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FINITE GROUPS IN WHICH EVERY SELF-CENTRALIZING SUBGROUP IS NILPOTENT OR SUBNORMAL OR A TI-SUBGROUP

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Abstract. Let G be a finite group. We prove that if every self-centralizing subgroup of G is nilpotent or subnormal or a TI-subgroup, then every subgroup of G is nilpotent or subnormal. Moreover, G has either a normal Sylow p-subgroup or a normal p-complement for each prime divisor p of $|G|$.

Keywords: self-centralizing; nilpotent; TI-subgroup; subnormal; p-complement

MSC 2020: 20D10

1. Introduction

In this paper all groups are considered to be finite. Let G be a group and N a subgroup of G. If $N^g \cap N = 1$ or N for each $g \in G$, then N is said to be a TI-subgroup of G. It is obvious that any normal subgroup of a group is a TI-subgroup but a TI-subgroup might not be a normal subgroup. The concept of subnormal subgroup is a natural generalization of the concept of normal subgroup. In [3] Shi and Zhang produced examples showing that a TI-subgroup might not be a subnormal subgroup and a subnormal subgroup might also not be a TI-subgroup, and they obtained a complete classification of groups in which every subgroup is subnormal or a TI-subgroup. As a generalization of [3], Shi in [2] proved that if every subgroup of a group G is abelian or subnormal or a TI-subgroup, then every subgroup of G is abelian or subnormal, and for every prime p dividing $|G|$, G must have either a normal Sylow p-subgroup or else a Sylow p-subgroup is abelian and there exists a normal p-complement.

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Recall that a subgroup H of a group G is said to be self-centralizing if $C_G(H) \leqslant H$, where $C_G(H)$ is the centralizer of H in G. It is clear that any self-normalizing subgroup of a group G is a self-centralizing subgroup of G. Moreover, if H is a self-centralizing subgroup of a group G, then for any subgroup K of G satisfying $K > H$ one has that K is also a self-centralizing subgroup of G. Sun, Lu and Meng in [5] showed that if every self-centralizing subgroup of a group G is abelian or subnormal or a TI-subgroup, then every subgroup of G is abelian or subnormal, which extended the research in [2].

Note that any abelian subgroup of a group must be nilpotent but a nilpotent subgroup might not be abelian. Motivated by the research presented above, we will give a complete characterization of the groups in which every self-centralizing subgroup is nilpotent or subnormal or a TI-subgroup. Our result is as follows.

Theorem 1. *Suppose that every self-centralizing subgroup of a group G is nilpotent or subnormal or a TI-subgroup. Then every subgroup of G is nilpotent or subnormal.*

Moreover, we obtain that the groups in Theorem 1 have the following structure.

Theorem 2. *Let G be a group in which every self-centralizing subgroup is nilpotent or subnormal or a TI-subgroup. Then G is solvable and for each prime divisor p of $|G|$ we have that G has either a normal Sylow p-subgroup or a normal p-complement.*

In [4], Theorem 1.1 we proved that if every subgroup of a group G is nilpotent or a TI-subgroup, then every subgroup of G is nilpotent or normal. As an extension, we have the following result.

Theorem 3. *Suppose that every self-centralizing subgroup of a group G is nilpotent or a TI-subgroup. Then every self-centralizing subgroup of G is nilpotent or normal.*

Remark 4. In Theorem 3, although we have that every subgroup of G is nilpotent or subnormal by Theorem 1, we cannot get that every subgroup of G is nilpotent or normal. For example, let $G = D_{24} = \langle a^{12} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be a dihedral group of order 24. It is clear that G only has the following three subgroups which are not nilpotent: $\langle a^4, b \rangle$, $\langle a^2, b \rangle$, G itself. Observe that both $\langle a^2, b \rangle$ and G are self-centralizing, and $\langle a^4, b \rangle$ is not self-centralizing. Moreover, both $\langle a^2, b \rangle$ and G are normal in G and both obviously are TI-subgroups of G, and $\langle a^4, b \rangle$ is subnormal in G. Then G satisfies the hypothesis of Theorem 3. However, $\langle a^4, b \rangle$ is not normal in G.

2. Proof of Theorem 1

Assume that G has subgroups which are neither nilpotent nor subnormal. We can assume that H is a subgroup of G which is non-nilpotent nor subnormal and for any subgroup $K > H$ we have that K is subnormal in G. Then $H = N_G(H)$. It follows that $C_G(H) \leqslant H$ and H is a self-centralizing subgroup of G which is non-nilpotent. By hypothesis, H is a TI-subgroup of G. Since $H = N_G(H)$, one has that G is a Frobenius group with H being its complement.

Assume $G = N \rtimes H$, where N is the Frobenius kernel. Let H_0 be any maximal subgroup of H, where $H_0 > 1$ since H is non-nilpotent. Then $N \rtimes H_0$ is maximal in $N \rtimes H = G$. Assume that $N \rtimes H_0$ is not normal in G. One has that $N \rtimes H_0$ is not subnormal in G and $N \rtimes H_0 = N_G(N \rtimes H_0)$. It follows that $N \rtimes H_0$ is a self-centralizing subgroup of G which is non-nilpotent. Then $N \rtimes H_0$ is a non-normal TI-subgroup of G by hypothesis. However, one has $(N \rtimes H_0)^g \cap (N \rtimes H_0) = (N^g \rtimes {H_0}^g) \cap (N \rtimes H_0) = (N \rtimes {H_0}^g) \cap (N \rtimes H_0) \geqslant N \neq 1$ for each $g \in G \backslash N_G(N \rtimes H_0) = G \setminus (N \rtimes H_0)$, this is a contradiction. Thus, $N \rtimes H_0$ is normal in G. Note that $H_0 = (N \cap H)H_0 = (N \rtimes H_0) \cap H$. It follows that H_0 is normal in H. And then H is nilpotent by the arbitrariness of H_0, a contradiction.

