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# AN EXAMPLE OF A REFLEXIVE LORENTZ GAMMA SPACE WITH TRIVIAL BOYD AND ZIPPIN INDICES

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*Abstract.* We show that for every $p \in (1, \infty)$ there exists a weight $w$ such that the Lorentz Gamma space $\Gamma_{p,w}$ is reflexive, its lower Boyd and Zippin indices are equal to zero and its upper Boyd and Zippin indices are equal to one. As a consequence, the Hardy-Littlewood maximal operator is unbounded on the constructed reflexive space $\Gamma_{p,w}$ and on its associate space $\Gamma'_{p,w}$.

*Keywords*: Lorentz Gamma space; reflexivity; Boyd indices; Zippin indices

*MSC 2020*: 46E30, 42B25

## 1. Introduction

Let $m$ denote the standard Lebesgue measure on $\mathbb{R}^n$. The Hardy-Littlewood maximal operator $M$ is one of the most important operators in harmonic analysis. For a function $f \in L^1_{\rm loc}(\mathbb{R}^n, m)$ it is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes. We are interested in the following question: Given a class $\mathfrak{S}$ of Banach function spaces (see [1], [9]), is it true that the Hardy-Littlewood maximal operator $M$ is bounded on each reflexive space $X(\mathbb{R}^n, m) \in \mathfrak{S}$ or on its associate space $X'(\mathbb{R}^n, m)$? It is well known that the answer is "yes" for the class of all Lebesgue spaces. We show

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that the answer is "no" for the class of Lorentz Gamma spaces $\Gamma_{p,w}(\mathbb{R}^n, m)$ (being a proper subclass of the class of all rearrangement-invariant Banach function spaces).

A measurable function $w\colon (0,\infty) \to [0,\infty)$ is called a *weight function*. Following [6], page 289, we say that a weight function belongs to the class $\mathcal{D}_p$, $1 \leqslant p < \infty$, whenever for all $x \in (0,\infty)$,

$$0 < W(x) := \int_0^x w(t)\,\mathrm{d}t < \infty, \quad W_p(x) := x^p \int_x^\infty t^{-p} w(t)\,\mathrm{d}t < \infty.$$

Let $(\mathcal{R}, \mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. The Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is the set of all $\mu$-measurable functions $f\colon \mathcal{R} \to \mathbb{C}$ such that

$$\|f\|_{\Gamma_{p,w}} := \left(\int_0^\infty (f^{**}(t))^p w(t)\,\mathrm{d}t\right)^{1/p} < \infty,$$

where

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(x)\,\mathrm{d}x$$

and $f^*$ is the decreasing rearrangement of $f$ (see e.g. [1], Chapter 2, Section 1 or [7], Chapter II, §2 for its definition and properties). The Lorentz Gamma spaces were introduced by Sawyer (see [10]), but they can be traced to earlier works of Calderón, Hunt, O'Neil and others. These spaces have been studied by many authors (see e.g. [3], [4], [5], [6] and [9], Chapter 10 and also the references in the above monograph). The Lorentz Gamma spaces are examples of rearrangement-invariant (or symmetric) Banach function spaces (see [1], Chapter 2 and [7], Chapter II for the theory of rearrangement-invariant Banach function spaces). We should also note that one can study quasi-Banach Lorentz Gamma spaces $\Gamma_{p,w}$ if one admits that $0 < p < \infty$, see e.g. [6].

It is well known that Boyd and Zippin indices play important role in many questions related to interpolation properties of rearrangement-invariant spaces and boundedness behavior of classical operators of harmonic analysis (like the Hardy-Littlewood maximal operator or the Hilbert transform) on rearrangement-invariant spaces. The Boyd indices

$$0 \leqslant \alpha(X) \leqslant \beta(X) \leqslant 1$$

and Zippin indices

$$0 \leqslant p(X) \leqslant q(X) \leqslant 1$$

of a rearrangment-invariant space $X(\mathcal{R}, \mu)$ were introduced in [2] and [11], respectively. We refer to [1], Chapter 3, [7], Chapter 2, and [8] for their definitions, properties, and applications in interpolation theory and harmonic analysis. Boyd indices of Lorentz Gamma spaces $\Gamma_{p,w}$ were studied in [4], [6].

