PROJECTIVELY EQUIVARIANT QUANTIZATION AND SYMBOL ON SUPERCIRCLE $S^{1|3}$

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Abstract. Let $\mathcal{D}_{\lambda,\mu}$ be the space of linear differential operators on weighted densities from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ as module over the orthosymplectic Lie superalgebra $\mathfrak{osp}(3|2)$, where $\mathcal{F}_\lambda$, $\lambda \in \mathbb{C}$ is the space of tensor densities of degree λ on the supercircle $S^{1|3}$. We prove the existence and uniqueness of projectively equivariant quantization map from the space of symbols to the space of differential operators. An explicite expression of this map is also given.

Keywords: differential operator; density; equivariant quantization and orthosymplectic algebra

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1. Introduction

The usual quantization procedure consists of building a map Q from the space $\mathrm{Pol}(T^*M)$ of polynomials on T^*M and the space $\mathcal{D}(M)$ of linear differential operators on M called a *quantization map*. Generally, there is no quantization and symbol map equivariant with respect to the action of the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M (or the group $\mathrm{Diff}(M)$ of diffeomorphisms of M) on the two spaces $\mathcal{D}(M)$ and $\mathrm{Pol}(T^*M)$. Thus, we restrict ourselves to equivariant symbols and quantization maps with respect to the action of a given subalgebra of $\mathrm{Vect}(M)$.

The concept of equivariant quantization over $\mathbb{R}^n$ was introduced by Lecomte and Ovsienko in [5]. In this seminal work, they considered spaces of differential operators acting between densities and the Lie algebra of projective vector fields over $\mathbb{R}^n$, $\mathrm{sl}(n+1)$. In this situation, they showed the existence and uniqueness of an equivariant quantization. These results were generalized in many references (see for instance [1], [3]). In [4], Lecomte globalized the problem of equivariant quantization by defining the problem of natural invariant quantization on arbitrary manifolds.

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In [7], [8], the existence and uniqueness of equivariant quantizations was proven in the context of supergeometry. Our motivation is to extend the results proved in [9] to that of the case of dimension $1|3$. Namely, we consider the supermanifold $S^{1|3}$ and $\mathcal{D}_{\lambda,\mu}(S^{1|3})$ the space of differential operators $A\colon \mathcal{F}_\lambda \to \mathcal{F}_\mu$, where $\mathcal{F}_\lambda, \lambda \in \mathbb{C}$, is the space of tensor densities on the supercircle $S^{1|3}$ of degree λ. The analogue, in the super setting, of the projective algebra sl(2) is the orthosymplectic Lie superalgebra $\mathfrak{osp}(3|2)$, which is the smallest simple Lie superalgebra, can be realized as a subalgebra of $\mathrm{Vect}_{\mathbb{C}}(S^{1|3})$. Naturally, the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|3})$, and therefore $\mathfrak{osp}(3|2)$, acts on $\mathcal{D}_{\lambda,\mu}$; the $\mathfrak{osp}(3|2)$-module $\mathcal{D}_{\lambda,\mu}$ is filtered as:

$$\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^{1/2}_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \mathcal{D}^{3/2}_{\lambda,\mu} \subset \ldots \subset \mathcal{D}^{k-1/2}_{\lambda,\mu} \subset \mathcal{D}^k_{\lambda,\mu} \subset \ldots$$

The graded module $\mathrm{gr}(\mathcal{D}_{\lambda,\mu})$, also called the *space of symbols* and denoted by $\mathcal{S}_{\lambda,\mu}$, depends only on the shift $\delta = \mu - \lambda$ of the weights. Moreover, as a $\mathrm{Vect}_{\mathbb{C}}(S^{1|3})$-module, $\mathcal{S}_{\lambda,\mu}$ is decomposed as $\bigoplus_{k\in\mathbb{N}/2} \mathcal{S}^k_{\lambda,\mu}$, where

$$\mathcal{S}^k_\delta = \mathcal{S}^k_{\lambda,\mu} = \bigoplus_{l=0}^{2k} \mathcal{D}^l_{\lambda,\mu}/\mathcal{D}^{l-1/2}_{\lambda,\mu}.$$

Moreover, in the main theorem of the paper, we prove that if $\delta = \mu - \lambda \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, k$, then $\mathcal{D}^k_{\lambda,\mu}$ is isomorphic to $\mathcal{S}^k_{\lambda,\mu}$ as an $\mathfrak{osp}(3|2)$-module. This isomorphism, called a *conformally equivariant quantization map*, is unique (once we fix a principal symbol). Explicit expressions of the quantization map, is also given.

2. Basic definitions and tools

2.1. Geometry of the supercircle $S^{1|3}$. The supercircle $S^{1|3}$ is the simplest supermanifold of dimension $1|3$ generalizing S^1. It can be defined in terms of its superalgebra of functions, denoted by $C^\infty(S^{1|3})$ and consisting of elements of the form

$$f(x,\theta_1,\theta_2,\theta_3) = f_0(x) + \sum_{i=1}^{3} \theta_i f_i(x) + \sum_{i<j} \theta_i\theta_j f_{i,j}(x) + \theta_1\theta_2\theta_3 f_{1,2,3}(x),$$

where x is an arbitrary parameter on S^1 (the even variable), θ_i ($1 \leqslant i \leqslant 3$) are the odd variables ($\theta_i^2 = 0$), and $f_0, f_i, f_{i,j}, f_{1,2,3} \in C^\infty(S^1)$. We denote by $p = |\cdot|$ the parity function by setting $p(x) = p(\theta_i\theta_j) = 0$ ($1 \leqslant i < j \leqslant 3$) and $p(\theta_i) = p(\theta_1\theta_2\theta_3) = 1$ ($1 \leqslant i \leqslant 3$).

Let $\mathrm{Vect}(S^{1|3})$ be the superspace of vector fields on $S^{1|1}$:

$$\mathrm{Vect}_{\mathbb{C}}(S^{1|3}) = \left\{ F_0\partial_x + \sum_{i=1}^{3} F_i\partial_{\theta_i} :\ F_i \in C^\infty_{\mathbb{C}}(S^{1|3}) \right\}, \tag{2.1}$$

where ∂_{θ_i} or ∂_x means the partial derivative $\partial/\partial\theta_i$ or $\partial/\partial x$, respectively.

The standard contact structure on $S^{1|3}$ is defined by the distribution generated by $\overline{D}_1$, $\overline{D}_2$ and $\overline{D}_3$, which is equivalently the kernel of differential 1-form

$$\alpha = \mathrm{d}x + \sum_{i=1}^{3} \theta_i\, \mathrm{d}\theta_i.$$

We recall that every contact vector field can be expressed for a given function $f \in C^\infty_{\mathbb{C}}(S^{1|3})$ by

$$X_f = f\partial_x - (-1)^{p(f)}\frac{1}{2}\sum_{i=1}^{3}\overline{D}_i(f)\overline{D}_i \tag{2.2}$$

such that $\overline{D}_i = \partial_{\theta_i} - \theta_i\partial_x$ for each $1 \leqslant i \leqslant 3$ (see [11] for the interpretation of this fields).

