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SCHATTEN CLASS GENERALIZED TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. Let μ be a finite positive measure on the unit disk and let $j \geqslant 1$ be an integer. D. Suárez (2015) gave some conditions for a generalized Toeplitz operator $T_\mu^{(j)}$ to be bounded or compact. We first give a necessary and sufficient condition for $T_\mu^{(j)}$ to be in the Schatten p-class for $1 \leqslant p < \infty$ on the Bergman space A^2, and then give a sufficient condition for $T_\mu^{(j)}$ to be in the Schatten p-class $(0 < p < 1)$ on A^2. We also discuss the generalized Toeplitz operators with general bounded symbols. If $\varphi \in L^\infty(D, \mathrm{d}A)$ and $1 < p < \infty$, we define the generalized Toeplitz operator $T_\varphi^{(j)}$ on the Bergman space A^p and characterize the compactness of the finite sum of operators of the form $T_{\varphi_1}^{(j)} \ldots T_{\varphi_n}^{(j)}$.

Keywords: generalized Toeplitz operator; Schatten class; compactness; Bergman space; Berezin transform

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1. Introduction and notations

Let $\mathrm{d}A$ denote the normalized Lebesgue area measure on the unit disk D. For $0 < p < \infty$, the space $L^p(D, \mathrm{d}A)$ consists of complex valued measurable functions on D such that

$$\|f\|_p := \left[\int_D |f(z)|^p \,\mathrm{d}A(z)\right]^{1/p} < \infty.$$

Let $L^\infty(D, \mathrm{d}A)$ be the space of measurable functions f on D such that

$$\|f\|_\infty := \operatorname{ess\,sup}\{|f(z)|\colon\ z \in D\} < \infty.$$

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For $1 \leqslant p < \infty$, the Bergman space A^p consists of all analytic functions on D that are also in $L^p(D, \mathrm{d}A)$. Let $\mathcal{L}(A^p)$ be the space of all linear bounded operators on A^p. For $z \in D$, let φ_z be the analytic automorphism of D defined by $\varphi_z(w) = (z-w)/(1-\overline{z}w)$. For $z \in D$, define the operator U_z on A^2 by $U_z f = (f \circ \varphi_z)\varphi_z'$, then U_z is unitary and self-adjoint on A^2. Let $K_z(w) = 1/(1-\overline{z}w)^2$ be the reproducing kernel of A^2 and let $k_z = K_z/\|K_z\|$. For any $f, g \in A^2$, let $f \otimes g$ be the rank-one operator on A^2 which is defined by

$$(f \otimes g)h = \langle h, g\rangle f \quad \forall\, h \in A^2.$$

Let $e_k = \sqrt{k+1}w^k$ $(k \geqslant 0)$, then $\{e_k\}_{k\geqslant 0}$ is an orthonormal basis of A^2. The operator $E_k := e_k \otimes e_k$ is in fact the orthogonal projection onto the subspace generated by e_k. For $z \in D$, it is easy to check that

$$\langle U_z E_0 U_z f, g\rangle = (1-|z|^2)^2 f(z)\overline{g(z)} \quad \forall\, f, g \in A^2. \tag{1.1}$$

Let $\mathrm{d}\tilde{A}(z) = (1-|z|^2)^{-2}\,\mathrm{d}A(z)$, then by (1.1), the traditional Toeplitz operator T_a on A^2 with the symbol $a \in L^\infty(D, \mathrm{d}A)$ can be written as

$$T_a = \int_D U_z E_0 U_z a(z)\,\mathrm{d}\tilde{A}(z),$$

where the integral converges in the weak operator topology. If R is a bounded linear operator on A^2 and $a \in L^\infty(D, \mathrm{d}A)$, Engliš in [2] considered the more general operators defined as

$$R_a := \int_D U_z R U_z a(z)\,\mathrm{d}\tilde{A}(z) \tag{1.2}$$

and showed that if R is in the trace class then $\|R_a\| \leqslant \|R\|_{\mathrm{tr}}\|a\|_\infty$. If the matrix of R in the orthonormal basis $\{e_k\}_{k\geqslant 0}$ is diagonal, then the operator R is an l^1 linear combination of the projections E_j, with the trace norm of R given by the corresponding l^1-norm of its eigenvalues, and then the above result is equivalent to $\|T_a^{(j)}\| \leqslant \|a\|_\infty$ for all integers $j \geqslant 0$, where the operator $T_a^{(j)}$ is defined by