In this paper we construct an example of a reflexive Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R},\mu)$, whose Boyd and Zippin indices behave "badly", (one says that the indices are trivial in this case) by rather elementary techniques:

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = 0, \quad \beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = 1.$$

It may well be that examples of reflexive rearrangement-invariant spaces $X(\mathcal{R},\mu)$ with both Boyd indices being trivial are known to experts or can be obtained from known results, but we were unable to find a published example of such space. Moreover, we believe that an example of a reflexive Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R},\mu)$ with trivial Boyd and Zippin indices is of interest because it leads to the negative answer to the question posed above not only for the whole class of rearrangement-invariant Banach function spaces, but to its narrower subclass of Lorentz Gamma spaces.

For a precise definition of what we mean by a "rearrangement-invariant Banach function space" and by its "associate space", see [1], Chapter 1, Definitions 1.1 and 2.1; Chapter 2, Definition 4.1 and Section 2 below. Definitions of Boyd and Zippin indices are also given in Section 2.

**Theorem 1.1** (Main result)**.** *Let $(\mathcal{R},\mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. If $p \in (1,\infty)$ and*

$$w(x) := \begin{cases} \dfrac{1}{x(1-\log x)^2}, & 0 < x < 1, \\ \dfrac{x^{p-1}}{(1+\log x)^2}, & x \geqslant 1, \end{cases} \tag{1.1}$$

*then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R},\mu)$ is a reflexive rearrangement-invariant Banach function space and its Boyd and Zippin indices are trivial, that is,*

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = 0, \quad \beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = 1.$$

Combining Theorem 1.1 with the Lorentz-Shimogaki theorem (see e.g. [1], Chapter 3, Theorem 5.17), we arrive at the following corollary.

**Corollary 1.2.** *Let $p \in (1,\infty)$ and a weight $w$ be given by* (1.1)*. Then the Lorentz Gamma space $\Gamma_{p,w}(\mathbb{R}^n, m)$ is a reflexive rearrangement-invariant Banach function space and the Hardy-Littlewood maximal operator $M$ is unbounded on the space $\Gamma_{p,w}(\mathbb{R}^n, m)$ and on its associate space $\Gamma'_{p,w}(\mathbb{R}^n, m)$.*

The paper is organized as follows. In Section 2, we recall the notions of a rearrangement-invariant Banach function space $X(\mathcal{R},\mu)$ and its associate space $X'(\mathcal{R},\mu)$, and give the definitions of its Boyd indices $\alpha(X)$, $\beta(X)$ and Zippin indices $p(X), q(X)$. In Section 3, we show that the Lorentz Gamma spaces $\Gamma_{p,q}(\mathcal{R},\mu)$ fall into the class of rearrangement-invariant Banach function spaces if $1 \leqslant p < \infty$ and $w \in \mathcal{D}_p$. Further, we formulate necessary and sufficient conditions for the reflexivity of $\Gamma_{p,w}(\mathcal{R},\mu)$. In Section 4, we prove Theorem 1.1 and Corollary 1.2.