2.2. The orthosymplectic Lie superalgebra $\mathfrak{osp}(3|2)$. We consider the orthosymplectic Lie superalgebra $\mathfrak{osp}(3|2)$, which is the smallest simple Lie superalgebra. This superalgebra defines a projective (conformal) structure on the supercircle $S^{1|3}$ (see [10]), and it is spanned by the contact vector X_f which are the elements of

$$\{X_1, X_{\theta_i}, X_{\theta_m\theta_n}, X_{x\theta_i}, X_{x^2};\ 1 \leqslant i \leqslant 3,\ 1 \leqslant m < n \leqslant 3\}.$$

The subalgebra $\mathrm{Aff}(3|2)$ of $\mathfrak{osp}(3|2)$ is called the *affine Lie superalgebra* spanned by

$$\{X_1, X_{\theta_i}, X_{\theta_m\theta_n};\ 1 \leqslant i \leqslant 3,\ 1 \leqslant m < n \leqslant 3\}.$$

2.3. The space of weighted densities on $S^{1|3}$. In the super setting, by replacing $\mathrm{d}x$ by the 1-form α, we get an analogous definition for weighted densities, i.e., we define the space of λ-densities as

$$\mathcal{F}_\lambda = \{F\alpha^\lambda :\ F \in C^\infty_{\mathbb{C}}(S^{1|3})\}. \tag{2.3}$$

As a vector space, $\mathcal{F}_\lambda$ is isomorphic to $C^\infty_{\mathbb{C}}(S^{1|3})$.

Let X_f be a contact vector field. We define a one-parameter family of the first order differential operators on $C^\infty_{\mathbb{C}}(S^{1|3})$

$$\mathcal{L}^\lambda_{X_f} = \mathcal{L}_{X_f} + \lambda F', \quad \lambda \in \mathbb{C}. \tag{2.4}$$

One easily checks that the map $X_F \mapsto \mathcal{L}^\lambda_{X_f}$ is a homomorphism of Lie superalgebra, that is, $[\mathcal{L}^\lambda_{X_f}, \mathcal{L}^\lambda_{X_G}] = \mathcal{L}^\lambda_{[X_f,\, X_G]}$ for every λ. Thus, $\mathcal{F}_\lambda$ becomes a $\mathcal{K}(3)$-module on $C^\infty_{\mathbb{C}}(S^{1|3})$. Evidently, the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_f in $\mathcal{K}(3)$ is given by

$$\mathcal{L}^\lambda_{X_f}(G\alpha^\lambda) = \Big(fG' - \frac{1}{2}(-1)^{|f|}\sum_{i=1}^{3}\overline{D}_i(f)\overline{D}_i(G) + \lambda f'G\Big)\alpha^\lambda. \tag{2.5}$$

2.4. Differential operators on weighted densities. In this section we consider the space of differential operators acting on the space of weighted densities

$$A\colon\ \mathcal{F}_\lambda \to \mathcal{F}_\mu,$$

where $\lambda, \mu \in \mathbb{R}$. This space is denoted by $\mathcal{D}_{\lambda,\mu}$. The space of differential operators of order less or equal than k, is denoted by $\mathcal{D}^k_{\lambda,\mu}$. For every integer or half integer, a linear differential operator $A \in \mathcal{D}^k_{\lambda,\mu}$ is of the form

$$A = \sum_{l+m/2+n/2+p/2 \leqslant k} a_{l,m,n,p}\partial^l_x \overline{D}^m_1 \overline{D}^n_2 \overline{D}^p_3, \tag{2.6}$$

where $a_{l,m,n,p} \in C^\infty(S^{1|3})$ and $m, n, p \leqslant 1$.

The Lie superalgebra $\mathcal{K}(3)$ acts on the space of linear differential operators as follows:

$$\mathcal{L}^{\lambda,\mu}_{X_f}(A) = \mathcal{L}^\mu_{X_f} \circ A - (-1)^{|A||f|} A \circ \mathcal{L}^\lambda_{X_f}. \tag{2.7}$$

This above $\mathcal{K}(3)$-module space has a $\mathcal{K}(3)$-invariant finer filtration:

$$\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^{1/2}_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \mathcal{D}^{3/2}_{\lambda,\mu} \subset \ldots \subset \mathcal{D}^{k-1/2}_{\lambda,\mu} \subset \mathcal{D}^k_{\lambda,\mu} \subset \ldots \tag{2.8}$$

2.5. Space of symbols of differential operators. Consider the graded $\mathcal{K}(3)$-module $\mathrm{gr}\mathcal{D}_{\lambda,\mu}$, associated with the finer filtration (2.6) and called the *space of symbols of differential operators*, is defined by the direct sum

$$\mathrm{gr}\mathcal{D}_{\lambda,\mu} = \bigoplus_{i=0}^{\infty} \mathrm{gr}^{i/2}\mathcal{D}_{\lambda,\mu},$$

where $\mathrm{gr}^k\mathcal{D}_{\lambda,\mu} = \mathcal{D}^k_{\lambda,\mu}/D^{k-1/2}_{\lambda,\mu}$ for every integer or half-integer k. The image of any differential operator through the natural projection

$$\sigma_{pr}\colon\ \mathcal{D}^k_{\lambda,\mu} \to \mathrm{gr}^k\mathcal{D}_{\lambda,\mu}$$

that is defined by the filtration (2.6), has been called the *principal symbol* (see [1], [2], [6] and [5]).

Proposition 2.1. *The action of contact vector field X_f on the space of symbols in the case of integer k ($A = F_1\partial_x^k + F_2\partial_x^{k-1}\overline{D}_1\overline{D}_2 + F_3\partial_x^{k-1}\overline{D}_1\overline{D}_3 + F_4\partial_x^{k-1}\overline{D}_2\overline{D}_3 +$ lower order terms) is given by $\sigma_{pr}(\mathcal{L}^{\lambda,\mu}_{X_f}(A)) = L^{\mu-\lambda-k}_{X_f}(F_1,F_2,F_3,F_4) = (\widetilde{F}_1,\widetilde{F}_2,\widetilde{F}_3,\widetilde{F}_4)$,*

$$\begin{aligned}
\widetilde{F}_1 &= L^{\mu-\lambda-k}_{X_f}(F_1),\\
\widetilde{F}_2 &= L^{\mu-\lambda-k}_{X_f}(F_2) - \tfrac{1}{2}\overline{D}_2\overline{D}_3(f)F_3 + \frac{1}{2}\overline{D}_1\overline{D}_3(f)F_4,\\
\widetilde{F}_3 &= L^{\mu-\lambda-k}_{X_f}(F_3) + \frac{1}{2}\overline{D}_2\overline{D}_3(f)F_2 - \frac{1}{2}\overline{D}_1\overline{D}_2(f)F_4,\\
\widetilde{F}_4 &= L^{\mu-\lambda-k}_{X_f}(F_4) - \frac{1}{2}\overline{D}_1\overline{D}_3(f)F_2 + \frac{1}{2}\overline{D}_1\overline{D}_2(f)F_3,
\end{aligned} \tag{2.9}$$

as for the half integer case ($A = \sum\limits_{i=1}^{3} F_i\partial_x^k\overline{D}_i + F_4\partial_x^{k-1}\overline{D}_1\overline{D}_2\overline{D}_3 +$ lower order terms), the $\mathcal{K}(3)$-action is given by $L^{\mu-\lambda-k-1/2}_{X_f}(F_1,F_2,F_3,F_4) = (\widetilde{F}_1,\widetilde{F}_2,\widetilde{F}_3,\widetilde{F}_4)$,

$$\begin{aligned}
\widetilde{F}_1 &= L^{\mu-\lambda-k-1/2}_{X_f}(F_1) - \frac{1}{2}\overline{D}_1\overline{D}_2(f)F_2 - \frac{1}{2}\overline{D}_1\overline{D}_3(f)F_3,\\
\widetilde{F}_2 &= L^{\mu-\lambda-k-1/2}_{X_f}(F_2) - \frac{1}{2}\overline{D}_2\overline{D}_1(f)F_1 - \frac{1}{2}\overline{D}_2\overline{D}_3(f)F_3,\\
\widetilde{F}_3 &= L^{\mu-\lambda-k-1/2}_{X_f}(F_3) - \frac{1}{2}\overline{D}_3\overline{D}_1(f)F_1 - \frac{1}{2}\overline{D}_3\overline{D}_2(f)F_2,\\
\widetilde{F}_4 &= L^{\mu-\lambda-k-1/2}_{X_f}(F_4).
\end{aligned} \tag{2.10}$$