$$T_a^{(j)} := \int_D U_z E_j U_z a(z)\,\mathrm{d}\tilde{A}(z). \tag{1.3}$$

More generally, let μ be a finite Borel measure on D and let $j \geqslant 0$, then Suárez defined the following generalized Toeplitz operator with symbol μ on the Bergman space, see [8]:

$$T_\mu^{(j)} := \int_D U_z E_j U_z (1-|z|^2)^{-2}\,\mathrm{d}\mu(z). \tag{1.4}$$

In [8], using Carleson measure conditions, Suárez characterized the boundedness and compactness of the operator $T_\mu^{(j)}$ on the Bergman space.

It is a natural problem to discuss when an operator $T_\mu^{(j)}$ is in the Schatten class operator on the Bergman space.

For any $0 < p < \infty$, the Schatten class S_p on a separable Hilbert space H consists of all the compact operators on H for which their singular numbers form a sequence belonging to l^p. The singular numbers of a compact operator T are defined by

$$s_n = s_n(T) = \inf\{\|T - K\|\colon \operatorname{rank} K \leqslant n - 1\}.$$

For any $T \in S_p$, the S_p norm of T is defined as

$$\|T\|_{S_p} = \left(\sum_{n=1}^{\infty} s_n^p\right)^{1/p}.$$

For more information one refers, for example, to [6] and [12].

Luecking was the first to study Toeplitz operators with measures as symbols on the Bergman space, see [3]. He gave a characterization of Schatten class Toeplitz operators based on l^p condition at a hyperbolic lattice of the unit disk. While the characterization in terms of the $L^p(\mathrm{d}\tilde{A})$ integrability of the averaging functions and the Berezin transform is proved in [9] in the situation of a bounded symmetric domain, Arazy, Fisher and Peetre in [1] studied Schatten class Hankel operators on the weighted Bergman spaces.

The organization of the paper is as follows. In Section 2, we consider the case of $1 \leqslant p < \infty$. Let $\varphi \in L^p(\mathrm{d}\tilde{A})$ be a nonnegative function, using the formula of Faá di Bruno, we then prove that $T_\varphi^{(j)} \in S_p$ on the Bergman space A^2 for any integer $j \geqslant 0$. Furthermore, we give a necessary and sufficient condition for $T_\mu^{(j)} \in S_p$ on A^2. In Section 3, we consider the situation of $0 < p < 1$. We give a sufficient condition for $T_\mu^{(j)} \in S_p$ on A^2. In Section 4, if $\varphi \in L^\infty(D, \mathrm{d}A)$ and $1 < p < \infty$, we introduce the generalized Toeplitz operator $T_\varphi^{(j)}$ on the Bergman space A^p and characterize the compactness of the finite sum of operators of the form $T_{\varphi_1}^{(j)} \dots T_{\varphi_n}^{(j)}$ on A^p. Throughout this paper, let j denote a fixed natural number.

2. The situation of $1 \leqslant p < \infty$

In this section, we use the Berezin transform and average function of the symbol to characterize the Schatten class property of generalized Toeplitz operators. For an operator S on A^2, with a dense domain containing H^∞, the Berezin transform of S is the function $\widetilde{S}$ defined on D by

$$\widetilde{S}(z) = \langle Sk_z, k_z\rangle.$$

Let $\beta(z,w)$ be the Bergman metric on D. For any $z \in D$ and $r > 0$, let

$$D(z,r) = \{w \in D\colon\ \beta(z,w) < r\}$$

be the hyperbolic disk with center z and radius r, and let $|D(z,r)|$ be the area of $D(z,r)$. By Proposition 4.5 of [11], there exists a constant C_r (depending only on r) such that

$$C_r^{-1} \leqslant |D(z,r)|K(w,w) \leqslant C_r, \quad w \in D(z,r). \tag{2.1}$$

Let μ be a finite positive Borel measure on D, $r > 0$, and $j \in \mathbb{N}$, then put

$$\widehat{\mu}_{r,j}(z) = \int_{D(z,r)} |\varphi_z(w)|^{2j} K(w,w)\,\mathrm{d}\mu(w).$$

When $j = 0$, by (2.1), $\widehat{\mu}_{r,j}$ is then equivalent to $\widehat{\mu}_r$ defined in [11].