## 2. Rearrangement-invariant Banach function spaces and their indices

**2.1. Banach function spaces.** Let $(\mathcal{R},\mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. The set of all $\mu$-measurable complex-valued functions on $\mathcal{R}$ is denoted by $\mathfrak{M}(\mathcal{R},\mu)$. Let $\mathfrak{M}^+(\mathcal{R},\mu)$ be the subset of functions in $\mathfrak{M}(\mathcal{R},\mu)$ whose values lie in $[0,\infty]$. The characteristic function of a $\mu$-measurable set $E \subset \mathcal{R}$ is denoted by $\chi_E$. Following [1], Chapter 1, Definition 1.1, a mapping

$$\varrho\colon \mathfrak{M}^+(\mathcal{R},\mu) \to [0,\infty]$$

is called a *Banach function norm* if, for all functions $f$, $g$, $f_n$ $(n \in \mathbb{N})$ in $\mathfrak{M}^+(\mathcal{R},\mu)$, for all constants $a \geqslant 0$, and for all $\mu$-measurable subsets $E$ of $\mathcal{R}$, the following axioms hold:

(A1) $\varrho(f) = 0 \Leftrightarrow f = 0$ a.e., $\varrho(af) = a\varrho(f)$, $\varrho(f+g) \leqslant \varrho(f) + \varrho(g)$,
(A2) $0 \leqslant g \leqslant f$ a.e. $\Rightarrow \varrho(g) \leqslant \varrho(f)$ (the lattice property),
(A3) $0 \leqslant f_n \uparrow f$ a.e. $\Rightarrow \varrho(f_n) \uparrow \varrho(f)$ (the Fatou property),
(A4) $\mu(E) < \infty \Rightarrow \varrho(\chi_E) < \infty$,
(A5) $\mu(E) < \infty \Rightarrow \int_E f(x)\,\mathrm{d}\mu(x) \leqslant C_E\varrho(f)$

with $C_E \in (0,\infty)$, which may depend on $E$ and $\varrho$ but is independent of $f$. When functions differing only on a set of $\mu$-measure zero are identified, the set $X(\mathcal{R},\mu)$ of all functions $f \in \mathfrak{M}(\mathcal{R},\mu)$ for which $\varrho(|f|) < \infty$ is called a Banach function space. For each $f \in X(\mathcal{R},\mu)$, the norm of $f$ is defined by

$$\|f\|_X := \varrho(|f|).$$

Under the natural linear space operations and under this norm, the set $X(\mathcal{R},\mu)$ becomes a Banach space (see [1], Chapter 1, Theorems 1.4 and 1.6). If $\varrho$ is a Banach function norm, its associate norm $\varrho'$ is defined on $\mathfrak{M}^+(\mathcal{R},\mu)$ by

$$\varrho'(g) := \sup\biggl\{\int_{\mathcal{R}} f(x)g(x)\,\mathrm{d}\mu(x)\colon\ f \in \mathfrak{M}^+(\mathcal{R},\mu),\ \varrho(f) \leqslant 1\biggr\}, \quad g \in \mathfrak{M}^+(\mathcal{R},\mu).$$

It is a Banach function norm itself, see [1], Chapter 1, Theorem 2.2. The Banach function space $X'(\mathcal{R},\mu)$ determined by the Banach function norm $\varrho'$ is called the *associate space* (Köthe dual) of $X(\mathcal{R},\mu)$. The associate space $X'(\mathcal{R},\mu)$ is naturally identified with a subspace of the (Banach) dual space $[X(\mathcal{R},\mu)]^*$.

**2.2. Rearrangement-invariant Banach function spaces.** Let $\mathfrak{M}_0(\mathcal{R},\mu)$ and $\mathfrak{M}_0^+(\mathcal{R},\mu)$ be the classes of a.e. finite functions in $\mathfrak{M}(\mathcal{R},\mu)$ and $\mathfrak{M}^+(\mathcal{R},\mu)$, respectively. The distribution function $\mu_f$ of $f\in\mathfrak{M}_0(\mathcal{R},\mu)$ is given by

$$\mu_f(\lambda) := \mu\{x\in\mathcal{R}\colon\ |f(x)|>\lambda\},\quad \lambda\geqslant 0.$$

Two functions $f,g\in\mathfrak{M}_0(\mathcal{R},\mu)$ are said to be equimeasurable if $\mu_f(\lambda)=\mu_g(\lambda)$ for all $\lambda\geqslant 0$. The decreasing rearrangement of $f\in\mathfrak{M}_0(\mathcal{R},\mu)$ is the function defined by

$$f^*(t) := \inf\{\lambda\colon\ \mu_f(\lambda)\leqslant t\},\quad t\geqslant 0.$$

We use here the standard convention that $\inf\emptyset=\infty$.