P r o o f. We calculate $\mathcal{L}^{\lambda,\mu}_{X_f}(A)$ for each case and by using the principal symbole map σ_{pr}, we easily get the $\mathcal{K}(3)$-action of both formulas (2.9) and (2.10). □

Remark 2.1. The space of symbols in the case of k integer (or $k + \frac{1}{2}$) are not isomorphic to the space of weighted densities $\mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k}$ (or $\mathcal{F}_{\delta-k-1/2} \oplus \mathcal{F}_{\delta-k-1/2} \oplus \mathcal{F}_{\delta-k-1/2} \oplus \mathcal{F}_{\delta-k-1/2}$), respectively.

3. Examples of Aff(3|2)-equivariant operators

In this section, we provide examples of affine equivariant differential operators on the space of symbols $\mathcal{S}_{\mu-\lambda}$. These expressions will be used to give a full description of the quantization map.

At first, we consider the case of differential operators of contact order k, where k is an integer.

3.1. The case of k-order Divergence operators. For this case, the k-order Divergence operates on the space of symbols $\mathcal{S}_{\mu-\lambda}$ in the following sense:

$$\mathrm{DIV}^{2n+1}(F) = (-1)^{p(F)}(\mathrm{DIV}_1^{2n+1}(F), \mathrm{DIV}_2^{2n+1}(F), \mathrm{DIV}_3^{2n+1}(F), \mathrm{DIV}_4^{2n+1}(F)) \tag{3.1}$$

and

$$\mathrm{DIV}^{2n}(F) = (\mathrm{DIV}_1^{2n}(F), \mathrm{DIV}_2^{2n}(F), \mathrm{DIV}_3^{2n}(F), \mathrm{DIV}_4^{2n}(F)), \tag{3.2}$$

where $\mathrm{DIV}_i^{2n+1}(F)$ and $\mathrm{DIV}_i^{2n}(F)$ for each $1 \leqslant i \leqslant 4$, are differential operators given by:

$$\begin{aligned}
\mathrm{DIV}_1^{2n+1}(F) &= \mathbf{a}_1^1\overline{D}_1(F_1)^{(n)} + \mathbf{a}_1^2\overline{D}_2(F_2)^{(n)} + \mathbf{a}_1^3\overline{D}_3(F_3)^{(n)} \\
&\quad + \mathbf{a}_1^4\overline{D}_1\overline{D}_2\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_2^{2n+1}(F) &= \mathbf{a}_2^1\overline{D}_2(F_1)^{(n)} + \mathbf{a}_2^2\overline{D}_1(F_2)^{(n)} + \mathbf{a}_2^3\overline{D}_1\overline{D}_2\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{a}_2^4\overline{D}_3(F_4)^{(n)}, \\
\mathrm{DIV}_3^{2n+1}(F) &= \mathbf{a}_3^1\overline{D}_3(F_1)^{(n)} + \mathbf{a}_3^2\overline{D}_1\overline{D}_2\overline{D}_3(F_2)^{(n-1)} + \mathbf{a}_3^3\overline{D}_1(F_3)^{(n)} \\
&\quad + \mathbf{a}_3^4\overline{D}_2(F_4)^{(n)}, \\
\mathrm{DIV}_4^{2n+1}(F) &= \mathbf{a}_4^1\overline{D}_1\overline{D}_2\overline{D}_3(F_1)^{(n-1)} + \mathbf{a}_4^2\overline{D}_3(F_2)^{(n)} + \mathbf{a}_4^3\overline{D}_2(F_3)^{(n)} \\
&\quad + \mathbf{a}_4^4\overline{D}_1(F_4)^{(n)}, \\
\mathrm{DIV}_1^{2n}(F) &= \mathbf{b}_1^1(F_1)^{(n)} + \mathbf{b}_1^2\overline{D}_1\overline{D}_2(F_2)^{(n-1)} + \mathbf{b}_1^3\overline{D}_1\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{b}_1^4\overline{D}_2\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_2^{2n}(F) &= \mathbf{b}_2^1\overline{D}_1\overline{D}_2(F_1)^{(n-1)} + \mathbf{b}_2^2(F_2)^{(n)} + \mathbf{b}_2^3\overline{D}_2\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{b}_2^4\overline{D}_1\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_3^{2n}(F) &= \mathbf{b}_3^1\overline{D}_1\overline{D}_3(F_1)^{(n-1)} + \mathbf{b}_3^2\overline{D}_2\overline{D}_3(F_2)^{(n-1)} + \mathbf{b}_3^3(F_3)^{(n)} \\
&\quad + \mathbf{b}_3^4\overline{D}_1\overline{D}_2(F_4)^{(n-1)}, \\
\mathrm{DIV}_4^{2n}(F) &= \mathbf{b}_4^1\overline{D}_2\overline{D}_3(F_1)^{(n-1)} + \mathbf{b}_4^2\overline{D}_1\overline{D}_3(F_2)^{(n-1)} + \mathbf{b}_4^3\overline{D}_1\overline{D}_2(F_3)^{(n-1)} \\
&\quad + \mathbf{b}_4^4(F_4)^{(n)},
\end{aligned} \tag{3.3}$$