The following lemma is Corollary 6.5 of [11].

Lemma 2.1. *If T is a trace class operator on A^2, then $\widetilde{T}$ is in $L^1(D, \mathrm{d}\tilde{A})$ and the formula*

$$\operatorname{tr}(T) = \int_D \langle TK_z, K_z\rangle\,\mathrm{d}A(z)$$

holds.

Theorem 2.2. *Suppose that μ is a finite positive Borel measure on D, $1 \leqslant p < \infty$, and $j \in \mathbb{N}$, then the following conditions are equivalent:*

(1) $T_\mu^{(j)} \in S_p$ *on* A^2;

(2) $\widetilde{T_\mu^{(j)}}(z) \in L^p(D, \mathrm{d}\tilde{A}(z))$;

(3) *there exists some* $r > 0$ *such that* $\widehat{\mu}_{r,j}(z) \in L^p(D, \mathrm{d}\tilde{A}(z))$.

Proof. (1) $\Rightarrow$ (2) Suppose $T_\mu^{(j)} \in S_p$ on A^2. Since $T_\mu^{(j)} \geqslant 0$, using Lemma 2.1, we get

$$\begin{aligned}\|T_\mu^{(j)}\|_{S_p}^p = \operatorname{tr}((T_\mu^{(j)})^p) &= \int_D \langle (T_\mu^{(j)})^p K_z, K_z\rangle\,\mathrm{d}A(z)\\ &= \int_D K(z,z)\langle (T_\mu^{(j)})^p k_z, k_z\rangle\,\mathrm{d}A(z).\end{aligned}$$

Since $1 \leqslant p < \infty$ and k_z is the unit vector in A^2, by Proposition 6.4 of [1], we have

$$\|T_\mu^{(j)}\|_{S_p}^p \geqslant \int_D K(z,z)\langle T_\mu^{(j)} k_z, k_z\rangle^p\,\mathrm{d}A(z)$$

and then $\widetilde{T_\mu^{(j)}}(z) \in L^p(D,\ \mathrm{d}\tilde{A}(z))$.

(2) $\Rightarrow$ (3). By Proposition 4.5 of [11], for $r > 0$, there exists a constant C_r (depending only on r) such that

$$1 - |w|^2 \geqslant C_r |1 - \overline{z}w|$$

for $w \in D(z, r)$ such that

$$\begin{aligned}
\widetilde{T_\mu^{(j)}}(z) = \langle T_\mu^{(j)} k_z, k_z \rangle &= \int_D |\langle U_w e_j, k_z \rangle|^2 K(w, w) \, \mathrm{d}\mu(w) \\
&= (j+1) \int_D (1 - |z|^2)^2 |\langle U_w \xi^j, K_z \rangle|^2 K(w, w) \, \mathrm{d}\mu(w) \\
&= (j+1) \int_D (1 - |z|^2)^2 |\varphi_w(z)|^{2j} |\varphi_w'(z)|^2 K(w, w) \, \mathrm{d}\mu(w) \\
&= (j+1) \int_D |\varphi_z(w)|^{2j} \frac{(1 - |z|^2)^2 (1 - |w|^2)^2}{|1 - \overline{z}w|^4} K(w, w) \, \mathrm{d}\mu(w) \\
&\geqslant C_r (j+1) \int_{D(z,r)} |\varphi_z(w)|^{2j} K(w, w) \, \mathrm{d}\mu(w)
\end{aligned}$$

and then we get

$$\widehat{\mu}_{r,j}(z) \in L^p(D, \mathrm{d}\tilde{A}(z)).$$

In order to prove that (3) $\Rightarrow$ (1), we need some preliminaries.

Let $1 \leqslant p < \infty$, $\varphi \in L^p(D, \mathrm{d}\tilde{A})$, and $j \in \mathbb{N}$. The generalized Toeplitz operator $T_\varphi^{(j)}$ on A^2 is defined as

$$T_\varphi^{(j)} = \int_D U_z E_j U_z \varphi(z) \, \mathrm{d}\tilde{A}(z), \tag{2.2}$$

where the integral converges in the weak operator topology.

