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SIGN CHANGES OF CERTAIN ARITHMETICAL FUNCTION AT PRIME POWERS

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Abstract. We examine an arithmetical function defined by recursion relations on the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ and obtain sufficient condition(s) for the sequence to change sign infinitely often. As an application we give criteria for infinitely many sign changes of Chebyshev polynomials and that of sequence formed by the Fourier coefficients of a cusp form.

Keywords: arithmetic function; Dirichlet series; Chebyschev polynomial; modular form

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1. Introduction

Sign changes of a sequence of real numbers have been considered in various aspects by many mathematicians. Many authors have studied the sign changes of real sequence of Fourier coefficients of a modular form with seemingly different techniques and some references are [2], [7], [8] and [9]. Let f be a real valued arithmetical function. An interesting question is to study the sign changes of the sequence $\{f(n)\}_{n\in\mathbb{N}}$ and in particular whether it offers infinitely many sign changes. In general, it may be possible that f does not have any change of sign. For example, Euler's φ-function does not have any sign change whereas Möbius function μ admits infinitely many sign changes. Here we discuss sign changes of a class of arithmetical functions satisfying certain recurrence relation at prime powers. Let f be a real valued arithmetical function with $f(1)=1$. Further, f satisfies the recurrence relation (for a fixed prime p)

$$f(p^{k+1}) = f(p)f(p^k) - g(p)f(p^{k-1}) \tag{1.1}$$

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for all positive integers k and for some real number $g(p)$. For example, let $g_m(n) = n^m$ for a positive integer m and

$$\sigma_m(n) = \sum_{d|n} d^m.$$

Then $\sigma_m(n)$ satisfy (1.1). Another natural example comes from Fourier coefficients of Hecke eigenforms for the full modular group $SL_2(\mathbb{Z})$ with suitable $g(p)$, see [5]. In this note, we study the sign change of f at prime powers. More precisely, we obtain sufficient condition(s) in terms of $f(p)$ and $g(p)$ such that $\{f(p^k)\}_{k\in\mathbb{N}}$ has infinitely many sign changes.

We prove the following results with f always being a real valued arithmetical function satisfying (1.1) for a fixed prime p.

Theorem 1.1. *If $f(p)^2 < 4g(p)$, then $\{f(p^k)\}_{k\in\mathbb{N}}$ admits infinitely many sign changes.*

Theorem 1.2. *If $f^2(p) \geqslant 4g(p)$, then:*

(1) *If $f^2(p) = 4g(p)$ and $f(p) < 0$, then $\{f(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely many times. Moreover, $f(p^k)$ has positive sign or negative sign when k is even or odd, respectively.*
(2) *If $f^2(p) = 4g(p)$ and $f(p) > 0$, then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ has constant sign with terms having positive sign.*
(3) *If $f^2(p) > 4g(p)$ and $f(p) > 0$, then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ has constant sign with all terms having positive sign.*
(4) *If $f^2(p) > 4g(p)$ and $f(p) < 0$, then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely many times. Also $f(p^k)$ has positive sign or negative sign when k is odd or even, respectively.*

Theorem 1.3. *If $f(p) = 0$, then:*

(1) *If $g(p) > 0$, then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely often.*
(2) *If $g(p) < 0$, then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ has constant positive sign.*
(3) *If $g(p) = 0$, then the sequence $f(p^k) = 0$ for all positive integers k.*

We consider the sign changes in the following sequences as application of the above results:

(i) Sequence defined by a recurrence relation.
(ii) Chebyshev polynomials.
(iii) Fourier coefficients of Hecke eigen cusp forms of general level.

2. Preliminary results

We begin by recalling/proving some results which will be used in the sequel.

Lemma 2.1 (Landau's Theorem, [1]). *Let $F(s)$ be represented in the half-plane $\sigma > c$ by the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s},$$

where c is finite, and assume that $\lambda(n) \geqslant 0$ for all $n \geqslant n_0$. If $F(s)$ is analytic in some disk about the point $s = c$, then the Dirichlet series converges in the half-plane $\sigma > c - \varepsilon$ for some $\varepsilon > 0$. Consequently, if the Dirichlet series has a finite abscissa of convergence σ_c, then $F(s)$ has a singularity on the real axis at the point $s = \sigma_c$.

