Czechoslovak Mathematical Journal, 71 (146) (2021), 1189–1198

BLOW-UP FOR 3-D COMPRESSIBLE ISENTROPIC NAVIER-STOKES-POISSON EQUATIONS

Shanshan Yang, Hangzhou, Hongbiao Jiang, Yinhe Lin, Lishui

Received August 12, 2020. Published online June 28, 2021.

Abstract. We study compressible isentropic Navier-Stokes-Poisson equations in $\mathbb{R}^3$. With some appropriate assumptions on the density, velocity and potential, we show that the classical solution of the Cauchy problem for compressible unipolar isentropic Navier-Stokes-Poisson equations with attractive forcing will blow up in finite time. The proof is based on a contradiction argument, which relies on proving the conservation of total mass and total momentum.

Keywords: compressible isentropic Navier-Stokes-Poisson equation; unipolar; energy solution; blow-up

MSC 2020: 35Q35, 35B44

1. Introduction

This paper is concerned with Cauchy problems for the compressible unipolar isentropic Navier-Stokes-Poisson equation in $\mathbb{R}^3$:

$$(1.1)\qquad \begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ (\varrho u)_t + \operatorname{div}(\varrho u \otimes u) + \nabla P(\varrho) = \mu \Delta u + (\mu+\lambda)\nabla \operatorname{div} u + a\varrho \nabla \Phi, \\ -\Delta \Phi = \varrho, \quad x \in \mathbb{R}^3, \quad t > 0, \\ \varrho(0,x) = \varrho_0(x), \quad u(0,x) = u_0(x). \end{cases}$$

The unknown functions ϱ, u and Φ denote the density, velocity field and potential of underlying force, respectively, and P is the pressure, which admits the form

$$(1.2)\qquad P(\varrho) = \varrho^\gamma,$$

where $\gamma > 1$ represents the adiabatic constant in the isentropic regime.

The third author was supported by NSFC (11771194) and Zhejiang Province Science Foundation (LY18A010008).

DOI: 10.21136/CMJ.2021.0347-20

The symbols μ and λ denote the coefficient of viscosity and the second coefficient of viscosity, respectively, which satisfy

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geqslant 0. \tag{1.3}$$

The coefficient a in $(1.1)_2$ can be used to describe the property of the forcing, thus, it is repulsive if $a > 0$ and attractive if $a < 0$, and we focus on the latter one in this paper.

If $a = 0$, then system $(1.1)_1$, $(1.1)_2$ and $(1.1)_4$ is called *compressible Navier-Stokes equations*, and there are many blow-up results. The first result is due to Xin (see [12]), in which he proved that smooth solutions for the compressible Navier-Stokes equations for nonbarotropic flows in the absence of heat conduction in any space dimension will blow up in finite time, under the assumption the initial data have compact support. And this result was generalized by Cho and Jin (see [1]) to fluids with positive heat conductivity. Without compact supported initial data assumption, but with the assumption that the data decrease rapidly when approaching the infinity, Rozanova in [10] could also prove finite time blow-up for $n \geqslant 3$ (n denotes the space dimension) and $\gamma \geqslant 2n/(n+2)$. Xin and Yan in [13] showed that any classical solution to the compressible Navier-Stokes system without heat conductivity will blow up in finite time, if the initial density has local vacuum in a bounded region. Lai in [8] established a blow-up result for the classical solution of the compressible isentropic Navier-Stokes system by assuming that the gradient of velocity satisfies some decay constraint and the initial total momentum does not vanish. Jiu in [7] studied the full compressible Navier-Stokes system and compressible Navier-Stokes system with constant viscosity or degenerate viscosity in any space dimension, and a blow-up result for smooth solution was established. Let us mention that he does not have to assume that the initial data should have compact support or that there is vacuum in bounded regions. However, there is some limitation for the adiabatic constant, which can be removed by [11]. For more detailed introduction about the blow-up result for the compressible Navier-Stokes system, we refer to [2] and [3].

Remark 1.1. We remark that there are many finite blow-up results for all other kinds of fluid models, see [4], [5], [6], [9] and so on.

In the present work we are devoted to proving a blow-up for the classical solution of the compressible isentropic Navier-Stokes-Poisson equations in $\mathbb{R}^3$ under some constraint on the density, gradient of velocity and initial momentum. We borrow the idea of Lai (see [8]) to establish the finite time blow-up result.