Lemma 2.3. *Let $\varphi \in L^p(D, \mathrm{d}\tilde{A})$ for $1 \leqslant p < \infty$ and let φ has a compact support in D, then $T_\varphi^{(j)}$ is a compact operator on A^2.*

Proof. The proof is similar to that of Lemma 4.6 of [8] and we omit it. □

Next lemma follows from Theorem 4.28 of [11].

Lemma 2.4. *Suppose that $p > 0$, $n \geqslant 1$, and f is a holomorphic function in D, then $f \in L^p(D, \mathrm{d}A)$ if and only if the function*

$$g(z) = (1 - |z|^2)^n f^{(n)}(z)$$

is in $L^p(D, \mathrm{d}A)$. Furthermore, the norm of $f \in L^p(D, \mathrm{d}A)$ is equivalent to the norm

$$|f(0)| + |f'(0)| + \ldots + |f^{(n-1)}(0)| + \|(1 - |z|^2)^n f^{(n)}(z)\|_{L^p}.$$

The following lemma is a formula of Faá di Bruno, see [5].

Lemma 2.5. *Let $l \geqslant 1$. If $f(t)$ and $g(t)$ are functions defined in some intervals for which all the necessary derivatives are defined, then*

$$[f\circ g]^{(l)}(x)=\sum\frac{l!}{k_1!\dots k_l!}f^{(k)}(g(x))\Big[\frac{g'(x)}{1!}\Big]^{k_1}\Big[\frac{g''(x)}{2!}\Big]^{k_2}\dots\Big[\frac{g^{(l)}(x)}{l!}\Big]^{k_l}, \tag{2.3}$$

where $k=k_1+k_2+\ldots+k_l$ and the sum is over all $k_1,\ldots,k_l$ for which $l=k_1+2k_2+\ldots+lk_l$. In particular, if f is a holomorphic function in D and $g=\varphi_z$, then

$$[f\circ\varphi_z]^{(l)}(0)=\sum\frac{l!}{k_1!\dots k_l!}f^{(k)}(z)(-1)^k\overline{z}^{l-k}(1-|z|^2)^k, \tag{2.4}$$

where $k=k_1+k_2+\ldots+k_l$ and the sum is over all $k_1,\ldots,k_l$ for which $l=k_1+2k_2+\ldots+lk_l$.

Theorem 2.6. *If $1\leqslant p<\infty$, and if $\varphi\in L^p(D,\mathrm{d}\tilde{A})$, $\varphi\geqslant 0$ and $j\in\mathbb{N}$, then $T_\varphi^{(j)}\in S_p$ on A^2.*

Note that this result is a particular case of Theorem 1 (d) in [2]. Using Marcinkiewicz interpolation, Engliš proved this result in a far more general form. For completeness, we present an elementary proof in some details here.

Proof. If $\varphi\in L^p(D,\mathrm{d}\tilde{A})$ has a compact support in D, then, by Lemma 2.3, $T_\varphi^{(j)}$ is a compact operator on A^2. Let

$$T_\varphi^{(j)}f=\sum_{n=1}^{\infty}\lambda_n\langle f,f_n\rangle g_n$$

be the canonical decomposition of $T_\varphi^{(j)}$, where $\{\lambda_n\}$ is the sequence of singular values of $T_\varphi^{(j)}$ repeated according to their multiplicity, and $\{f_n\}$ and $\{g_n\}$ are two orthonormal sets in A^2. Hence,

$$\begin{aligned}\lambda_n=\langle T_\varphi^{(j)}f_n,g_n\rangle&=\int_D\langle U_zE_jU_zf_n,g_n\rangle\varphi(z)\,\mathrm{d}\tilde{A}(z)\\&\leqslant\int_D|\langle U_zf_n,e_j\rangle||\langle U_zg_n,e_j\rangle||\varphi(z)|\,\mathrm{d}\tilde{A}(z).\end{aligned} \tag{2.5}$$