A Banach function norm $\varrho\colon\ \mathfrak{M}^+(\mathcal{R},\mu)\to[0,\infty]$ is called *rearrangement-invariant* if for every pair of equimeasurable functions $f,g\in\mathfrak{M}_0^+(\mathcal{R},\mu)$, the equality $\varrho(f)=\varrho(g)$ holds. In that case, the Banach function space $X(\mathcal{R},\mu)$ generated by $\varrho$ is said to be a rearrangement-invariant Banach function space (or simply a rearrangement-invariant space). Lebesgue spaces $L^p(\mathcal{R},\mu)$, $1\leqslant p\leqslant\infty$, Orlicz spaces $L^\Phi(\mathcal{R},\mu)$, and Lorentz spaces $L^{p,q}(\mathcal{R},\mu)$ are classical examples of rearrangement-invariant Banach function spaces (see e.g. [1] and the references therein). By [1], Chapter 2, Proposition 4.2, if a Banach function space $X(\mathcal{R},\mu)$ is rearrangement-invariant, then its associate space $X'(\mathcal{R},\mu)$ is also rearrangement-invariant.

**2.3. Boyd and Zippin indices.** A measurable function $\varrho\colon\ (0,\infty)\to(0,\infty)$ is said to be submultiplicative if

$$\varrho(x_1x_2)\leqslant\varrho(x_1)\varrho(x_2)\quad \forall\, x_1,x_2\in(0,\infty).$$

The behavior of a measurable submultiplicative function $\varrho$ in neighborhoods of zero and infinity is described by the quantities

$$\alpha(\varrho) := \sup_{x\in(0,1)}\frac{\log\varrho(x)}{\log x}=\lim_{x\to 0}\frac{\log\varrho(x)}{\log x},\qquad (2.1)$$

$$\beta(\varrho) := \inf_{x\in(1,\infty)}\frac{\log\varrho(x)}{\log x}=\lim_{x\to \infty}\frac{\log\varrho(x)}{\log x},$$

where

$$-\infty<\alpha(\varrho)\leqslant\beta(\varrho)<\infty,\qquad (2.2)$$

see [7], Chapter 2, Theorem 1.3. The numbers $\alpha(\varrho)$ and $\beta(\varrho)$ are called the *lower* and *upper indices* of the measurable submultiplicative function $\varrho$.

Let $(\mathcal{R},\mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R})=\infty$. Suppose $X(\mathcal{R},\mu)$ is a rearrangement-invariant Banach function space generated by a rearrangement-invariant Banach function norm $\varrho$. In this case, the Luxemburg representation theorem (see [1], Chapter 2, Theorem 4.10) provides a unique rearrangement-invariant Banach function norm $\bar{\varrho}$ over the half-line $\mathbb{R}_+=(0,\infty)$ equipped with the Lebesgue measure $m$, defined by

$$\bar{\varrho}(h):=\sup\left\{\int_0^\infty g^*(t)h^*(t)\,\mathrm{d}t\colon\ \varrho'(g)\leqslant 1\right\},$$

and such that $\varrho(f)=\bar{\varrho}(f^*)$ for all $f\in\mathfrak{M}_0^+(\mathcal{R},\mu)$. The rearrangement-invariant Banach function space generated by $\bar{\varrho}$ is denoted by $\overline{X}(\mathbb{R}_+,m)$.