such that $\mathbf{b}_1^1 = 1$,

$$\mathbf{a}_1^1 = \mathbf{a}_2^1 = \mathbf{a}_3^1 = \frac{(n+1)(2\delta - 2k + n + 1)}{(k - n - 1)(2\lambda + k - n - 1)},$$
$$\mathbf{a}_1^2 = \mathbf{a}_1^3 = \mathbf{a}_2^4 = -\mathbf{a}_2^2 = -\mathbf{a}_3^3 = -\mathbf{a}_3^4 = \frac{(n+1)(2\lambda + k)(2\delta - 2k + n + 2)(2\delta - 2k)}{k(k - n - 1)(2\lambda + k - n - 1)(2\delta - 2k + 2)},$$
$$\mathbf{a}_1^4 = -\mathbf{a}_2^3 = \mathbf{a}_3^2 = \frac{-(n+1)n(2\lambda + k)(2\delta - 2k)}{k(k - n - 1)(2\lambda + k - n - 1)(2\delta - 2k + 2)},$$
$$\mathbf{a}_4^1 = \frac{(n+1)n}{(2\lambda + k - n)(2\lambda + k - n - 1)},$$
$$\mathbf{a}_4^2 = -\mathbf{a}_4^3 = \mathbf{a}_4^4 = \frac{(n+1)(2\lambda + k)(2\delta - 2k + n + 1)(2\delta - 2k)}{k(2\lambda + k - n)(2\lambda + k - n - 1)(2\delta - 2k + 2)},$$
$$\mathbf{b}_2^2 = \mathbf{b}_3^3 = \mathbf{b}_4^4 = \frac{(k - n)(2\lambda + k)(2\delta - 2k + n + 2)(2\delta - 2k)}{k(2\lambda + k - n)(2\delta - 2k + n)(2\delta - 2k + 2)},$$
$$\mathbf{b}_1^2 = \mathbf{b}_1^3 = \mathbf{b}_1^4 = \frac{-n(2\lambda + k)(2\delta - 2k)}{k(2\delta - 2k + n)(2\delta - 2k + 2)},$$
$$\mathbf{b}_2^1 = \mathbf{b}_3^1 = \mathbf{b}_4^1 = \frac{n(k - n)}{(2\lambda + k - n)(2\delta - 2k + n)},$$
$$\mathbf{b}_4^2 = \mathbf{b}_4^3 = -\mathbf{b}_2^3 = -\mathbf{b}_2^4 = -\mathbf{b}_3^4 = \mathbf{b}_3^2 = \frac{-n(k - n)(2\lambda + k)(2\delta - 2k)}{k(2\lambda + k - n)(2\delta - 2k + n)(2\delta - 2k + 2)},$$

and

$$\operatorname{div}^{2k-(2n+1)} = (\partial_x^{k-n-1}\overline{D}_1, \partial_x^{k-n-1}\overline{D}_2, \partial_x^{k-n-1}\overline{D}_3, \partial_x^{k-n-2}\overline{D}_1\overline{D}_2\overline{D}_3)^t,$$
$$\operatorname{div}^{2k-(2n)} = (\partial_x^{k-n}, \partial_x^{k-n-1}\overline{D}_1\overline{D}_2, \partial_x^{k-n-1}\overline{D}_1\overline{D}_3, \partial_x^{k-n-1}\overline{D}_2\overline{D}_3)^t.$$

Lemma 3.1. *The Divergence operators* (3.1) *and* (3.2) *commute with the* Aff(3|2)*-action.*

P r o o f. This is a direct consequence of projectively equivariant symbol calculus. We take $\mathrm{DIV}^{2n}(F)$ and $\mathrm{DIV}^{2n+1}(F)$ as they are written above, where $\mathbf{a}_p^l$ and $\mathbf{b}_p^l$ ($1 \leqslant l \leqslant 4$, $1 \leqslant p \leqslant 4$) are arbitrary constants. From the commutation relation $[X_f, \mathrm{DIV}]$ for $f \in \mathrm{Aff}(3|2)$ we easily get the Aff(3|2)-equivariance of Divergence operators. □

3.2. The case of $k + \frac{1}{2}$-order Divergence operators. In this case, we also define the Divergence as Affine equivariant differential operators on the space of symbols $\mathcal{S}_{\mu-\lambda}$. In each component $\mathcal{S}_{\mu-\lambda}^{k+1/2}$ we have

$$\mathrm{DIV}^{2n+1}(F) = (-1)^{p(F)}(\mathrm{DIV}_1^{2n+1}(F), \mathrm{DIV}_2^{2n+1}(F), \mathrm{DIV}_3^{2n+1}(F), \mathrm{DIV}_4^{2n+1}(F)), \tag{3.4}$$
$$\mathrm{DIV}^{2n}(F) = (\mathrm{DIV}_1^{2n}(F), \mathrm{DIV}_2^{2n}(F), \mathrm{DIV}_3^{2n}(F), \mathrm{DIV}_4^{2n}(F)), \tag{3.5}$$

where $\mathrm{DIV}_i^{2n+1}(F)$ and $\mathrm{DIV}_i^{2n}(F)$, for each $1 \leqslant i \leqslant 4$, are differential operators given by:

$$
\begin{aligned}
(3.6)\quad \mathrm{DIV}_1^{2n+1}(F) &= \mathbf{c}_1^1\overline{D}_1(F_1)^{(n)} + \mathbf{c}_1^2\overline{D}_2(F_2)^{(n)} + \mathbf{c}_1^3\overline{D}_3(F_3)^{(n)} \\
&\quad + \mathbf{c}_1^4\overline{D}_1\overline{D}_2\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_2^{2n+1}(F) &= \mathbf{c}_2^1\overline{D}_2(F_1)^{(n)} + \mathbf{c}_2^2\overline{D}_1(F_2)^{(n)} + \mathbf{c}_2^3\overline{D}_1\overline{D}_2\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{c}_2^4\overline{D}_3(F_4)^{(n)}, \\
\mathrm{DIV}_3^{2n+1}(F) &= \mathbf{c}_3^1\overline{D}_3(F_1)^{(n)} + \mathbf{c}_3^2\overline{D}_1\overline{D}_2\overline{D}_3(F_2)^{(n-1)} + \mathbf{c}_3^3\overline{D}_1(F_3)^{(n)} \\
&\quad + \mathbf{c}_3^4\overline{D}_2(F_4)^{(n)}, \\
\mathrm{DIV}_4^{2n+1}(F) &= \mathbf{c}_4^1\overline{D}_1\overline{D}_2\overline{D}_3(F_1)^{(n-1)} + \mathbf{c}_4^2\overline{D}_3(F_2)^{(n)} + \mathbf{c}_4^3\overline{D}_2(F_3)^{(n)} \\
&\quad + \mathbf{c}_4^4\overline{D}_1(F_4)^{(n)}, \\
\mathrm{DIV}_1^{2n}(F) &= \mathbf{d}_1^1(F_1)^{(n)} + \mathbf{d}_1^2\overline{D}_1\overline{D}_2(F_2)^{(n-1)} + \mathbf{d}_1^3\overline{D}_1\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{d}_1^4\overline{D}_2\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_2^{2n}(F) &= \mathbf{d}_2^1\overline{D}_1\overline{D}_2(F_1)^{(n-1)} + \mathbf{d}_2^2(F_2)^{(n)} + \mathbf{d}_2^3\overline{D}_2\overline{D}_3(F_3)^{(n-1)} \\
&\quad + \mathbf{d}_2^4\overline{D}_1\overline{D}_3(F_4)^{(n-1)}, \\
\mathrm{DIV}_3^{2n}(F) &= \mathbf{d}_3^1\overline{D}_1\overline{D}_3(F_1)^{(n-1)} + \mathbf{d}_3^2\overline{D}_2\overline{D}_3(F_2)^{(n-1)} + \mathbf{d}_3^3(F_3)^{(n)} \\
&\quad + \mathbf{d}_3^4\overline{D}_1\overline{D}_2(F_4)^{(n-1)}, \\
\mathrm{DIV}_4^{2n}(F) &= \mathbf{d}_4^1\overline{D}_2\overline{D}_3(F_1)^{(n-1)} + \mathbf{d}_4^2\overline{D}_1\overline{D}_3(F_2)^{(n-1)} + \mathbf{d}_4^3\overline{D}_1\overline{D}_2(F_3)^{(n-1)} \\
&\quad + \mathbf{d}_4^4(F_4)^{(n)},
\end{aligned}
$$