When $p=1$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n&\leqslant\int_D\sum_{n=1}^{\infty}|\langle f_n,U_ze_j\rangle||\langle g_n,U_ze_j\rangle||\varphi(z)|\,\mathrm{d}\tilde{A}(z)\\&\leqslant\int_D\|U_ze_j\|^2|\varphi(z)|\,\mathrm{d}\tilde{A}(z)=\int_D|\varphi(z)|\,\mathrm{d}\tilde{A}(z)<\infty.\end{aligned} \tag{2.6}$$

If $1 < p < \infty$, it follows from Hölder's inequality that

$$\lambda_n^p \leqslant \int_D |\langle U_z f_n, e_j\rangle||\langle U_z g_n, e_j\rangle||\varphi(z)|^p \, \mathrm{d}\tilde{A}(z) \times \Big(\int_D |\langle U_z f_n, e_j\rangle||\langle U_z g_n, e_j\rangle| \, \mathrm{d}\tilde{A}(z)\Big)^{p/q}. \tag{2.7}$$

Let $F(z) = \int_0^z f(u)\,\mathrm{d}u$ for a function $f \in A^2$. We can calculate that

$$\begin{aligned} \int_D |\langle U_z f, e_j\rangle|^2 \, \mathrm{d}\tilde{A}(z) &= (j+1)\int_D |\langle (f\circ\varphi_z)\varphi_z', w^j\rangle|^2 \, \mathrm{d}\tilde{A}(z) \\ &= (j+1)\int_D |\langle (F\circ\varphi_z)', w^j\rangle|^2 \, \mathrm{d}\tilde{A}(z) \\ &= (j+1)\int_D \Big|\frac{(F\circ\varphi_z)^{(j+1)}(0)}{(j+1)!}\Big|^2 \, \mathrm{d}\tilde{A}(z) \\ &= (j+1)\int_D \Big|\sum \frac{1}{k_1!\ldots k_{j+1}!} F^{(k)}(z)(-1)^k \overline{z}^{j+1-k}(1-|z|^2)^k\Big|^2 \, \mathrm{d}\tilde{A}(z) \\ &\leqslant (j+1)^2 \int_D \sum_{k=1}^{j+1} |F^{(k)}(z)(1-|z|^2)^k|^2 \, \mathrm{d}\tilde{A}(z) \\ &= (j+1)^2 \int_D \sum_{k=1}^{j+1} |f^{(k-1)}(z)(1-|z|^2)^k|^2 \, \mathrm{d}\tilde{A}(z) \\ &= (j+1)^2 \sum_{k=0}^{j} \int_D |f^{(k)}(z)(1-|z|^2)^k|^2 \, \mathrm{d}A(z) \leqslant C_j\|f\|, \end{aligned} \tag{2.8}$$

where the fourth equality follows from Lemma 2.5, and the last inequality follows from Lemma 2.4, and C_j is a constant depending only on j. Let $C = C_j^{p/q}$, by (2.7) and (2.8), we then have

$$\lambda_n^p \leqslant C\int_D |\langle U_z f_n, e_j\rangle||\langle U_z g_n, e_j\rangle||\varphi(z)|^p \, \mathrm{d}\tilde{A}(z), \quad n \geqslant 1. \tag{2.9}$$

Therefore, like in the proof of (2.6),

$$\|T_\varphi^{(j)}\|_{S_p}^p = \sum_{n=1}^{\infty} \lambda_n^p \leqslant C\|\varphi\|_{L^p(\mathrm{d}\tilde{A})}^p.$$

In the general case, for $0 < r < 0$, let $\varphi_r = \chi_{rD}\varphi$, where χ_{rD} is the characteristic function of $rD := \{z\colon\ |z| \leqslant r\}$. The argument in the preceding paragraph shows that $\{T_{\varphi_r}^{(j)}\}$ is a Cauchy net in S_p-norm, so it converges to some $T \in S_p$ in S_p-norm as $r \to 1^-$.

