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ON TWO SUPERCONGRUENCES INVOLVING ALMKVIST-ZUDILIN SEQUENCES

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Abstract. We prove two supercongruences involving Almkvist-Zudilin sequences, which were originally conjectured by Z.-H. Sun (2020).

Keywords: supercongruence; Euler number; Almkvist-Zudilin sequence

MSC 2020: 11A07, 11B68, 05A19

1. Introduction

In 1979, Apéry (see [3]) in his ingenious proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ introduced the following two kinds of numbers:

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad\text{and}\quad A'_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

These numbers are now known as the famous *Apéry numbers*. It is well-known that the Apéry numbers satisfy the following recurrences (see [5]):

$$(n+1)^3A_{n+1}=(2n+1)(17n(n+1)+5)A_n-n^3A_{n-1},$$

and

$$(n+1)^2A'_{n+1}=(11n(n+1)+3)A'_n+n^2A'_{n-1}.$$

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For integers a, b and $c \neq 0$, the *Apéry-like numbers of the first kind* $\{u_n\}$ satisfy

$$u_0 = 1, \quad u_1 = b, \quad (n+1)^3 u_{n+1} = (2n+1)(an(n+1)+b)u_n - cn^3 u_{n-1},$$

and the *Apéry-like numbers of the second kind* $\{u'_n\}$ satisfy the recurrence (see [33]):

$$u'_0 = 1, \quad u'_1 = b, \quad (n+1)^2 u'_{n+1} = (an(n+1)+b)u'_n - cn^2 u'_{n-1}.$$

In 2006, Almkvist and Zudilin in [1] introduced many interesting Apéry-like numbers such as

$$G_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k} 4^{n-k},$$

and

$$\gamma_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{3^{n-3k}(3k)!}{(k!)^3} \binom{n}{3k} \binom{n+k}{k}.$$

Note that the numbers γ_n and G_n are Apéry-like numbers of the first kind with $(a,b,c) = (-7,-3,81)$ and Apéry-like numbers of the second kind with $(a,b,c) = (32,12,256)$, respectively. We remark that the numbers γ_n are also called *Almkvist-Zudilin numbers*.

A *supercongruence* is a p-adic congruence which happens to hold not just modulo a prime p as predicted by formal group laws or other considerations but a higher power of p. Since the appearance of the Apéry numbers and Apéry-like numbers, some interesting supercongruences for these numbers have been gradually discovered (see, for instance, [2], [4], [5], [6], [8], [10], [12], [13], [19], [20], [22], [23], [25], [29]). A typical example is

$$A_{np^r} \equiv A_{np^{r-1}} \pmod{p^{3r}}$$

for any prime $p \geqslant 5$, which was proved by Coster (see [7]) in a more general form.

Another example due to Amdeberhan and Tauraso (see [2]) is the beautiful supercongruence

$$\gamma_{np} \equiv \gamma_n \pmod{p^3}$$

for any prime $p \geqslant 5$. It is worth mentioning that Sun in [27], Conjecture 2.5 conjectured a similar supercongruence for G_n

$$G_{np^r} \equiv G_{np^{r-1}} \pmod{p^{2r}}$$

for positive integers n, r, and an odd prime p.

The motivation of this paper is to prove the following two supercongruences involving the Almkvist-Zudilin sequence $\{G_n\}$, which were originally conjectured by Sun, see [27], Conjectures 2.1 and 2.2.

Theorem 1.1. *For any prime $p \geqslant 5$ we have*

$$G_{p-1} \equiv (-1)^{(p-1)/2} 256^{p-1} + 3p^2 E_{p-3} \pmod{p^3}. \tag{1.1}$$

Here the Euler numbers are defined as

$$\frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

Theorem 1.2. *For any prime $p \geqslant 5$ we have*

$$\sum_{k=0}^{p-1} \frac{G_k}{16^k} \equiv p^2 (4(-1)^{(p-1)/2} - 3) \pmod{p^3}. \tag{1.2}$$

For Theorem 1.1, that would be comparatively easy from the definition as all the terms in the sum for G_{p-1} are divisible by p except for the central term. Hence, the expected congruence is

$$G_{p-1} \equiv \binom{p-1}{\frac{1}{2}(p-1)}^3 4^{(p-1)/2} \equiv (-1)^{(p-1)/2} 256^{p-1} \pmod{p}.$$

For Theorem 1.2 we can easily deduce from (3.11) that

$$\sum_{k=0}^{p-1} \frac{G_k}{16^k} \equiv 0 \pmod{p}.$$

However, Theorems 1.1 and 1.2 prove that they not only hold modulo a higher power p^2, but also refine them further modulo p^3.

In the next section, we first recall some auxiliary results. We shall prove Theorems 1.1 and 1.2 in the final section.