Hence, every subgroup of G is nilpotent or subnormal. □

3. Proof of Theorem 2

By Theorem 1, every subgroup of G is nilpotent or subnormal. First we show that such a group G is solvable. Let G be a counterexample of minimal order. Then G is a non-solvable group in which every proper subgroup is solvable. It follows that $G/\Phi(G)$ is a minimal non-abelian simple group.

(1) Assume that G has maximal subgroups which are non-nilpotent. Let M be a maximal subgroup of G which is non-nilpotent. By assumption M is subnormal in G. Then M is normal in G, which implies that $M/\Phi(G)$ is normal in $G/\Phi(G)$, a contradiction.

(2) Assume that every maximal subgroup of G is nilpotent. Then G is either a nilpotent group or a non-nilpotent group in which every proper subgroup is nilpotent. By Schmidt Theorem (see [1], Theorem 9.1.9), one has that G is solvable, also a contradiction. Hence, the counterexample of minimal order does not exist. One has that G is solvable.

Next we prove that G must have a normal Sylow subgroup. Let G be a counterexample of minimal order. Since G is solvable, one has that G has a minimal normal subgroup N which is an elementary abelian subgroup of prime-power order. Assume

$|N| = p^m$ for a prime divisor p of $|G|$ and a positive integer m. By the minimality of G, one has that G/N has a normal Sylow subgroup. Let QN/N be a normal Sylow q-subgroup of G/N, where $Q \in \mathrm{Syl}_q(G)$ for a prime divisor q of $|G|$.

(1) Suppose $q = p$. Then $N \leqslant Q$ and Q is a normal Sylow subgroup of G, a contradiction.

(2) Suppose $q \neq p$. Then $N_G(Q)N/N = N_{G/N}(QN/N) = G/N$. It follows that $G = N_G(Q)N$. By the hypothesis, $N_G(Q) < G$. Let R be a maximal subgroup of G such that $R \geqslant N_G(Q)$. Then $G = RN$.

(i) Assume that R is non-nilpotent. Then R is normal in G. By Frattini argument, one has $G = N_G(Q)R = R$, a contradiction.

(ii) Assume that R is nilpotent. Let R_p be a Sylow p-subgroup of R. Then R_pN is a Sylow p-subgroup of G, since $G = RN$. Since R is nilpotent, the subgroup R_p is normal in R and so R_pN is normal in $RN = G$, also a contradiction. Thus, the counterexample of minimal order does not exist and G must have a normal Sylow subgroup.

In the following we give the final conclusion. Suppose that not all Sylow subgroups of G are normal (otherwise there is nothing to be proven). Let $P_1, P_2, \ldots, P_{s-1}$ and P_s be all normal Sylow subgroups of G. Since G is solvable, there is a subgroup K of G such that $G = (P_1 \times P_2 \times \ldots \times P_s) \rtimes K$ by Schur-Zassenhaus Theorem, see [1], Theorem 9.1.2. Note that for any prime divisor r of $|K|$ the Sylow r-subgroup of K which is also a Sylow r-subgroup of G is not normal in G. Assume that K is non-nilpotent. Let K_0 be any non-nilpotent subgroup of K. Then $(P_1 \times P_2 \times \ldots \times P_s) \rtimes K_0$ is a non-nilpotent subgroup of G. By the hypothesis, $(P_1 \times P_2 \times \ldots \times P_s) \rtimes K_0$ is subnormal in G. It follows that K_0 is subnormal in K. Then K is a non-nilpotent group in which every subgroup is nilpotent or subnormal. Arguing as above, K has a normal Sylow subgroup T. Then $K \leqslant N_G(T)$. Note that T is not normal in G by the definition of K and so $N_G(T) < G$. Let L be a maximal subgroup of G such that $N_G(T) \leqslant L$.

(i) Suppose that L is nilpotent. It follows that K is nilpotent since $K \leqslant N_G(T) \leqslant L$, a contradiction.

(ii) Suppose that L is non-nilpotent. Then L is normal in G. The Frattini argument gives $G = LN_G(T) = L$. This is already a contradiction (L is a maximal subgroup). Thus K is nilpotent. For each prime divisor p of $|G|$, if $p = p_i$ for $1 \leqslant i \leqslant s$, one has that G has a normal Sylow p-subgroup by our assumption. Suppose $p \mid |K|$. Let $P \in \mathrm{Syl}_p(K)$. Since K is nilpotent, one has $K = P \times K_1$, where K_1 is a normal nilpotent Hall-subgroup of K. Then $(P_1 \times P_2 \times \ldots \times P_s) \rtimes K_1$ is a normal p-complement of P in G. □

4. Proof of Theorem 3

Assume that G has self-centralizing subgroups which are neither nilpotent nor normal. Then we can assume that H is a self-centralizing subgroup which is neither nilpotent nor normal such that for any subgroup $M > H$ one has that M is normal in G. It is clear that $H < G$. Let N be a subgroup of G such that H is maximal in N. Then N is normal in G. By Theorem 1, H is subnormal in G. It follows that H is normal in N, since it is maximal in N. Since H is not normal in G, there exists $g \in G$ such that $H^g \neq H$. By hypothesis, H is a TI-subgroup and so $H^g \cap H = 1$. Note that $H^g < N^g = N$ and H is maximal in N. It follows that $N = H \times H^g$. Then $H^g \cong N/H$ is a cyclic group of prime order, which contradicts that H is non-nilpotent. Thus, every self-centralizing subgroup of G is nilpotent or normal. □

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