Next, we prove that $T_\varphi^{(j)} \in \mathcal{L}(A^2)$ and $T_{\varphi_r}^{(j)} \to T_\varphi^{(j)}$ in the operator norm as $r \to 1^-$. In fact, for any $f, g \in A^2$, similarly to the proof of (2.6) and (2.9), it is easy to check that

$$
\begin{aligned}
|\langle (T_{\varphi_r}^{(j)} - T_\varphi^{(j)}) f, g\rangle| &\leqslant \int_D |\langle U_z f, e_j\rangle \langle e_j, U_z g\rangle| |\varphi_r(z) - \varphi(z)| \,\mathrm{d}\tilde{A}(z) \\
&\leqslant C \|\varphi_r - \varphi\|_{L^p(\mathrm{d}\tilde{A})} \|f\| \|g\|.
\end{aligned} \tag{2.10}
$$

Then $T_\varphi^{(j)} \in \mathcal{L}(A^2)$ and $T_{\varphi_r}^{(j)} \to T_\varphi^{(j)}$ in the operator norm as $r \to 1^-$. □

Now we prove that (3) ⇒ (1) in Theorem 2.2. Let $r > 0$ be such that

$$\widehat{\mu}_{r,j}(z) \in L^p(D, \mathrm{d}\tilde{A}(z)),$$

then by Theorem 2.6, $T_{\widehat{\mu}_{r,j}}^{(j)} \in S_p$. By Lemma 14 of [9], it is sufficient to show that there exists a positive constant C such that $T_\mu^{(j)} \leqslant C T_{\widehat{\mu}_{r,j}}^{(j)}$. In fact, for any $f \in A^2$, by Fubini's theorem,

$$
\begin{aligned}
\langle T_{\widehat{\mu}_{r,j}}^{(j)} f, f\rangle &= \int_D \langle U_z E_j U_z f, f\rangle \widehat{\mu}_{r,j}(z) \,\mathrm{d}\tilde{A}(z) \\
&= \int_D \langle U_z E_j U_z f, f\rangle \int_D |\varphi_z(w)|^{2j} \chi_{D(z,r)}(w) K(w,w) \,\mathrm{d}\mu(w) \,\mathrm{d}\tilde{A}(z) \\
&= \int_D \langle U_z E_j U_z f, f\rangle \int_D |\varphi_w(z)|^{2j} \chi_{D(w,r)}(z) K(w,w) \,\mathrm{d}\mu(w) \,\mathrm{d}\tilde{A}(z) \\
&= \int_D \left(\int_{D(w,r)} |\varphi_w(z)|^{2j} \langle U_z E_j U_z f, f\rangle \,\mathrm{d}\tilde{A}(z) \right) K(w,w) \,\mathrm{d}\mu(w) \\
&\geqslant \int_D \left(\int_{D(w,r)/D(w,r/2)} |\varphi_w(z)|^{2j} \langle U_z E_j U_z f, f\rangle \,\mathrm{d}\tilde{A}(z) \right) K(w,w) \,\mathrm{d}\mu(w) \\
&\geqslant \left(\tanh \frac{r}{2}\right)^{2j} \int_D \left(\int_{D(w,r)/D(w,r/2)} \langle U_z E_j U_z f, f\rangle \,\mathrm{d}\tilde{A}(z) \right) K(w,w) \,\mathrm{d}\mu(w).
\end{aligned}
$$

Next we need to prove that the inequality

$$\int_{D(w,r)/D(w,r/2)} \langle U_z E_j U_z f, f\rangle \,\mathrm{d}\tilde{A}(z) \geqslant C_{r,j} |\langle U_w f, e_j\rangle|^2 \tag{2.11}$$

holds for some constant $C_{r,j} > 0$. For any $F(\xi) = \sum a_m e_m(\xi) \in A^2$ and $0 \leqslant t \leqslant 2\pi$, $0 \leqslant s < 1$, it is easy to check that

$$
\begin{aligned}
|\langle F, U_{s\mathrm{e}^{\mathrm{i}t}} e_j\rangle|^2 &= |\langle F(\xi), (U_s e_j)(\mathrm{e}^{-\mathrm{i}t}\xi)\rangle|^2 = |\langle F(\mathrm{e}^{\mathrm{i}t}\xi), (U_s e_j)(\xi)\rangle|^2 \\
&= \sum_{m,l} a_m \overline{a_l} \langle e_m(\mathrm{e}^{\mathrm{i}t}\xi), (U_s e_j)(\xi)\rangle \overline{\langle e_l(\mathrm{e}^{\mathrm{i}t}\xi), (U_s e_j)(\xi)\rangle} \\
&= \sum_{m,l} a_m \overline{a_l} \mathrm{e}^{\mathrm{i}(m-l)t} \langle e_m, U_s e_j\rangle \overline{\langle e_l, U_s e_j\rangle}.
\end{aligned}
$$