2. Auxiliary results

Let

$$H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r}$$

denote the nth generalized harmonic number of order r with the convention that $H_n = H_n^{(1)}$. The Fermat quotient of an integer a with respect to an odd prime p is given by $q_p(a) = (a^{p-1} - 1)/p$.

In order to prove Theorems 1.1 and 1.2, we need the following identities.

Lemma 2.1. *For any non-negative integer n and positive integer r we have*

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}4^{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}^2 16^{n-k}, \tag{2.1}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+r}\binom{n}{k}\binom{n+k}{k}=\frac{(-1)^n}{r}\prod_{j=1}^{n}\Big(\frac{r-j}{r+j}\Big), \tag{2.2}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}H_k^2=2(-1)^n\Big(2H_n^2+\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\Big), \tag{2.3}$$

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}\binom{n+k}{k}H_k=\frac{(-1)^n-1}{n(n+1)}. \tag{2.4}$$

P r o o f. Identities (2.1)–(2.4) have already been proved by Sun in [27], Theorem 2.1, Mortenson in [21], Lemma 3.1, Wang in [31], Lemma 2.2, and the first author in [16], Lemma 2.2, respectively.

In fact, all of these identities can be proved by the symbolic summation package `Sigma` developed by Schneider, see [26]. By using `Sigma`, we find that both sides of (2.1)–(2.4) satisfy the same recurrences. More specifically, we list out these recurrences:

(2.1): $$256(n+1)^2S_n-4(8n^2+24n+19)S_{n+1}+(n+2)^2S_{n+2}=0,$$

(2.2): $$(r-n-1)S_n+(n+r+1)S_{n+1}=0,$$

(2.3): $$\begin{aligned}&(2n+5)(n+1)^2S_n+(2n+3)(3n^2+12n+11)S_{n+1}\\&\quad+(2n+5)(3n^2+12n+11)S_{n+2}+(2n+3)(n+3)^2S_{n+3}\\&\quad=\frac{4(2n+3)(2n+5)}{n+2},\end{aligned}$$

(2.4): $$nS_n+(n+2)S_{n+1}=-\frac{2}{n+1}.$$

It is trivial to verify that both sides of (2.1)–(2.4) are equal for initial values. One can also refer to [9], [14], [15], [17], [18], [24] for the computerized approach to proving such identities. □

We also require some known congruences.

Lemma 2.2. *For any prime $p\geqslant 5$ we have*

$$H_{(p-1)/2}\equiv -2q_p(2) \pmod p, \tag{2.5}$$

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\equiv 2(-1)^{(p-1)/2}E_{p-3} \pmod p, \tag{2.6}$$

$$\sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2 \equiv (-1)^{(p-1)/2} + p^2E_{p-3} \pmod{p^3}, \tag{2.7}$$

$$\sum_{k=(p+1)/2}^{p-1} \frac{1}{16^k}\binom{2k}{k}^2 \equiv -2p^2E_{p-3} \pmod{p^3}, \tag{2.8}$$

$$\sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2 H_k \equiv (-1)^{(p-1)/2}(-4q_p(2) + 2pq_p(2)^2) \pmod{p^2}, \tag{2.9}$$

$$\sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2 H_k^{(2)} \equiv -4E_{p-3} \pmod{p}. \tag{2.10}$$

P r o o f. See [11], (30), (45), [28], (1.7), (1.9), Lemma 2.4 and [30], (4.8), (4.9). □

3. Proof of Theorems 1.1 and 1.2

P r o o f of Theorem 1.1. By (2.1), we have

$$G_{p-1} = \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}\binom{2k}{k}^2 16^{p-1-k}. \tag{3.1}$$

Now we split the sum on the right-hand side of (3.1) into two pieces:

$$S_1 = \sum_{k=0}^{(p-1)/2}(-1)^k\binom{p-1}{k}\binom{2k}{k}^2 16^{p-1-k}, \tag{3.2}$$

and

$$S_2 = \sum_{k=(p+1)/2}^{p-1}(-1)^k\binom{p-1}{k}\binom{2k}{k}^2 16^{p-1-k}. \tag{3.3}$$

We first evaluate S_1 modulo p^3. Note that

$$\binom{p-1}{k} \equiv (-1)^k\Big(1 - pH_k + \frac{p^2}{2}(H_k^2 - H_k^{(2)})\Big) \pmod{p^3}. \tag{3.4}$$

It follows from (3.2) and (3.4) that

$$S_1 \equiv 16^{p-1}\sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2\Big(1 - pH_k + \frac{p^2}{2}(H_k^2 - H_k^{(2)})\Big) \pmod{p^3}. \tag{3.5}$$