such that

$$
\begin{aligned}
\mathbf{c}_1^1 = \mathbf{c}_1^2 = \mathbf{c}_1^3 &= \frac{-(2\lambda+k-n)(2\delta-2k-1)}{(2\delta-2k+n-1)(2\delta-2k+1)}, \\
\mathbf{c}_2^1 = \mathbf{c}_3^1 = \mathbf{c}_4^2 = -\mathbf{c}_2^2 = -\mathbf{c}_3^3 = -\mathbf{c}_4^3 &= \frac{-(k-n)(2\delta-2k+n+1)(2\delta-2k-1)}{(2\delta-2k+n)(2\delta-2k+n+1)(2\delta-2k+1)}, \\
\mathbf{c}_4^1 = \mathbf{c}_3^2 = \mathbf{c}_2^3 &= \frac{-(k-n)n(2\delta-2k-1)}{(2\delta-2k+n)(2\delta-2k+n-1)(2\delta-2k+1)}, \\
\mathbf{c}_2^4 = -\mathbf{c}_3^4 = \mathbf{c}_4^4 &= \frac{-(k-n)(2\lambda+k+1)}{k(2\delta-2k+n-1)}, \\
\mathbf{c}_1^4 &= \frac{n(2\lambda+k+1)(2\lambda+k-n)}{k(2\delta-2k+n)(2\delta-2k+n-1)}, \\
\mathbf{d}_1^1 = \mathbf{d}_2^2 = \mathbf{d}_3^3 &= \frac{-(2\delta-2k+n+1)(2\delta-2k-1)}{(2\lambda+k-n)(2\delta-2k+1)}, \\
\mathbf{d}_2^1 - \mathbf{d}_1^2 = -\mathbf{d}_1^3 = -\mathbf{d}_2^3 = \mathbf{d}_3^1 = \mathbf{d}_3^2 &= \frac{-n(2\delta-2k-1)}{(2\lambda+k-n)(2\delta-2k+1)}, \\
\mathbf{d}_4^4 &= \frac{-(k-n)(2\lambda+k+1)(2\delta-2k+n-1)}{k(2\lambda+k-n+1)(2\lambda+k-n)},
\end{aligned}
$$

$$\mathbf{d}_4^1 = \mathbf{d}_4^2 = \mathbf{d}_4^3 = \frac{-n(k-n)(2\delta-2k-1)}{(2\lambda+k-n)(2\lambda+k-n+1)(2\delta-2k+1)},$$
$$\mathbf{d}_1^4 = -\mathbf{d}_2^4 = \mathbf{d}_3^4 = \frac{n(2\lambda+k+1)}{k(2\lambda+k-n)},$$

and

$$\operatorname{div}^{2k+1-(2n+1)} = (\partial_x^{k-n}, \partial_x^{k-n-1}\overline{D}_1\overline{D}_2, \partial_x^{k-n-1}\overline{D}_1\overline{D}_3, \partial_x^{k-n-1}\overline{D}_2\overline{D}_3)^t,$$
$$\operatorname{div}^{2k+1-(2n)} = (\partial_x^{k-n}\overline{D}_1, \partial_x^{k-n}\overline{D}_2, \partial_x^{k-n}\overline{D}_3, \partial_x^{k-n-1}\overline{D}_1\overline{D}_2\overline{D}_3)^t.$$

Lemma 3.2. *The Divergence operators* (3.4) *and* (3.5) *commute with the* $\mathrm{Aff}(3|2)$-*action.*

P r o o f. Straightforward calculus. □

4. Projectively equivariant quantization on $S^{1|3}$

A map $Q\colon \mathcal{S}_{\mu-\lambda} \to \mathcal{D}_{\lambda,\mu}$, is called *quantization map* if it is linear bijection and preserves the principal symbol of every differential operator. The main result of this paper is the existence and uniqueness of an $\mathfrak{osp}(3|2)$-equivariant quantization map in dimension $1|3$. We calculate its explicit formula.

4.1. Statement of the main result. Let us give the explicit formula of the projectively equivariant quantization map. We will give the proof in the next section.

Theorem 4.1. *The unique* $\mathfrak{osp}(3|2)$-*equivariant quantization map associates the following differential operator with a symbol* $F = (F_1, F_2, F_3, F_4) \in \mathcal{S}^k_{\mu-\lambda}$, *where* k *is (even or odd) integer:*

(4.1)
$$Q(F) = \sum_{n=0}^{k} \binom{[\frac{1}{2}k]}{[\frac{1}{4}(2n+1+(-1)^k)]}\binom{2\lambda+[\frac{1}{2}(k-1)]}{[\frac{1}{4}(2n+1-(-1)^k)]}\Xi^{-1}\mathrm{DIV}^n(F)\operatorname{div}^{k-n}$$

provided $\delta = \mu - \lambda \neq 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, *where*

$$\Xi = \binom{k-2(\mu-\lambda)}{[\frac{1}{2}(n+1)]};$$

DIV *and* div *are defined in each particular case of even or odd contact order and the binomial coefficients are defined by*

$$\binom{\nu}{q} = \frac{\nu(\nu-1)\ldots(\nu-q+1)}{q!}.$$

This expression makes sense for arbitrary $\nu \in \mathbb{C}$.

4.2. Proof of the theorem in the case of k-differential operators.

Proof. Let us first consider the case of k-differential operators, where k is an integer. The quantization map (4.1) is indeed $\mathfrak{osp}(3|2)$-equivariant. Now, we consider a differentiable linear map $Q\colon \mathcal{S}^k_{\mu-\lambda} \to \mathcal{D}^k_{\lambda,\mu}$ for $k \geqslant 1$, preserving the principal symbol. Such a map is of the form

$$\begin{aligned} Q(F) = {} & F_1\partial_x^k + F_2\partial_x^{k-1}\overline{D}_1\overline{D}_2 + F_3\partial_x^{k-1}\overline{D}_1\overline{D}_3 + F_4\partial_x^{k-1}\overline{D}_2\overline{D}_3 + \ldots \\ & + \widetilde{Q}_1^{(m)}(F_1) + \widetilde{Q}_2^{(m)}(F_2) + \widetilde{Q}_3^{(m)}(F_3) + Q_4^{(m)}(F_4) + \ldots \\ & + (a\partial_x^k(F_1) + b\partial_x^{k-1}\overline{D}_1\overline{D}_2(F_2) + c\partial_x^{k-1}\overline{D}_1\overline{D}_3(F_3) + d\partial_x^{k-1}\overline{D}_2\overline{D}_3(F_4)), \end{aligned}$$

where $\widetilde{Q}_1^{(m)}$, $\widetilde{Q}_2^{(m)}$, $\widetilde{Q}_3^{(m)}$ and $\widetilde{Q}_4^{(m)}$ are differential operators with coefficients in $\mathcal{F}_{\mu-\lambda}$, see (4.2). We obtain the following:

This map commutes with the action of the vector fields in $\mathrm{Aff}(3|2)$, if and only if the differential operators $\widetilde{Q}_1^{(m)}$, $\widetilde{Q}_2^{(m)}$, $\widetilde{Q}_3^{(m)}$ and $\widetilde{Q}_4^{(m)}$ are with constant coefficients.