Then

$$\text{(2.12)} \qquad \int_0^{2\pi} |\langle F, U_{se^{it}} e_j\rangle|^2 \frac{dt}{2\pi} = \sum_m |a_m|^2 |\langle e_m, U_s e_j\rangle|^2$$
$$\geqslant |a_j|^2 |\langle e_j, U_s e_j\rangle|^2 = |\langle F, e_j\rangle|^2 |\langle e_j, U_s e_j\rangle|^2$$
$$= |\langle F, e_j\rangle|^2 \int_0^{2\pi} |\langle e_j, U_{se^{it}} e_j\rangle|^2 \frac{dt}{2\pi}.$$

Hence,

$$\text{(2.13)} \qquad \int_{D(0,r)/D(0,r/2)} |\langle F, U_z e_j\rangle|^2 K(z,z)\, dA(z)$$
$$= \int_{\tanh r/2}^{\tanh r} \frac{2s}{(1-s^2)^2} \left(\int_0^{2\pi} |\langle F, U_{se^{it}} e_j\rangle|^2 \frac{dt}{2\pi} \right) ds$$
$$\geqslant |\langle F, e_j\rangle|^2 \int_{\tanh r/2}^{\tanh r} \frac{2s}{(1-s^2)^2} \left(\int_0^{2\pi} |\langle e_j, U_{se^{it}} e_j\rangle|^2 \frac{dt}{2\pi} \right) ds$$
$$= |\langle F, e_j\rangle|^2 \int_{D(0,r)/D(0,r/2)} |\langle e_j, U_z e_j\rangle|^2 K(z,z)\, dA(z).$$

In particular, let $F(\xi) = (U_w f)(\xi)$, by (2.13), we then have

$$\text{(2.14)} \qquad \int_{D(0,r)/D(0,r/2)} |\langle U_w f, U_z e_j\rangle|^2 K(z,z)\, dA(z)$$
$$\geqslant |\langle U_w f, e_j\rangle|^2 \int_{D(0,r)/D(0,r/2)} |\langle e_j, U_z e_j\rangle|^2 K(z,z)\, dA(z).$$

Let $f(z) = |\langle e_j, U_z e_j\rangle|^2 K(z,z)$, $z \in D$. By Lemma 4.3 of [7], the function $z \mapsto \langle e_j, U_z e_j\rangle$ is uniformly continuous on compact sets of D, then $f(z)$ is continuous on D. Note that $f(0) = 1$, we assume that $f(z) \neq 0$ on $D(0,r)$. Then by (2.8),

$$\int_{D(0,r)/D(0,r/2)} |\langle e_j, U_z e_j\rangle|^2 K(z,z)\, dA(z) < \infty$$

is a finite positive constant depending on r and j. On the other hand, note that $U_w U_z = U_{\varphi_w(z)} V_\lambda$, where $\lambda = (z\overline{w} - 1)/(1 - w\overline{z})$, $(V_\lambda h)(w) = \lambda h(\lambda w)$ for any $h \in A^2$. Consequently, $|\langle U_w f, U_z e_j\rangle| = |\langle f, U_{\varphi_w(z)} e_j\rangle|$ and the change of variable $\nu = \varphi_w(z)$ on the left hand side of (2.14) yields

$$\text{(2.15)} \qquad \int_{D(w,r)/D(w,r/2)} |\langle f, U_\nu e_j\rangle|^2 K(\nu,\nu)\, dA(\nu)$$
$$\geqslant |\langle U_w f, e_j\rangle|^2 \int_{D(0,r)/D(0,r/2)} |\langle e_j, U_z e_j\rangle|^2 K(z,z)\, dA(z).$$

Hence, (2.11) holds and the proof is complete. $\square$

Corollary 2.7. *If $1 \leqslant p < \infty$ and if $\varphi \in L^\infty(D, \mathrm{d}A)$, is a nonnegative function on D, then the following conditions are equivalent:*