Lemma 2.2. *Let f be a real valued arithmetical function satisfying* (1.1) *for a fixed prime p. Then*

$$\sum_{n=0}^{\infty} f(p^n)x^n = \frac{1}{1 - f(p)x + g(p)x^2}. \tag{2.1}$$

Proof. We show

$$\left(\sum_{n=0}^{\infty} f(p^n)x^n\right)(1 - f(p)x + g(p)x^2) = 1.$$

Upon expanding the left-hand side of the above,

$$1+(f(p)-f(p))x+(f(1)g(p)-f(p)^2+f(p^2))x^2+(f(p^3)-f(p^2)f(p)+f(p)g(p))x^3+\dots$$

The above infinite sum equals 1 by simply using the recurrence relation of f and $f(1) = 1$. □

Lemma 2.3. *Let f be as before. Write*

$$1 - f(p)x + g(p)x^2 = (1 - \alpha(p)x)(1 - \beta(p)x).$$

Then

$$f(p^k) = \frac{\alpha^{k+1}(p) - \beta^{k+1}(p)}{\alpha(p) - \beta(p)}.$$

If $\alpha(p) = \beta(p)$, then $f(p^k) = (k+1)\alpha^k(p)$.

Proof. The proof of the lemma is an elementary exercise using recurrence relation. □

3. Proof of the theorems

P r o o f of Theorem 1.1. Consider the Dirichlet series

$$F(s) = \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}.$$

Note that the Dirichlet series defined above converges in some right half-plane due to (1.1). Therefore, we can apply Lemma 2.2 in that region, thus,

$$F(s) = \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = \frac{1}{1 - f(p)p^{-s} + g(p)p^{-2s}}. \tag{3.1}$$

Note that the denominator of (3.1) is a quadratic polynomial in p^{-s}. Let $\alpha(p)$ and $\beta(p)$ be the roots of

$$\frac{1}{p^{2s}}((p^s)^2 - f(p)p^s + g(p)),$$

then $\alpha(p) + \beta(p) = f(p)$ and $\alpha(p)\beta(p) = g(p)$. Thus, (3.1) becomes

$$F(s) = \frac{1}{1 - f(p)p^{-s} + g(p)p^{-2s}} = \frac{1}{(1 - \alpha(p)p^{-s})(1 - \beta(p)p^{-s})}.$$

By assumption, the discriminant $(f(p)^2 - 4g(p))$ of the quadratic polynomial in the denominator is negative, so $\alpha(p)$ and $\beta(p)$ are purely complex numbers. Hence, $F(s)$ does not have any real pole although $F(s)$ has complex poles. This implies that $F(s)$ is not an entire function, thus it has finite abscissa of convergence. Assume that $\{f(p^k)\}_{k\in\mathbb{N}}$ has constant sign except for finitely many terms. Then by Landau's theorem (see Lemma 2.2) $F(s)$ must have a real pole, which contradicts the fact that $F(s)$ does not have any real pole. □

P r o o f of Theorem 1.2. Let $f^2(p) \geqslant 4g(p)$, then $x^2 - f(p)x + g(p)$ has real roots say $\{\alpha(p), \beta(p)\}$, given by

$$\alpha(p) = \frac{f(p) + \sqrt{f^2(p) - 4g(p)}}{2}, \quad \beta(p) = \frac{f(p) - \sqrt{f^2(p) - 4g(p)}}{2} \tag{3.2}$$

and

$$\alpha(p) - \beta(p) = \sqrt{f^2(p) - 4g(p)} \geqslant 0.$$

Note that if $f^2(p) = 4g(p)$, then $\alpha(p) = \beta(p)$ and by Lemma 2.3

$$f(p^k) = (k+1)(\alpha(p))^k = (k+1)\frac{(f(p))^k}{2^k}.$$

Thus, the proof of part (1) and (2) follows.

