2.8, all the statements stay to hold.

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