For each $t>0$ let $E_t$ denote the dilation operator defined on $\mathfrak{M}(\mathbb{R}_+,m)$ by

$$(E_tf)(s)=f(st),\quad 0<s<\infty.$$

With $X(\mathcal{R},\mu)$ and $\overline{X}(\mathbb{R}_+,m)$ as above, let $h(t,X)$ denote the norm of $E_{1/t}$ as an operator on $\overline{X}(\mathbb{R}_+,m)$. By [1], Chapter 3, Proposition 5.11, for each $t>0$, the operator $E_t$ is bounded on $\overline{X}(\mathbb{R}_+,m)$ and the function $h(\cdot,X)$ is increasing (and hence, measurable) and submultiplicative on $(0,\infty)$. The indices of $h(\cdot,X)$ are called the *Boyd indices* of the rearrangement-invariant Banach function space $X(\mathcal{R},\mu)$ and are denoted by

$$\alpha(X):=\alpha(h(\cdot,X)),\quad \beta(X):=\beta(h(\cdot,X)).$$

Following [1], Chapter 2, Definition 5.1, for each finite value $t$ let $E\subset\mathcal{R}$ be such that $\mu(E)=t$ and let

$$\varphi_X(t):=\|\chi_E\|_X.$$

The function $\varphi_X$ so defined is called the fundamental function of the rearrangement-invariant Banach function space $X(\mathcal{R},\mu)$. Following [11], page 271 (see also [8], page 28), for a given rearrangement-invariant Banach function space $X(\mathcal{R},\mu)$ with the fundamental function $\varphi_X$, let us consider the function

$$M(t,X):=\sup_{0<x<\infty}\frac{\varphi_X(tx)}{\varphi_X(x)},\quad t\in(0,\infty).$$

It is easy to check that this function is increasing (and hence, measurable) and submultiplicative on $(0,\infty)$. The indices of $M(\cdot,X)$ are called the *Zippin* (or fundamental) *indices* of the rearrangement-invariant Banach function space $X(\mathcal{R},\mu)$ and are denoted by

$$p(X):=\alpha(M(\cdot,X)),\quad q(X):=\beta(M(\cdot,X)).$$

It is easy to see that $M(t,X) \leqslant h(t,X)$ for $t \in (0,\infty)$. Combining this inequality with [1], Chapter 3, Proposition 5.13, we conclude that

$$M(t,X) \leqslant h(t,X) \leqslant \max\{1,t\}, \quad t \in (0,\infty). \tag{2.3}$$

It follows from (2.1)–(2.3) that

$$0 \leqslant \alpha(X) \leqslant p(X) \leqslant q(X) \leqslant \beta(X) \leqslant 1.$$

The lower Boyd (or Zippin) index is said to be trivial if $\alpha(X) = 0$ (or $p(X) = 0$). Analogously, the upper Boyd (or Zippin) index is said to be trivial if $\beta(X) = 1$ (or $q(X) = 1$).

Note that for the Lebesgue spaces $L^p(\mathcal{R},\mu)$, $1 \leqslant p \leqslant \infty$, all these indices are equal to $1/p$. Hence the Lebesgue space $L^p(\mathcal{R},\mu)$ is reflexive if and only if its Boyd and Zippin indices are nontrivial. In contrast, Theorem 1.1 asserts that an analogous result fails for more general Lorentz Gamma spaces $\Gamma_{p,w}(\mathcal{R},\mu)$.

## 3. Lorentz Gamma spaces

**3.1. Lorentz Gamma spaces are rearrangement-invariant Banach function spaces.** The following lemma contains well known information on Lorentz Gamma spaces (see e.g. [4], Section 2 and [6], Section 0). We give its proof here for completeness.