Definition 1.1. Let $T>0$. We say $(\varrho(t,x),u(t,x),\Phi(t,x))$ is a classical solution to the compressible isentropic Navier-Stokes-Poisson system (1.1) over $[0,T)\times\mathbb{R}^3$ if $\varrho\in C^1([0,T)\times\mathbb{R}^3)$, $u\in C^1([0,T),C^2(\mathbb{R}^3))$, and $\Phi\in C^1([0,T),C^2(\mathbb{R}^3))$ satisfy system (1.1) pointwise over $[0,T)\times\mathbb{R}^3$.

Our main result reads:

Theorem 1.1. *Let* (μ,λ) *satisfy* (1.3) *and* $a<0$. *Assume that* ϱ, $\sum_{\beta\leqslant 1}|\partial^\beta u|$, *and* $\sum_{\beta\leqslant 2}|\partial^\beta \Phi|$ *decay to* 0 *when* $|x|\to\infty$ *and* (ϱ,u) *satisfy*

$$\int_{\mathbb{R}^3}\varrho(t,x)\,\mathrm{d}x<\infty,\qquad \int_{\mathbb{R}^3}|\nabla u(t,x)|\,\mathrm{d}x<\infty,\quad t\geqslant 0. \tag{1.4}$$

For the initial data, we assume that

$$\varrho_0(x)\in L^1(\mathbb{R}^3)\cap L^\gamma(\mathbb{R}^3),\quad \sqrt{\varrho_0(x)}u_0(x)\in L^2(\mathbb{R}^3),\quad \Phi_0(x)\in \dot H^1(\mathbb{R}^3), \tag{1.5}$$

and the initial total momentum does not vanish,

$$\left|\int_{\mathbb{R}^3}\varrho_0(x)u_0(x)\,\mathrm{d}x\right|\neq 0. \tag{1.6}$$

Then the classical solution of (1.1) *will blow up in finite time.*

2. Preliminaries

Before demonstrating the proof of our main result, we give four preliminary lemmas.

Lemma 2.1 (Energy estimate)**.** *Let* $a<0$. *Assuming* (ϱ,u,Φ) *are classical solutions of system* (1.1) *as stated in Theorem* 1.1, *then it holds for* $t\in(0,T^*)$

$$\begin{aligned}E(t)&+\mu\int_0^t\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}s+(\mu+\lambda)\int_0^t\int_{\mathbb{R}^3}|\operatorname{div}u|^2\,\mathrm{d}x\,\mathrm{d}s\\&=\frac12\int_{\mathbb{R}^3}\varrho_0(x)|u_0(x)|^2\,\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\varrho_0^\gamma(x)\,\mathrm{d}x-\frac a2\int_{\mathbb{R}^3}|\nabla\Phi(0,x)|^2\,\mathrm{d}x,\end{aligned} \tag{2.1}$$

where

$$E(t)=\frac12\int_{\mathbb{R}^3}\varrho|u|^2\,\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\varrho^\gamma\,\mathrm{d}x-\frac a2\int_{\mathbb{R}^3}|\nabla\Phi|^2\,\mathrm{d}x \tag{2.2}$$

denotes the total energy and T^* *is the maximum existing time for the solution.*

P r o o f. Multiplying $(1.1)_2$ by u,

$$(\varrho u)_t\cdot u+\operatorname{div}(\varrho u\otimes u)\cdot u+\nabla P(\varrho)\cdot u=\mu\Delta u\cdot u+(\mu+\lambda)\nabla\operatorname{div}u\cdot u+a\varrho\nabla\Phi\cdot u. \tag{2.3}$$

By using $(1.1)_1$ we have

$$(\varrho u)_t\cdot u=-\operatorname{div}(\varrho u)|u|^2+\varrho u_t\cdot u. \tag{2.4}$$

Hence, we deduce that

$$\operatorname{div}(\varrho u\otimes u)\cdot u=\operatorname{div}(\varrho u)|u|^2+\frac{1}{2}\varrho u\cdot\nabla u^2.$$

Further,

$$(\varrho u)_t\cdot u+\operatorname{div}(\varrho u\otimes u)\cdot u=\frac{1}{2}\varrho\partial_t u^2+\frac{1}{2}\varrho u\cdot\nabla u^2=\frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}\varrho|u|^2+\frac{1}{2}\operatorname{div}(\varrho u|u|^2). \tag{2.5}$$