$$\begin{aligned} \widetilde{Q}_1^{(2i+1)}(F_1) &= \tau_1^{i,1,0,0}\overline{D}_1(F_1)^{(k-i-1)}\partial_x^i\overline{D}_1 + \tau_1^{i,0,1,0}\overline{D}_2(F_1)^{(k-i-1)}\partial_x^i\overline{D}_2 \\ &\quad + \tau_1^{i,0,0,1}\overline{D}_3(F_1)^{(k-i-1)}\partial_x^i\overline{D}_3 \\ &\quad + \tau_1^{i,1,1,1}\overline{D}_1\overline{D}_2\overline{D}_3(F_1)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\ \widetilde{Q}_1^{(2i)}(F_1) &= \tau_1^{i,0,0,0}(F_1)^{(k-i)}\partial_x^i + \tau_1^{i,1,1,0}\overline{D}_1\overline{D}_2(F_1)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2 \\ &\quad + \tau_1^{i,1,0,1}\overline{D}_1\overline{D}_3(F_1)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_3 \\ &\quad + \tau_1^{i,0,1,1}\overline{D}_2\overline{D}_3(F_1)^{(k-i-2)}\partial_x^i\overline{D}_2\overline{D}_3, \\ \widetilde{Q}_2^{(2i+1)}(F_2) &= \tau_2^{i,1,0,0}\overline{D}_2(F_2)^{(k-i-1)}\partial_x^i\overline{D}_1 + \tau_2^{i,0,1,0}\overline{D}_1(F_2)^{(k-i-1)}\partial_x^i\overline{D}_2 \\ &\quad + \tau_2^{i,0,0,1}\overline{D}_1\bar{D}_2\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_3 \\ &\quad + \tau_2^{i,1,1,1}\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\ \widetilde{Q}_2^{(2i)}(F_2) &= \tau_2^{i,0,0,0}\overline{D}_1\overline{D}_2(F_2)^{(k-i-1)}\partial_x^i + \tau_2^{i,1,1,0}(F_2)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2 \\ &\quad + \tau_2^{i,1,0,1}\overline{D}_2\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_3 \\ &\quad + \tau_2^{i,0,1,1}\overline{D}_1\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_2\overline{D}_3, \\ \widetilde{Q}_3^{(2i+1)}(F_3) &= \tau_3^{i,1,0,0}\overline{D}_3(F_3)^{(k-i-1)}\partial_x^i\overline{D}_1 + \tau_3^{i,0,1,0}\overline{D}_1\overline{D}_2\overline{D}_3(F_3)^{(k-i-1)}\partial_x^i\overline{D}_2 \\ &\quad + \tau_3^{i,0,0,1}\overline{D}_1(F_3)^{(k-i-1)}\partial_x^i\overline{D}_3 \\ &\quad + \tau_3^{i,1,1,1}\overline{D}_2(F_3)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\ \widetilde{Q}_3^{(2i)}(F_3) &= \tau_1^{i,0,0,0}\overline{D}_1\overline{D}_3(F_3)^{(k-i-1)}\partial_x^i + \tau_3^{i,1,1,0}\overline{D}_2\overline{D}_3(F_3)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2 \\ &\quad + \tau_3^{i,1,0,1}(F_3)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_3 \\ &\quad + \tau_3^{i,0,1,1}\overline{D}_1\overline{D}_2(F_3)^{(k-i-2)}\partial_x^i\overline{D}_2\overline{D}_3, \end{aligned} \tag{4.2}$$

$$\begin{aligned}
\widetilde{Q}_4^{(2i+1)}(F_4) &= \tau_4^{i,1,0,0}\overline{D}_1\overline{D}_2\overline{D}_3(F_4)^{(k-i-2)}\partial_x^i\overline{D}_1 + \tau_4^{i,0,1,0}\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i\overline{D}_2 \\
&\quad + \tau_4^{i,0,0,1}\overline{D}_2(F_4)^{(k-i-1)}\partial_x^i\overline{D}_3 \\
&\quad + \tau_4^{i,1,1,1}\overline{D}_1(F_4)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_4^{(2i)}(F_4) &= \tau_4^{i,0,0,0}\overline{D}_2\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i + \tau_4^{i,1,1,0}\overline{D}_1\overline{D}_3(F_4)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2 \\
&\quad + \tau_4^{i,1,0,1}\overline{D}_1\overline{D}_2(F_4)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_3 \\
&\quad + \tau_4^{i,0,1,1}(F_4)^{(k-i-1)}\partial_x^i\overline{D}_2\overline{D}_3,
\end{aligned}$$

where the coefficients τ_j^{i,r_1,r_2,r_3} are arbitrary constants for each i, r_1, r_2, r_3, and j. The above quantization map commutes with the action of X_{x^2} and $X_{\theta_i x}$ if and only if any of the coefficients τ_j^{i,r_1,r_2,r_3} verify the following conditions:

$$\begin{aligned}
&\tau_2^{i,0,0,0} = \tau_3^{i,0,0,0} = \tau_4^{i,0,0,0}, \\
&\tau_1^{i,1,0,0} = \tau_1^{i,0,1,0} = \tau_1^{i,0,0,1}, \\
&\tau_2^{i,0,0,1} = -\tau_3^{i,0,1,0} = \tau_4^{i,1,0,0}, \\
&\tau_2^{i,1,1,0} = \tau_3^{i,1,0,1} = \tau_4^{i,0,1,1}, \\
&\tau_2^{i,1,1,1} = -\tau_3^{i,1,1,1} = \tau_4^{i,1,1,1}, \\
&\tau_1^{i,1,1,0} = \tau_1^{i,1,0,1} = \tau_1^{i,0,1,1}, \\
&\tau_2^{i,1,0,0} = \tau_3^{i,1,0,0} = \tau_4^{i,0,1,0} = -\tau_2^{i,0,1,0} = -\tau_3^{i,0,0,1} = -\tau_4^{i,0,0,1}, \\
&\tau_4^{i,1,1,0} = \tau_3^{i,0,1,1} = \tau_2^{i,1,0,1} = -\tau_2^{i,0,1,1} = -\tau_3^{i,1,1,0} = -\tau_4^{i,1,0,1},
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
(i+1)(2\lambda+i+3)\tau_2^{i+1,1,1,1} + (k-i-2)(2\delta-k-i-2)\tau_2^{i,1,1,1} &= 0, \\
(-1)^{p(F)}(i+1)\tau_2^{i+1,1,1,0} - (2\delta-k-i)\tau_2^{i,1,1,1} &= 0, \\
(-1)^{p(F)}(2\lambda+i+2)\tau_2^{i,1,1,1} + (2\delta-k-i-1)\tau_2^{i,1,0,1} &= 0, \\
(-1)^{p(F)}(2\lambda+i+1)\tau_2^{i,1,0,1} + (2\delta-k-i)\tau_2^{i,0,0,1} &= 0, \\
(-1)^{p(F)}(2\lambda+i+1)\tau_2^{i,1,1,0} - (2\delta-k-i)\tau_2^{i,1,0,0} &= 0, \\
(-1)^{p(F)}(2\lambda+i)\tau_2^{i,1,0,0} - (2\delta-k-i+1)\tau_2^{i,0,0,0} &= 0, \\
(i+1)(2\lambda+i)\tau_1^{i+1,0,0,0} + (k-i)(2\delta-k-i-1)\tau_1^{i,0,0,0} &= 0, \\
(-1)^{p(F)}(i+1)\tau_1^{i+1,0,0,0} - (2\delta-k-i-1)\tau_1^{i,1,0,0} &= 0, \\
(-1)^{p(F)}(i+1)\tau_1^{i+1,1,0,0} + (2\delta-k-i-1)\tau_1^{i,1,1,0} &= 0, \\
(-1)^{p(F)}(i+1)\tau_1^{i+1,1,1,0} - (2\delta-k-i-1)\tau_1^{i,1,1,1} &= 0.
\end{aligned}$$