Letting $n = \frac{1}{2}(p-1)$ in (2.3) and noting that

$$(3.6)\qquad (-1)^k \binom{\frac{1}{2}(p-1)}{k}\binom{\frac{1}{2}(p-1)+k}{k} \equiv \frac{1}{16^k}\binom{2k}{k}^2 \pmod{p^2},$$

we obtain

$$(3.7)\qquad \sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2 H_k^2 \equiv 2(-1)^{(p-1)/2}\biggl(2H_{(p-1)/2}^2 + \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2}\biggr) \pmod{p}.$$

Applying (2.5) and (2.6) to the right-hand side of (3.7) gives

$$(3.8)\qquad \sum_{k=0}^{(p-1)/2} \frac{1}{16^k}\binom{2k}{k}^2 H_k^2 \equiv 16(-1)^{(p-1)/2} q_p(2)^2 + 4E_{p-3} \pmod{p}.$$

Furthermore, substituting (2.7), (2.9), (2.10) and (3.8) into the right-hand side of (3.5), we arrive at

$$(3.9)\qquad S_1 \equiv (-1)^{(p-1)/2} 16^{p-1}(1 + 4pq_p(2) + 6p^2 q_p(2)^2) + 5p^2 E_{p-3} \pmod{p^3}.$$

Next, we evaluate S_2 modulo p^3. For $\frac{1}{2}(p+1) \leqslant k \leqslant p-1$ we have

$$\binom{2k}{k}^2 \equiv 0 \pmod{p^2}, \qquad \binom{p-1}{k} \equiv (-1)^k \pmod{p},$$

and so

$$(3.10)\quad S_2 \equiv \sum_{k=(p+1)/2}^{p-1} \binom{2k}{k}^2 16^{p-1-k} \equiv \sum_{k=(p+1)/2}^{p-1} \frac{1}{16^k}\binom{2k}{k}^2 \equiv -2p^2 E_{p-3} \pmod{p^3},$$

where we have used Fermat's little theorem in the second step and (2.8) in the last step. Then the proof of (1.1) follows from (3.9) and (3.10). □

Proof of Theorem 1.2. Using (2.1) and exchanging the summation order, we directly deduce that

$$\begin{aligned}(3.11)\qquad \sum_{k=0}^{p-1} \frac{G_k}{16^k} &= \sum_{j=0}^{p-1} \frac{1}{(-16)^j}\binom{2j}{j}^2\binom{p}{j+1} \\ &= \sum_{j=0}^{p-2} \frac{1}{(-16)^j}\binom{2j}{j}^2\binom{p}{j+1} + \frac{1}{16^{p-1}}\binom{2p-2}{p-1}^2 \\ &\equiv p\sum_{j=0}^{(p-1)/2} \frac{1}{(-16)^j(j+1)}\binom{2j}{j}^2\binom{p-1}{j} \\ &\quad + \frac{1}{16^{p-1}}\binom{2p-2}{p-1}^2 \pmod{p^3},\end{aligned}$$

where we have used the fact that $\binom{2j}{j}^2/(j+1) \equiv 0 \pmod{p^2}$ for $\frac{1}{2}(p-1) < j \leqslant p-2$ in the last step.

On one hand, by Fermat's little theorem and Wolstenholme's theorem (see [32]), we have

$$\frac{1}{16^{p-1}}\binom{2p-2}{p-1}^2 = \frac{p^2}{16^{p-1}(2p-1)^2}\binom{2p-1}{p-1}^2 \equiv p^2 \pmod{p^3}. \tag{3.12}$$

On the other hand, by using (3.4), we have

$$\begin{aligned}\sum_{j=0}^{(p-1)/2} \frac{1}{(-16)^j(j+1)}\binom{2j}{j}^2\binom{p-1}{j} \\ \equiv \sum_{j=0}^{(p-1)/2} \frac{1}{16^j(j+1)}\binom{2j}{j}^2(1-pH_j) \pmod{p^2}.\end{aligned} \tag{3.13}$$

In view of (3.6), we obtain

$$\begin{aligned}\sum_{j=0}^{(p-1)/2} \frac{1}{16^j(j+1)}\binom{2j}{j}^2 \\ \equiv \sum_{j=0}^{(p-1)/2} \frac{(-1)^j}{j+1}\binom{\frac{1}{2}(p-1)}{j}\binom{\frac{1}{2}(p-1)+j}{j} = 0 \pmod{p^2},\end{aligned} \tag{3.14}$$

where we have used the case $r=1$ of identity (2.2) in the last step.

Furthermore, letting $n = \frac{1}{2}(p-1)$ in (2.4) and using (3.6), we obtain

$$\sum_{k=0}^{(p-1)/2} \frac{1}{16^k(k+1)}\binom{2k}{k}^2 H_k \equiv 4(1-(-1)^{(p-1)/2}) \pmod{p}. \tag{3.15}$$

Finally, combining (3.11)–(3.15), we reach the desired result (1.2). □

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