From $(1.1)_1$ and (1.2) one has

$$\begin{aligned}\nabla P(\varrho)\cdot u&=\nabla(\varrho^\gamma)\cdot u=\gamma\varrho^{\gamma-1}\nabla\varrho\cdot u=\frac{\gamma}{\gamma-1}\nabla(\varrho^{\gamma-1})\cdot(\varrho u)\\&=\frac{\gamma}{\gamma-1}\operatorname{div}(\varrho^\gamma u)-\frac{\gamma}{\gamma-1}\varrho^{\gamma-1}\operatorname{div}(\varrho u)\\&=\frac{\gamma}{\gamma-1}\operatorname{div}(\varrho^\gamma u)+\frac{\gamma}{\gamma-1}\varrho^{\gamma-1}\varrho_t=\frac{\gamma}{\gamma-1}\operatorname{div}(\varrho^\gamma u)+\frac{\mathrm{d}}{\mathrm{dt}}\frac{\varrho^\gamma}{\gamma-1}.\end{aligned} \tag{2.6}$$

By integration by parts we come to

$$\mu\Delta u\cdot u+(\mu+\lambda)\nabla\operatorname{div}u\cdot u=\mu\operatorname{div}(\nabla uu)-\mu|\nabla u|^2+\operatorname{div}(\operatorname{div}uu)-(\mu+\lambda)|\operatorname{div}u|^2. \tag{2.7}$$

From $(1.1)_3$ we have

$$\begin{aligned}a\varrho\nabla\Phi\cdot u&=a\operatorname{div}(\varrho\Phi u)-a\Phi\operatorname{div}(\varrho u)=a\operatorname{div}(\varrho\Phi u)+a\Phi\varrho_t\\&=a\operatorname{div}(\varrho\Phi u)-a\Phi\Delta\Phi_t=a\operatorname{div}(\varrho\Phi u)-a\operatorname{div}(\Phi\nabla\Phi_t)+\frac{\mathrm{d}}{\mathrm{dt}}\frac{a}{2}|\nabla\Phi|^2.\end{aligned} \tag{2.8}$$

By combing (2.3)–(2.8) and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\frac{1}{2}\varrho|u|^2\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\frac{\varrho^\gamma}{\gamma-1}\,\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^3}\frac{a}{2}|\Phi|^2\,\mathrm{d}x\\&\qquad=-\int_{\mathbb{R}^3}\mu|u|^2\,\mathrm{d}x-(\mu+\lambda)\int_{\mathbb{R}^3}|\operatorname{div}u|^2\,\mathrm{d}x.\end{aligned} \tag{2.9}$$

Integrating this over $[0,T]$ yields

$$\begin{aligned}&E(T)+\mu\int_0^T\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t+(\mu+\lambda)\int_0^T\int_{\mathbb{R}^3}|\operatorname{div}u|^2\,\mathrm{d}x\,\mathrm{d}t\\&\qquad=\frac{1}{2}\int_{\mathbb{R}^3}\varrho_0(x)|u_0(x)|^2\,\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\varrho_0^\gamma(x)\,\mathrm{d}x-\frac{a}{2}\int_{\mathbb{R}^3}|\nabla\Phi(0,x)|^2\,\mathrm{d}x\end{aligned} \tag{2.10}$$

and the proof of Lemma 2.1 is finished. □

Lemma 2.2. *Let $f(x)$ be a continuous and nonnegative function. If*

$$\int_0^{\infty} f(x)\,\mathrm{d}x < \infty,$$

then there exists a sequence $x_n \to \infty$ such that

$$f(x_n) \to 0. \tag{2.11}$$

P r o o f. Since $\int_0^{\infty} f(x)\,\mathrm{d}x < \infty$, then for any integer $n > 0$, there exists a constant $M(n) > 0$ such that if $A_1, A_2 > M(n)$, then we have

$$\int_{A_1}^{A_2} f(x)\,\mathrm{d}x < \frac{1}{n}.$$

In particular, we may choose $A_1 = n + M(n)$ and $A_2 = 2n + M(n)$ to get

$$\int_{n+M(n)}^{2n+M(n)} f(x) < \frac{1}{n}. \tag{2.12}$$

By the mean value theorem for integrals, we get a number $x_n \in [n+M(n), 2n+M(n)]$ such that

$$0 \leqslant f(x_n) < \frac{1}{n^2}, \tag{2.13}$$

which in turn leads to $x_n \to \infty$ $(n \to \infty)$ and

$$f(x_n) \to 0, \tag{2.14}$$

and we finish the proof of Lemma 2.2. □

Next we give two lemmas stating the conservation of the total mass and momentum, respectively.