For the remaining cases, $f^2(p) > 4g(p)$. Hence $\alpha(p) - \beta(p) > 0$ and thus by Lemma 2.3, $f(p^k)$ and $\alpha^{k+1}(p) - \beta^{k+1}(p)$ have the same sign since the denominator $\alpha(p) - \beta(p)$ has positive sign. Further, if $f(p) > 0$, then by (3.2), $\alpha(p) > \beta(p)$ and consequently $\alpha^{k+1}(p) > \beta^{k+1}(p)$. If $f(p) < 0$, then $\alpha(p) > \beta(p)$ but $|\alpha(p)| < |\beta(p)|$ and consequently $\alpha^{k+1}(p) > \beta^{k+1}(p)$ or $\alpha^{k+1}(p) < \beta^{k+1}(p)$ according to whether k is odd or even, respectively. This completes the proof. □

Remark 3.1. It may seem that the proof of Theorem 1.2 is dependent on the choice of $\alpha(p)$ and $\beta(p)$ but it is not the case since Lemma 2.3 is symmetric with respect to $\alpha(p)$ and $\beta(p)$.

Proof of Theorem 1.3. Note that if $f(p) = 0$, then by (1.1) we have

$$f(p^{4k+i}) = \begin{cases} (-1)^{i/2} g(p)^{2k+i/2}, & i \equiv 0, 2 \pmod 4, \\ 0, & i \equiv \pm 1 \pmod 4. \end{cases}$$

This completes the proof of Theorem 1.3. □

Corollary 3.1. *Let f be a real valued arithmetical function satisfying* (1.1). *Let $f(p) > 0$. Then the sequence $\{f(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely often if and only if $f(p)^2 < 4g(p)$.*

Proof. This is an immediate consequence of the combination of Theorem 1.1 and Theorem 1.2. □

Corollary 3.2. *Let f be a multiplicative function. Let $p_1, p_2, \ldots, p_r$ be r distinct primes such that the sequence $\{f(p_i^k)\}_{k\in\mathbb{N}}$ satisfies* (1.1) *for each i with some real number $g(p_i)$. Assume that $f(p_i)$ and $g(p_i)$ satisfy either the first or the fourth statement of Theorem* 1.2. *If $n = p_1, p_2, \ldots, p_r$, then the sequence $\{f(n^k)\}_{k\in\mathbb{N}}$ change signs infinitely often, if in the first case r is odd, or if in the second case, r is even, and otherwise has a constant sign.*

Proof. It follows from part (1), part (4) of Theorem 1.2. □

4. Applications

4.1. Sequence defined by recurrence relation. Let $\{a(n)\}_{n\in\mathbb{N}}$ be a sequence of real numbers defined by the following recurrence relation. For all $n \geqslant 0$

$$a(n+1) = \alpha a(n) + \beta a(n-1), \tag{4.1}$$

$a(-n) = 0$ for all $n \in \mathbb{N}$ and $a(0) = 1$, where α and β are real numbers. We want sufficient condition(s) in terms of α and β so that the sequence $\{a(n)\}_{n\in\mathbb{N}}$ has infinitely many sign changes.

First note that $a(1) = \alpha$ (by putting $n = 0$ in (4.1)) and so

$$a(n+1) = a(1)a(n) + \beta a(n-1).$$

In order to apply Theorems 1.1, 1.2 and 1.3, we need an arithmetical function f satisfying (1.1) for a prime p. For a fixed prime p and for a positive integer k define

$$f(p^k) = a(k) \quad \text{and} \quad g(p) = -\beta,$$

and for $n \neq p^k$, set $f(n) = 1$. Then the recurrence relation $a(k+1) = a(1)a(k) + \beta a(k-1)$ is equivalent to the recurrence relation $f(p^{k+1}) = f(p)f(p^{k-1}) - g(p)f(p^{k-1})$. Now $\{a(n)\}_{n\in\mathbb{N}}$ have the same nature as that of $\{f(p^n)\}_{n\in\mathbb{N}}$. In particular by Theorems 1.1 and 1.2, we have the following:

Proposition 4.1. *Let $\{a(n)\}_{n\in\mathbb{N}}$ be a sequence as defined above. Then:*

(1) *If $\alpha^2 + 4\beta < 0$ then the sequence $\{a(n)\}_{n\in\mathbb{N}}$ has infinitely many sign changes.*
(2) *If $\alpha^2 + 4\beta = 0$ and $\alpha < 0$, then $\{a(n)\}_{n\in\mathbb{N}}$ changes sign infinitely often. In particular, $a(n)$ has positive sign or negative sign according to whether k is even or odd, respectively.*
(3) *If $\alpha^2 + 4\beta = 0$ and $\alpha > 0$, then $\{a(n)\}_{n\in\mathbb{N}}$ has constant sign. In particular, all terms have positive sign.*
(4) *If $\alpha^2 + 4\beta > 0$ and $\alpha > 0$, then $\{a(n)\}_{n\in\mathbb{N}}$ has constant sign. In particular, all terms have positive sign.*
(5) *If $\alpha^2 + 4\beta > 0$ and $\alpha < 0$, then $\{a(n)\}_{n\in\mathbb{N}}$ changes sign infinitely often. In particular, $a(n)$ has positive sign or negative sign according to whether n is odd or even, respectively.*

4.2. Chebyshev polynomials. Now we will give a particular example of Proposition 4.1. Chebyshev polynomials are defined by the following recurrence relation:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geqslant 1.$$

We consider the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$. By an easy calculation, we see that $\{T_n(0)\}_{n\in\mathbb{N}}$ has infinitely many sign changes. To examine the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ for $x \in \mathbb{R}$ we apply Proposition 4.1. In order to do this we put $\alpha = 2x$ and $\beta = -1$ in Proposition 4.1 and we get:

(1) If $x^2 < 1$, then the sequence $\{T_n(x)\}_{n\in\mathbb{N}}$ has infinitely many sign changes.
(2) If $x^2 = 1$ and $x < 0$, then $\{T_n(x)\}_{n\in\mathbb{N}}$ changes sign infinitely many often. In particular, $a(n)$ has positive sign or negative sign according to whether n is even or odd, respectively.
(3) If $x^2 = 1$ and $x > 0$, then $\{T_n(x)\}_{n\in\mathbb{N}}$ has constant sign. In particular, all terms of the sequence have positive sign.
(4) If $x^2 > 1$ and $x > 0$, then $\{T_n(x)\}_{n\in\mathbb{N}}$ has constant sign. In particular, all terms of the sequence have positive sign.
(5) If $x^2 > 1$ and $x < 0$, then $\{T_n(x)\}_{n\in\mathbb{N}}$ changes sign infinitely often. In particular, $a(n)$ has positive sign or negative sign according to whether n is odd or even, respectively.

4.3. Modular forms for $\Gamma = SL_2(\mathbb{Z})$. Sign changes in the sequences formed by the coefficients of a modular forms of various weights and levels are an interesting area to explore and many authors have contributed towards this question in various situations. Some more references that we would like to list other than the ones mentioned in the introduction are [3], [4], [6]. The results mentioned here are already known.

For a positive integer l and $\Gamma = SL_2(\mathbb{Z})$, let $S_l(\Gamma)$ denote the space of modular cusp form of weight l on Γ. Let $F \in S_l(\Gamma)$. We have the following Fourier series expansion for F:

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n.$$

Assume that the coefficients $a(n)$ are real numbers. We consider the sequence $\{a(p^k)\}_{k\in\mathbb{N}}$. Hecke theory ensures that this sequence satisfies (1.1) for all primes p with $f(p^k) = a(p^k)$ and $g(p) = p^{l-1}$, see [5]. Also by Deligne's bound, $a(p)^2 < 4p^{l-1}$. Then $\{a(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely often by Theorem 1.1.

Further, using Theorems 1.2 and 1.3, we have sufficient condition(s) as to when $\{a(p^k)\}_{k\in\mathbb{N}}$ changes sign infinitely often. Analogous results hold for Hecke eigenforms of higher level, too.

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