**Lemma 3.1.** *Let $(\mathcal{R},\mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. If $1 \leqslant p < \infty$ and $w \in \mathcal{D}_p$, then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R},\mu)$ is a rearrangement-invariant Banach function space. The fundamental function of $\Gamma_{p,w}(\mathcal{R},\mu)$ is given by*

$$\varphi_{\Gamma_{p,w}}(t) = (W(t) + W_p(t))^{1/p}, \quad t \in (0,\infty). \tag{3.1}$$

Proof. Axioms (A1)–(A3) in the definition of a Banach function space are satisfied in view of [1], Chapter 2, Proposition 3.2 and Theorem 3.4.

If $\mu(E) = 0$, then $\chi_E^{**}(x) = 0$ for $x \in (0,\infty)$ and $\|\chi_E\|_{\Gamma_{p,w}} = 0$. Let $E \subset \mathcal{R}$ be a set of measure $t > 0$. Then $\chi_E^* = \chi_{[0,t]}$ and $\chi_E^{**}(x) = \max\{1, t/x\}$. Hence

$$\varphi_{\Gamma_{p,w}}(t) = \|\chi_E\|_{\Gamma_{p,w}} = \left(\int_0^t w(x)\,\mathrm{d}x + \int_t^\infty (t/x)^p w(x)\,\mathrm{d}x\right)^{1/p} = (W(t) + W_p(t))^{1/p}.$$

If $w \in \mathcal{D}_p$, then the right-hand side of the above inequality is finite. Therefore $\mu(E) < \infty \Rightarrow \|\chi_E\|_{\Gamma_{p,w}} < \infty$. Thus, Axiom (A4) is satisfied and the fundamental function of the space $\Gamma_{p,w}$ is given by (3.1).

Since $w \in \mathcal{D}_p$, we have $0 < W(t) < \infty$ for all $t > 0$. Let $E \subset \mathcal{R}$ be a set of positive measure $t$ and $f \in \Gamma_{p,w}(\mathcal{R}, \mu)$. By the Hardy-Littlewood inequality (see [1], Chapter 2, inequality (3.1) and Theorem 2.2),

$$\int_E |f(y)| \,\mathrm{d}\mu(y) \leqslant \int_0^t f^*(x) \,\mathrm{d}x = tf^{**}(t). \tag{3.2}$$

Taking into account that $f^{**}$ is decreasing (see [1], Chapter 2, Proposition 3.2), we have

$$\begin{aligned} tf^{**}(t) &= \frac{t}{(W(t))^{1/p}} f^{**}(t) \Big( \int_0^t w(x) \,\mathrm{d}x \Big)^{1/p} \\ &\leqslant \frac{t}{(W(t))^{1/p}} \Big( \int_0^t (f^{**}(x))^p w(x) \,\mathrm{d}x \Big)^{1/p} \leqslant \frac{t}{(W(t))^{1/p}} \|f\|_{\Gamma_{p,w}}. \end{aligned} \tag{3.3}$$

It follows from (3.2)–(3.3) that

$$\mu(E) < \infty \Rightarrow \int_E |f(y)| \,\mathrm{d}\mu(y) \leqslant C_E \|f\|_{\Gamma_{p,w}},$$

where $C_E = \mu(E)/(W(\mu(E)))^{1/p}$. Thus, Axiom (A5) is satisfied. □

**3.2. Reflexivity of Lorentz Gamma spaces.** Now we recall necessary and sufficient conditions for the reflexivity of Lorentz Gamma spaces.

**Lemma 3.2.** *Let $(\mathcal{R}, \mu)$ be a totally $\sigma$-finite nonatomic measure space such that $\mu(\mathcal{R}) = \infty$. Suppose that $w \in \mathcal{D}_p$ is a weight such that*

$$\int_0^x w(t) t^{-p} \,\mathrm{d}t = \infty \quad \forall\, x \in (0, \infty). \tag{3.4}$$

*Then the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R}, \mu)$ is reflexive if and only if*

$$W(\infty) = \int_0^\infty w(t) \,\mathrm{d}t = \infty \quad \text{and} \quad V(\infty) = \int_0^\infty v(t) \,\mathrm{d}t = \infty,$$