Lemma 2.3 (Conservation of mass). *Let (μ, λ) satisfy* (1.3), *then for system* (1.1) *we have*

$$\int_{\mathbb{R}^3} \varrho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \varrho_0(x)\,\mathrm{d}x, \quad t > 0. \tag{2.15}$$

P r o o f. Let B_R represent a ball in $\mathbb{R}^3$ centered at the origin with radius R. Integrating $(1.1)_1$ over B_R, we have

$$\int_{|x|\leqslant R} \varrho_t(t,x)\,\mathrm{d}x = -\int_{|x|\leqslant R} \operatorname{div}(\varrho u)\,\mathrm{d}x. \tag{2.16}$$

Integrating this over $[0,t]$ yields

$$\Big|\int_{|x|\leqslant R} \varrho(t,x) - \varrho_0(x)\,\mathrm{d}x\Big| = \Big|\int_0^t \int_{|x|\leqslant R} \operatorname{div}(\varrho u)\,\mathrm{d}x\,\mathrm{d}t\Big|. \tag{2.17}$$

Noting that

$$\int_0^\infty \Big|\int_{|x|\leqslant R} \operatorname{div}(\varrho u)\,\mathrm{d}x\Big|\,\mathrm{d}R = \int_0^\infty \Big|\int_{|x|=R} \varrho u\cdot\frac{x}{R}\,\mathrm{d}x\Big|\,\mathrm{d}R,$$

the Hölder inequality implies

$$\int_0^\infty \Big|\int_{|x|=R} \varrho u\,\mathrm{d}x\Big|\,\mathrm{d}R \leqslant \int_0^\infty \Big(\int_{|x|=R} (\sqrt{\varrho})^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{|x|=R} (\sqrt{\varrho})^2 u^2\,\mathrm{d}x\Big)^{1/2}\,\mathrm{d}R,$$

which means

$$\int_0^\infty \Big|\int_{|x|\leqslant R} \operatorname{div}(\varrho u)\,\mathrm{d}x\Big|\,\mathrm{d}R \leqslant \|\sqrt{\varrho}\|_{L^2(\mathbb{R}^3)}\|\sqrt{\varrho}u\|_{L^2(\mathbb{R}^3)}$$

and

$$\int_0^t\int_0^\infty \Big|\int_{|x|\leqslant R} \operatorname{div}(\varrho u)\,\mathrm{d}x\Big|\,\mathrm{d}R\,\mathrm{d}t \leqslant t\|\sqrt{\varrho}\|_{L^2(\mathbb{R}^3)}\|\sqrt{\varrho}u\|_{L^2(\mathbb{R}^3)}.$$

Thus

$$\int_0^\infty \Big|\int_{|x|\leqslant R} \varrho(t,x) - \varrho_0(x)\,\mathrm{d}x\Big|\,\mathrm{d}R \leqslant t\|\sqrt{\varrho}\|_{L^2}\|\sqrt{\varrho}u\|_{L^2}$$

and hence, by Lemma 2.2, there exists a sequence $R_n \to \infty$ such that

$$\lim_{n\to\infty} \int_{|x|\leqslant R_n} (\varrho(t,x) - \varrho_0(x))\,\mathrm{d}x = 0.$$

The proof of Lemma 2.3 is finished. □

Lemma 2.4 (Conservation of momentum)**.** *Let* (μ,λ) *satisfy* (1.3), *then for system* (1.1) *we have*

$$\int_{\mathbb{R}^3} \varrho(t,x)u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \varrho_0(x)u_0(x)\,\mathrm{d}x, \quad t>0. \tag{2.18}$$

P r o o f. The proof is parallel to that of the mass conservation. Integrating $(1.1)_2$ over B_R, we have

$$\int_{|x|\leqslant R}(\varrho u)_t\,\mathrm{d}x+\int_{|x|\leqslant R}\operatorname{div}(\varrho u\otimes u)\,\mathrm{d}x+\int_{|x|\leqslant R}\nabla P(\varrho)\,\mathrm{d}x = \mu\int_{|x|\leqslant R}\Delta u\,\mathrm{d}x+(\mu+\lambda)\int_{|x|\leqslant R}\nabla\operatorname{div} u\,\mathrm{d}x+a\int_{|x|\leqslant R}\varrho\nabla\Phi\,\mathrm{d}x. \tag{2.19}$$