If $\delta = \mu - \lambda \neq 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, this system has been solved and the solutions are the following:

(4.4)

$$\tau_1^{i,0,0,0} = \binom{k}{k-i}\binom{2\lambda+k-1}{k-i}\Upsilon^{-1}, \text{ where } \Upsilon = \binom{2k-2(\mu-\lambda)}{k-i},$$
$$\tau_2^{i,1,1,1} = \binom{k-2}{k-i-2}\binom{2\lambda+k}{k-i-2}\Theta^{-1}\tau_2^{k-2,1,1,1}, \text{ where } \Theta = \binom{2k-1-2(\mu-\lambda)}{k-i-2},$$
$$\tau_2^{k-2,1,1,1} = (-1)^{p(F)}\frac{(k-1)}{2(\mu-\lambda)-2k+2}\tau_2^{k-1,1,1,0},$$
$$\tau_1^{k,0,0,0} = \tau_2^{k-1,1,1,0} = \tau_3^{k-1,1,0,1} = \tau_4^{k-1,0,1,1} = 1,$$

and

$$(4.5)\quad \tau_2^{i,1,1,0} = (-1)^{p(F)}\frac{2\delta-k-i+1}{i}\tau_2^{i-1,1,1,1},$$
$$\tau_2^{i,1,0,1} = (-1)^{p(F)+1}\frac{2\lambda+i+2}{2\delta-k-i-1}\tau_2^{i,1,1,1},$$
$$\tau_2^{i,0,0,1} = \frac{(2\lambda+i+1)(2\lambda+i+2)}{(2\delta-k-i)(2\delta-k-i-1)}\tau_2^{i,1,1,1},$$
$$\tau_2^{i,1,0,0} = -\frac{(2\lambda+i+1)(2\delta-k-i+1)}{i(2\delta-k-i)}\tau_2^{i-1,1,1,1},$$
$$\tau_2^{i,0,0,0} = (-1)^{p(F)+1}\frac{(2\lambda+i)(2\lambda+i+1)}{i(2\delta-k-i)}\tau_2^{i-1,1,1,1},$$
$$\tau_1^{i,1,0,0} = (-1)^{p(F)}\frac{i+1}{2\delta-k-i-1}\tau_1^{i+1,0,0,0},$$
$$\tau_1^{i,1,1,0} = -\frac{(i+1)(i+2)}{(2\delta-k-i-1)(2\delta-k-i-2)}\tau_1^{i+2,0,0,0},$$
$$\tau_1^{i,1,1,1} = (-1)^{p(F)+1}\frac{(i+1)(i+2)(i+3)}{(2\delta-k-i-1)(2\delta-k-i-2)(2\delta-k-i-3)}\tau_1^{i+3,0,0,0}.$$

That allows us to obtain formula (4.1). □

4.3. Proof of the theorem in the case of $(k+\frac{1}{2})$-differential operators.

P r o o f. In the case of $(k+\frac{1}{2})$-differential operators, where k is an integer, we get an Aff(3|2)-equivariant quantization map by a straightforward calculation which is given by

$$\begin{aligned} Q(F) = {} & F_1\partial_x^k\overline{D}_1 + F_2\partial_x^k\overline{D}_2 + F_3\partial_x^k\overline{D}_3 + F_4\partial_x^{k-1}\overline{D}_1\overline{D}_2\overline{D}_3 + \ldots \\ & + \widetilde{Q}_1^{(m)}(F_1) + \widetilde{Q}_2^{(m)}(F_2) + \widetilde{Q}_3^{(m)}(F_3) + \widetilde{Q}_4^{(m)}(F_4) + \ldots \\ & (a\partial_x^k\overline{D}_1(F_1) + b\partial_x^k\overline{D}_2(F_2) + c\partial_x^k\overline{D}_3(F_3) + d\partial_x^{k-1}\overline{D}_1\overline{D}_2\overline{D}_3(F_4)), \end{aligned}$$

where the differential operators $\widetilde{Q}_1^{(m)}$,$\widetilde{Q}_2^{(m)}$, $\widetilde{Q}_3^{(m)}$ and $\widetilde{Q}_4^{(m)}$ have the form:

$$
\begin{aligned}
(4.6)\quad \widetilde{Q}_1^{(2i+1)}(F_1) &= \tau_1^{i,1,0,0}(F_1)^{(k-i)}\partial_x^i\overline{D}_1 + \tau_1^{i,0,1,0}\overline{D}_1\overline{D}_2(F_1)^{(k-i-1)}\partial_x^i\overline{D}_2 \\
&\quad + \tau_1^{i,0,0,1}\overline{D}_1\overline{D}_3(F_1)^{(k-i-1)}\partial_x^i\overline{D}_3 \\
&\quad + \tau_1^{i,1,1,1}\overline{D}_2\overline{D}_3(F_1)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_1^{(2i)}(F_1) &= \tau_1^{i,0,0,0}\overline{D}_1(F_1)^{(k-i)}\partial_x^i + \tau_1^{i,1,1,0}\overline{D}_2(F_1)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_2 \\
&\quad + \tau_1^{i,1,0,1}\overline{D}_3(F_1)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_3 \\
&\quad + \tau_1^{i,0,1,1}\overline{D}_1\overline{D}_2\overline{D}_3(F_1)^{(k-i-2)}\partial_x^i\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_2^{(2i+1)}(F_2) &= \tau_2^{i,1,0,0}\overline{D}_1\overline{D}_2(F_2)^{(k-i-1)}\partial_x^i\overline{D}_1 + \tau_2^{i,0,1,0}(F_2)^{(k-i-1)}\partial_x^i\overline{D}_2 \\
&\quad + \tau_2^{i,0,0,1}\overline{D}_2\overline{D}_3(F_2)^{(k-i-1)}\partial_x^i\overline{D}_3 \\
&\quad + \tau_2^{i,1,1,1}\overline{D}_1\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_2^{(2i)}(F_2) &= \tau_2^{i,0,0,0}\overline{D}_2(F_2)^{(k-i)}\partial_x^i + \tau_2^{i,1,1,0}\overline{D}_1(F_2)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_2 \\
&\quad + \tau_2^{i,1,0,1}\overline{D}_1\overline{D}_2\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_3 \\
&\quad + \tau_2^{i,0,1,1}\overline{D}_3(F_2)^{(k-i-2)}\partial_x^i\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_3^{(2i+1)}(F_3) &= \tau_3^{i,1,0,0}\overline{D}_1\overline{D}_3(F_3)^{(k-i-1)}\partial_x^i\overline{D}_1 + \tau_3^{i,0,1,0}\overline{D}_2\overline{D}_3(F_3)^{(k-i-1)}\partial_x^i\overline{D}_2 \\
&\quad + \tau_3^{i,0,0,1}(F_3)^{(k-i)}\partial_x^i\overline{D}_3 \\
&\quad + \tau_3^{i,1,1,1}\overline{D}_1\overline{D}_2(F_3)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_3^{(2i)}(F_3) &= \tau_1^{i,0,0,0}\overline{D}_3(F_3)^{(k-i)}\partial_x^i + \tau_3^{i,1,1,0}\overline{D}_1\overline{D}_2\overline{D}_3(F_3)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2 \\
&\quad + \tau_3^{i,1,0,1}\overline{D}_1(F_3)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_3 \\
&\quad + \tau_3^{i,0,1,1}\overline{D}_2(F_3)^{(k-i-1)}\partial_x^i\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_4^{(2i+1)}(F_4) &= \tau_4^{i,1,0,0}\overline{D}_2\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i\bar{D}_1 + \tau_4^{i,0,1,0}\overline{D}_1\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i\overline{D}_2 \\
&\quad + \tau_4^{i,0,0,1}\overline{D}_1\bar{D}_2(F_4)^{(k-i-1)}\partial_x^i\overline{D}_3 \\
&\quad + \tau_4^{i,1,1,1}(F_4)^{(k-i-2)}\partial_x^i\overline{D}_1\overline{D}_2\overline{D}_3, \\
\widetilde{Q}_4^{(2i)}(F_4) &= \tau_4^{i,0,0,0}\overline{D}_1\overline{D}_2\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i + \tau_4^{i,1,1,0}\overline{D}_3(F_4)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_2 \\
&\quad + \tau_4^{i,1,0,1}\overline{D}_2(F_4)^{(k-i-1)}\partial_x^i\overline{D}_1\overline{D}_3 + \tau_4^{i,0,1,1}\overline{D}_1(F_4)^{(k-i-1)}\partial_x^i\overline{D}_2\overline{D}_3.
\end{aligned}
$$