*where $1/p + 1/p' = 1$ and*

$$v(t) = \frac{t^{p'-1} W(t) W_p(t)}{(W(t) + W_p(t))^{p'+1}}, \quad t \in (0, \infty).$$

The above lemma is proved in [3], Lemma 6.4 in the case of $(\mathcal{R},\mu) = (\mathbb{R}_+, m)$. Its proof is a combination of three ingredients. Two of these ingredients, namely, see [1], Chapter 1.1, Corollary 4.4 and [4], Theorem A, are proved for the measure spaces $(\mathcal{R},\mu)$ under consideration. It remains to observe that, although the last ingredient (see [6], Proposition 1.1 (1)) is proved for $(\mathbb{R}_+, m)$, an inspection of its proof shows that it is also valid for $(\mathcal{R},\mu)$.

## 4. Proof of the main results

**4.1. Proof of Theorem 1.1.** Taking into account (1.1), a straightforward calculation gives that

$$W(x) = \frac{1}{1-\log x} \quad \text{for } x \in (0,1], \tag{4.1}$$

$$W_p(x) = \frac{x^p}{1+\log x} \quad \text{for } x \in [1,\infty). \tag{4.2}$$

Thus $w \in \mathcal{D}_p$. By Lemma 3.1, the Lorentz Gamma space $\Gamma_{p,w}(\mathcal{R},\mu)$ is a rearrangement-invariant Banach function space.

Let us show that the space $\Gamma_{p,w}(\mathcal{R},\mu)$ is reflexive. Using L'Hôpital's rule, one can easily derive from (1.1) that

$$\lim_{x\to\infty} \frac{W(x)}{x^p(1+\log x)^{-2}} = \frac{1}{p}, \tag{4.3}$$

$$\lim_{x\to 0} \frac{W_p(x)}{(1-\log x)^{-2}} = \lim_{x\to 0} \frac{\int_x^\infty t^{-p} w(t)\,dt}{x^{-p}(1-\log x)^{-2}} = \frac{1}{p}. \tag{4.4}$$

It follows from (4.2) and (4.3) that

$$W(\infty) = \lim_{x\to\infty} W(x) = \infty \tag{4.5}$$

and

$$\begin{aligned} v(x) &= \frac{x^{p'-1} W(x) W_p(x)}{(W(x)+W_p(x))^{p'+1}} \\ &= x^{p'-1+2p-p(p'+1)} \frac{1+o(1)}{p(1+\log x)^3} \Big( \frac{1+o(1)}{p(1+\log x)^2} + \frac{1}{(1+\log x)} \Big)^{-(p'+1)} \\ &= \frac{1}{px} (1+\log x)^{p'-2} (1+o(1)) \Big( \frac{1+o(1)}{p(1+\log x)} + 1 \Big)^{-(p'+1)} \\ &= \frac{1}{px} (1+\log x)^{p'-2} (1+o(1)) \quad \text{as } x \to \infty. \end{aligned} \tag{4.6}$$

Since $p' > 1$, i.e., $2 - p' < 1$, it follows from (4.6) that

$$V(\infty) = \int_0^\infty v(x)\,\mathrm{d}x \geqslant \text{const} \int_1^\infty \frac{1}{x(1+\log x)^{2-p'}}\,\mathrm{d}x = \infty. \tag{4.7}$$

It is easy to see that (3.4) holds for $w$ defined by (1.1). Then it follows from (4.5) and (4.7) that the conditions of Lemma 3.2 are satisfied and hence the space $\Gamma_{p,w}(\mathcal{R},\mu)$ is reflexive.