Note that

$$\int_0^\infty\biggl|\int_{|x|\leqslant R}\operatorname{div}(\varrho u\otimes u)\,\mathrm{d}x\biggr|\,\mathrm{d}R\leqslant\int_0^\infty\biggl|\int_{|x|=R}\varrho|u|^2\,\mathrm{d}x\biggr|\,\mathrm{d}R\leqslant\|\varrho|u|^2\|_{L^1(\mathbb{R}^3)}, \tag{2.20}$$

$$\int_0^\infty\biggl|\int_{|x|\leqslant R}\nabla P(\varrho)\,\mathrm{d}x\biggr|\,\mathrm{d}R\leqslant\|P\|_{L^1(\mathbb{R}^3)}, \tag{2.21}$$

$$\begin{aligned}\int_0^\infty\biggl|\mu\int_{|x|\leqslant R}\Delta u\,\mathrm{d}x+(\mu+\lambda)\int_{|x|\leqslant R}\nabla\operatorname{div} u\,\mathrm{d}x\biggr|\,\mathrm{d}R\\ \leqslant\int_0^\infty(\lambda+2\mu)\biggl|\int_{|x|=R}\nabla u\,\mathrm{d}x\biggr|\,\mathrm{d}R\leqslant(\lambda+2\mu)\|\nabla u\|_{L^1(\mathbb{R}^3)},\end{aligned} \tag{2.22}$$

Particularly, for $(1.1)_3$ we have

$$a\int_{|x|\leqslant R}\varrho\nabla\Phi\,\mathrm{d}x=-a\int_{|x|\leqslant R}\Delta\Phi\nabla\Phi\,\mathrm{d}x$$

and hence

$$\begin{aligned}a\int_0^\infty\biggl|\int_{|x|\leqslant R}\varrho\nabla\Phi\,\mathrm{d}x\biggr|\,\mathrm{d}R&\leqslant-a\int_0^\infty\int_{|x|\leqslant R}\varrho|\nabla\Phi|\,\mathrm{d}x\,\mathrm{d}R\\&=-a\int_0^\infty\int_{|x|\leqslant R}\Delta\Phi|\nabla\Phi|^2\,\mathrm{d}x\,\mathrm{d}R\\&=-Ca\int_0^\infty\int_{|x|=R}|\nabla\Phi|\,\mathrm{d}\sigma\,\mathrm{d}R\\&\leqslant C\|\nabla\Phi\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{2.23}$$

Therefore, by combining (2.20)–(2.23) and integrating (2.19) over $[0,t]$, we get

$$\begin{aligned}\int_0^\infty\biggl|\int_{|x|\leqslant R}\varrho(t,x)u(t,x)-\varrho_0(x)u_0(x)\,\mathrm{d}x\biggr|\,\mathrm{d}R\\ \leqslant t\|\varrho|u|^2\|_{L^1(\mathbb{R}^3)}+t\|P\|_{L^1(\mathbb{R}^3)}+t(\lambda+2\mu)\|\nabla u\|_{L^1(\mathbb{R}^3)}+t\|\nabla\Phi\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{2.24}$$

Then for a fixed $t>0$, (2.1) yields

$$\int_0^\infty\biggl|\int_{|x|\leqslant R}\varrho(t,x)u(t,x)-\varrho_0(x)u_0(x)\,\mathrm{d}x\biggr|\,\mathrm{d}R<\infty,$$

which implies by combining with Lemma 2.2 that there exists a sequence $R_n \to \infty$ such that

$$\lim_{n\to\infty} \int_{|x|\leqslant R_n} (\varrho(t,x)u(t,x) - \varrho_0(x)u_0(x))\,\mathrm{d}x = 0.$$

Lemma 2.4 follows. $\square$

3. The proof of Theorem 1.1

Now we are going to prove Theorem 1.1. We first consider the case $1 < \gamma < \frac{6}{5}$. Note that for $0 < \alpha < 1$, it holds

$$\varrho u = (\varrho^{1/2} u)^{\alpha} u^{1-\alpha} \varrho^{(2-\alpha)/2}.$$

Integrating this over $\mathbb{R}^3$ yields

$$\int_{\mathbb{R}^3} \varrho u\,\mathrm{d}x = \int_{\mathbb{R}^3} (\varrho^{1/2} u)^{\alpha} u^{1-\alpha} \varrho^{(2-\alpha)/2}\,\mathrm{d}x.$$