The above quantization map commutes with the action of X_{x^2} if and only if the coefficients τ_j^{i,r_1,r_2,r_3} $(j = 1,2,3,4)$ verify the following system of linear equations:

$$
\begin{aligned}
(4.7)\quad & \tau_1^{i,0,0,0} = \tau_2^{i,0,0,0} = \tau_3^{i,0,0,0}, \qquad && \tau_4^{i,1,0,0} = -\tau_4^{i,0,1,0} = \tau_4^{i,0,0,1}, \\
& \tau_1^{i,1,0,0} = \tau_2^{i,0,1,0} = \tau_3^{i,0,0,1}, && \tau_4^{i,1,1,0} = -\tau_4^{i,1,0,1} = \tau_4^{i,0,1,1}, \\
& \tau_1^{i,0,1,1} = -\tau_2^{i,1,0,1} = \tau_3^{i,1,1,0}, && \tau_1^{i,1,1,1} = -\tau_2^{i,1,1,1} = \tau_3^{i,1,1,1} \\
& \tau_2^{i,1,0,0} = \tau_3^{i,1,0,0} = \tau_3^{i,0,1,0} = -\tau_1^{i,0,1,0} = -\tau_1^{i,0,0,1} = -\tau_2^{i,0,0,1}, \\
& \tau_2^{i,0,1,1} = \tau_1^{i,1,1,0} = \tau_1^{i,1,0,1} = -\tau_3^{i,1,0,1} = -\tau_3^{i,0,1,1} = -\tau_2^{i,1,1,0},
\end{aligned}
$$

and

$$\begin{aligned}
(i+1)(2\lambda+i)\tau_1^{i+1,0,0,0}+(k-i)(2\delta-k-i-1)\tau_1^{i,0,0,0}&=0,\\
(-1)^{p(F)}(2\lambda+i)\tau_1^{i,1,0,0}+(2\delta-k-i+1)\tau_1^{i,0,0,0}&=0,\\
(-1)^{p(F)}(i+1)\tau_1^{i+1,1,1,0}-(2\delta-k-i-1)\tau_1^{i,1,1,1}&=0,\\
(-1)^{p(F)}(i+1)\tau_1^{i+1,0,0,0}-(2\delta-k-i-1)\tau_1^{i,0,1,0}&=0,\\
(-1)^{p(F)}(i+1)\tau_1^{i+1,1,0,0}+(2\delta-k-i-1)\tau_1^{i,1,1,0}&=0,\\
(-1)^{p(F)}(i+1)\tau_1^{i+1,0,1,0}+(2\delta-k-i-1)\tau_1^{i,0,1,1}&=0,\\
(-1)^{p(F)}(2\lambda+i)\tau_4^{i,1,0,0}+(2\delta-k-i)\tau_4^{i,0,0,0}&=0,\\
(i+1)(2\lambda+i+3)\tau_4^{i+1,1,1,1}+(k-i-1)(2\delta-k-i-3)\tau_4^{i,1,1,1}&=0,\\
(-1)^{p(F)}(2\lambda+i+1)\tau_4^{i,1,1,0}+(2\delta-k-i-1)\tau_4^{i,1,0,0}&=0,\\
(-1)^{p(F)}(2\lambda+i)\tau_4^{i,1,0,0}+(2\delta-k-i)\tau_4^{i,0,0,0}&=0.
\end{aligned}$$

By solving this system, we obtain formula (4.1). □

References

[1] *C. Duval, P. Lecomte, V. Ovsienko*: Conformally equivariant quantization: Existence and uniqueness. Ann. Inst. Fourier *49* (1999), 1999–2029. zbl MR doi

[2] *P. Grozman, D. Leites, I. Shchepochkina*: Invariant operators on supermanifolds and standard models. Multiple Facets of Quantization and Supersymmetry. World Scientific, River Edge, 2002, pp. 508–555. zbl MR doi

[3] *P. B. A. Lecomte*: Classification projective des espaces d'opérateurs différentiels agissant sur les densités. C. R. Acad. Sci. Paris., Sér. I, Math. *328* (1999), 287–290. (In French.) zbl MR doi

[4] *P. B. A. Lecomte*: Towards projectively equivariant quantization. Prog. Theor. Phys., Suppl. *144* (2001), 125–132. zbl MR doi

[5] *P. B. A. Lecomte, V. Y. Ovsienko*: Projectively equivariant symbol calculus. Lett. Math. Phys. *49* (1999), 173–196. zbl MR doi

[6] *D. A. Leites, Y. Kochetkov, A. Weintrob*: New invariant differential operators on supermanifolds and pseudo-(co)homology. General Topology and Applications. Lecture Notes in Pure and Applied Mathematics 134. Marcel Dekker, New York, 1991, pp. 217–238. zbl MR

[7] *T. Leuther, P. Mathonet, F. Radoux*: One $osp(p+1,q+1|2r)$-equivariant quantizations. J. Geom. Phys. *62* (2012), 87–99. zbl MR doi

[8] *P. Mathonet, F. Radoux*: Projectively equivariant quantizations over the superspace $R^{p|q}$. Lett. Math. Phys. *98* (2011), 311–331. zbl MR doi

[9] *N. Mellouli*: Projectively equivariant quantization and symbol calculus in dimension 1|2. Available at `https://arxiv.org/abs/1106.5246v1` (2011), 9 pages.

[10] *V. Y. Ovsienko, O. D. Ovsienko, Y. V. Chekanov*: Classification of contact-projective structures on the supercircles. Russ. Math. Surv. *44* (1989), 212–213. zbl MR doi

[11] *I. Shchepochkina*: How to realize a Lie algebra by vector fields. Theor. Math. Phys. *147* (2006), 821–838. zbl MR doi

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