Let us calculate the Boyd and Zippin indices of the space $\Gamma_{p,w}(\mathcal{R},\mu)$. By Lemma 3.1, the fundamental function of $\Gamma_{p,w}(\mathcal{R},\mu)$ is given by the formula

$$\varphi_{\Gamma_{p,w}}(x) = (W(x) + W_p(x))^{1/p}, \quad x \in (0,\infty). \tag{4.8}$$

If $t \in [1,\infty)$, then it follows from (4.2) and (4.3) that

$$\begin{aligned} \sup_{0<x<\infty} \frac{W(tx)+W_p(tx)}{W(x)+W_p(x)} &\geqslant \lim_{x\to\infty} \frac{W(tx)+W_p(tx)}{W(x)+W_p(x)} \\ &= t^p \lim_{x\to\infty}\Big(\frac{1+\log x}{1+\log(tx)}\Big)^2 \lim_{x\to\infty} \frac{1+o(1)+p(1+\log(tx))}{1+o(1)+p(1+\log x)} = t^p. \end{aligned} \tag{4.9}$$

If $t \in (0,1)$, then it follows from (4.1) and (4.4) that

$$\begin{aligned} \sup_{0<x<\infty} \frac{W(tx)+W_p(tx)}{W(x)+W_p(x)} &\geqslant \lim_{x\to 0} \frac{W(tx)+W_p(tx)}{W(x)+W_p(x)} \\ &= \lim_{x\to 0}\Big(\frac{1-\log x}{1-\log(tx)}\Big)^2 \lim_{x\to 0} \frac{p(1-\log(tx))+1+o(1)}{p(1-\log x)+1+o(1)} = 1. \end{aligned} \tag{4.10}$$

Combining (4.8)–(4.10), we conclude that for $t \in (0,\infty)$,

$$\begin{aligned} M(t,\Gamma_{p,w}) &= \sup_{0<x<\infty} \frac{\varphi_{\Gamma_{p,w}}(tx)}{\varphi_{\Gamma_{p,w}}(x)} = \sup_{0<x<\infty}\Big(\frac{W(tx)+W_p(tx)}{W(x)+W_p(x)}\Big)^{1/p} \\ &\geqslant \max\{1,t\}. \end{aligned} \tag{4.11}$$

Then it follows from (2.3) and (4.11) that

$$M(t,\Gamma_{p,w}) = h(t,\Gamma_{p,w}) = \max\{1,t\}, \quad t \in (0,\infty).$$

Applying (2.1) to the submultiplicative functions $M(\cdot,\Gamma_{p,w})$ and $h(\cdot,\Gamma_{p,w})$, we get

$$\alpha(\Gamma_{p,w}) = p(\Gamma_{p,w}) = \lim_{t\to 0}\frac{\log\max\{1,t\}}{\log t} = 0,$$
$$\beta(\Gamma_{p,w}) = q(\Gamma_{p,w}) = \lim_{t\to\infty}\frac{\log\max\{1,t\}}{\log t} = 1,$$

which completes the proof. □

**4.2. Proof of Corollary 1.2.** In view of Theorem 1.1, the Lorentz Gamma space $\Gamma_{p,w}(\mathbb{R}^n, m)$ is a reflexive rearrangement-invariant Banach function space such that its Boyd indices are trivial, that is, $\alpha(\Gamma_{p,w}) = 0$ and $\beta(\Gamma_{p,w}) = 1$. Since $\beta(\Gamma_{p,w}) = 1$, the Hardy-Littlewood maximal operator $M$ is unbounded on the space $\Gamma_{p,w}(\mathbb{R}^n, m)$ in view of the Lorentz-Shimogaki theorem (see e.g. [1], Chapter 3, Theorem 5.17). By [1], Chapter 3, Proposition 5.13, $\beta(\Gamma'_{p,w}) = 1 - \alpha(\Gamma_{p,w}) = 1$. Now, applying [1], Chapter 3, Theorem 5.17 to the associate space $\Gamma'_{p,w}(\mathbb{R}^n, m)$, we conclude that the operator $M$ is unbounded on the space $\Gamma'_{p,w}(\mathbb{R}^n, m)$. □

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