The Hölder inequality implies

$$\begin{aligned} (3.1)\qquad \left|\int_{\mathbb{R}^3} \varrho u\,\mathrm{d}x\right| &\leqslant \left(\int_{\mathbb{R}^3} (\sqrt{\varrho} u)^2\,\mathrm{d}x\right)^{\alpha/2} \left(\int_{\mathbb{R}^3} u^6\,\mathrm{d}x\right)^{(1-\alpha)/6} \left(\int_{\mathbb{R}^3} \varrho^{\gamma}\,\mathrm{d}x\right)^{(2-\alpha)/2\gamma} \\ &= \|\sqrt{\varrho} u\|^{\alpha}_{L^2(\mathbb{R}^3)} \|u\|^{1-\alpha}_{L^6(\mathbb{R}^3)} \|\varrho\|^{(2-\alpha)/2}_{L^{\gamma}(\mathbb{R}^3)}. \end{aligned}$$

We use the Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C\|Du\|_{L^p(\mathbb{R}^n)},$$

where $1/p^* = 1/p - 1/n$, $p^* > p$. Taking $p = 2$, $n = 3$ we have

$$\|u\|^{1-\alpha}_{L^6(\mathbb{R}^3)} \leqslant C\|\nabla u\|^{1-\alpha}_{L^2(\mathbb{R}^3)}$$

and hence

$$(3.2)\qquad \left|\int_{\mathbb{R}^3} \varrho u\,\mathrm{d}x\right| \leqslant C\|\sqrt{\varrho} u\|^{\alpha}_{L^2(\mathbb{R}^3)} \|\nabla u\|^{1-\alpha}_{L^2(\mathbb{R}^3)} \|\varrho\|^{(2-\alpha)/2}_{L^{\gamma}(\mathbb{R}^3)},$$

where $C > 0$ denotes a positive constant which may have different values from line to line.

We now come to the case $\gamma \geqslant \frac{6}{5}$. Obviously we have

$$(3.3)\qquad \left|\int_{\mathbb{R}^3} \varrho u\,\mathrm{d}x\right| \leqslant \left(\int_{\mathbb{R}^3} |\varrho|^{6/5}\,\mathrm{d}x\right)^{5/6} \left(\int_{\mathbb{R}^3} |u|^6\,\mathrm{d}x\right)^{1/6} = \|\varrho\|_{L^{6/5}(\mathbb{R}^3)} \|u\|_{L^6(\mathbb{R}^3)}.$$

By the interpolation inequality one has

$$\|\varrho\|_{L^{6/5}(\mathbb{R}^3)} \leqslant \|\varrho\|_{L^1(\mathbb{R}^3)}^{(5\gamma/6-1)/(\gamma-1)} \|\varrho\|_{L^\gamma(\mathbb{R}^3)}^{\gamma/(6(\gamma-1))}$$

and we then conclude that

$$\left|\int_{\mathbb{R}^3} \varrho u \,\mathrm{d}x\right| \leqslant C\|\varrho\|_{L^1(\mathbb{R}^3)}^{(5\gamma/6-1)/(\gamma-1)} \|\varrho\|_{L^\gamma(\mathbb{R}^3)}^{\gamma/(6(\gamma-1))} \|\nabla u\|_{L^2(\mathbb{R}^3)}. \tag{3.4}$$

From (3.2) and (3.4) we obtain

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^{1-\alpha} \geqslant \frac{1}{C\|\sqrt{\varrho}u\|_{L^2(\mathbb{R}^3)}^{\alpha} \|\varrho\|_{L^\gamma(\mathbb{R}^3)}^{(2-\alpha)/2}} \left|\int_{R^3} \varrho u \,\mathrm{d}x\right| > 0 \tag{3.5}$$

and

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \geqslant \frac{1}{C\|\varrho\|_{L^1(\mathbb{R}^3)}^{(5\gamma/6-1)/(\gamma-1)} \|\varrho\|_{L^\gamma(\mathbb{R}^3)}^{\gamma/(6(\gamma-1))}} \left|\int_{R^3} \varrho u \,\mathrm{d}x\right| > 0, \tag{3.6}$$

respectively, which means by the energy estimate (2.1) that

$$\|\nabla u\|_{L^2(\mathbb{R}^3)}^2 \geqslant C_0 > 0, \tag{3.7}$$

where C_0 denotes a positive constant depending on

$$\frac{1}{2}\int_{\mathbb{R}^3} \varrho_0(x)|u_0(x)|^2 \,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}^3} \varrho_0^\gamma(x) \,\mathrm{d}x - \frac{a}{2}\int_{\mathbb{R}^3} |\nabla\Phi_0(x)|^2 \,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^3} \varrho_0 u_0 \,\mathrm{d}x.$$

We then conclude from the energy estimate (2.1) that

$$E(t) + C_0\mu t \leqslant \frac{1}{2}\int_{\mathbb{R}^3} \varrho_0(x)|u_0(x)|^2 \,\mathrm{d}x + \frac{1}{\gamma-1}\int_{\mathbb{R}^3} \varrho_0^\gamma(x) \,\mathrm{d}x - \frac{a}{2}\int_{\mathbb{R}^3} |\nabla\Phi_0(x)|^2 \,\mathrm{d}x,$$

which implies that there exists $T^* > 0$ such that $E(T^*) = 0$, which contradicts the mass and momentum conservation, and we finish the proof of Theorem 1.1.

References

[1] *Y. Cho, B. J. Jin*: Blow-up of viscous heat-conducting compressible flows. J. Math. Anal. Appl. *320* (2006), 819–826. zbl MR doi
[2] *J. Dong, Q. Ju*: Blow-up of smooth solutions to compressible quantum Navier-Stokes equations. Sci. Sin., Math. *50* (2020), 873–884. (In Chinese.) doi
[3] *J. Dong, J. Zhu, Y. Wang*: Blow-up for the compressible isentropic Navier-Stokes-Poisson equations. Czech. Math. J. *70* (2020), 9–19. zbl MR doi
[4] *I. M. Gamba, M. P. Gualdani, P. Zhang*: On the blowing up of solutions to quantum hydrodynamic models on bounded domains. Monatsh Math. *157* (2009), 37–54. zbl MR doi
[5] *B. Guo, G. Wang*: Blow-up of the smooth solution to quantum hydrodynamic models in $\mathbb{R}^d$. J. Differ. Equations *261* (2016), 3815–3842. zbl MR doi
[6] *B. Guo, G. Wang*: Blow-up of solutions to quantum hydrodynamic models in half space. J. Math. Phys. *58* (2017), Article ID 031505, 11 pages. zbl MR doi
[7] *Q. Jiu, Y. Wang, Z. Xin*: Remarks on blow-up of smooth solutions to the compressible fluid with constant and degenerate viscosities. J. Differ. Equations *259* (2015), 2981–3003. zbl MR doi
[8] *N.-A. Lai*: Blow up of classical solutions to the isentropic compressible Navier-Stokes equations. Nonlinear Anal., Real World Appl. *25* (2015), 112–117. zbl MR doi
[9] *Z. Lei, Y. Du, Q. Zhang*: Singularities of solutions to compressible Euler equations with vacuum. Math. Res. Lett. *20* (2013), 41–50. zbl MR doi
[10] *O. Rozanova*: Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations. J. Differ. Equations *245* (2008), 1762–1774. zbl MR doi
[11] *G. Wang, B. Guo, S. Fang*: Blow-up of the smooth solutions to the compressible Navier-Stokes equations. Math. Methods Appl. Sci. *40* (2017), 5262–5272. zbl MR doi
[12] *Z. Xin*: Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. Commun. Pure Appl. Math. *51* (1998), 229–240. zbl MR doi
[13] *Z. Xin, W. Yan*: On blowup of classical solutions to the compressible Navier-Stokes equations. Commun. Math. Phys. *321* (2013), 529–541. zbl MR doi

Authors' addresses: Shanshan Yang, School of Science, Zhejiang Sci-Tech University, 5 Second Avenue, Xiasha Higher Education Zone, Hangzhou 310018, P. R. China, e-mail: `yss960501@163.com`; Hongbiao Jiang, Yinhe Lin (corresponding author), Institute of Nonlinear Analysis and Department of Mathematics, Lishui University, Lishui 323000, P. R. China, e-mail: `scumatlyh@163.com`, `lsxyhbj@126.com`.