(i) $T_\varphi^{(j)} \in S_p$ *on* A^2;

(ii) $\widetilde{T_\varphi^{(j)}}(z) \in L^p(D, \mathrm{d}\tilde{A}(z))$;

(iii) *there exists some $r > 0$ such that*

$$\int_{D(z,r)} |\varphi_z(w)|^{2j} K(w,w)\varphi(w)\,\mathrm{d}A(w) \in L^p(D, \mathrm{d}\tilde{A}(z)).$$

A sequence $\{a_k\}_{k=1}^\infty$ in D is called an *r-lattice in the Bergman metric* if

$$D = \bigcup_{k=1}^\infty D(a_k, r)$$

and $\beta(a_i, a_j) \geqslant \frac{1}{2}r$ for $i \neq j$. For more information about lattices, see [11].

Theorem 2.8. *Suppose that μ is a finite positive Borel measure on D and $j \in \mathbb{N}$, then the following conditions are equivalent:*

(i) $T_\mu^{(j)} \in S_1$ *on* A^2;

(ii) $\widetilde{\mu} \in L^1(D, \mathrm{d}\tilde{A})$;

(iii) $\widehat{\mu}_r \in L^1(D, \mathrm{d}\tilde{A})$ *for all (or some)* $r > 0$;

(iv) $\sum\limits_{n=1}^\infty \widehat{\mu}_r(a_n) < \infty$, *where $\{a_n\}_{n=1}^\infty$ is an r-lattice in the Bergman metric.*

Proof. For any $j \geqslant 1$, $T_\mu^{(j)} \in S_1$ if and only if $T_\mu \in S_1$, since

$$\begin{aligned}\operatorname{tr}(T_\mu^{(j)}) &= \int_D \langle T_\mu^{(j)} K_z, K_z\rangle \,\mathrm{d}A(z) = \int_D \int_D \langle U_w E_j U_w K_z, K_z\rangle K(w,w)\,\mathrm{d}\mu(w)\,\mathrm{d}A(z)\\ &= \int_D \int_D |\langle U_w K_z, e_j\rangle|^2 \,\mathrm{d}A(z) K(w,w)\,\mathrm{d}\mu(w) = \int_D K(w,w)\,\mathrm{d}\mu(w) = \operatorname{tr}(T_\mu).\end{aligned}$$

By Theorem C of [9], the proof is complete. □

3. The situation of $0 < p < 1$

For $0 < p < \infty$, the sequence space l^p is defined by

$$l^p = \left\{ \{a_i\}_{i=1}^\infty \colon \left(\sum_{i=1}^\infty |a_i|^p \right)^{1/p} < \infty \right\}.$$

The atomic decomposition for Bergman spaces turns out to be a powerful theorem in the theory of Bergman spaces. The following lemma is related to [11]. For more information about atomic decomposition, see [10].

Lemma 3.1. *Suppose that $p > 0$ and*

$$b > \max\Bigl(1, \frac{1}{p}\Bigr) + \frac{1}{p}. \tag{3.1}$$

Then there exists a constant $\sigma > 0$ such that for any r-lattice $\{a_k\}$ in the Bergman metric, where $0 < r < \sigma$, the space A^p consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1 - |a_k|^2)^{(pb-2)/p}}{(1 - z\bar{a}_k)^b}, \tag{3.2}$$

where $\{c_k\} \in l^p$, the series in (3.2) converges in A^p, and the norm of f in A^p is comparable to

$$\inf\biggl\{\Bigl[\sum_{k=1}^{\infty} |c_k|^p\Bigr]^{1/p} : \{c_k\} \text{ satisfies } (3.2)\biggr\}.$$

The following lemma is Proposition 4.13 of [11] which reflects the subharmonic property of a holomorphic function in the Bergman metric.

Lemma 3.2. *Suppose that $p > 0$, $r > 0$, then there exists a positive constant C such that*

$$|f(z)|^p \leqslant \frac{C}{(1 - |z|^2)^2} \int_{D(z,r)} |f(w)|^p \, dA(w),$$

where f is a holomorphic function in D and $z \in D$.

Theorem 3.3. *Suppose that μ is a finite positive Borel measure on D, $0 < p < 1$, $j \in \mathbb{N}$. There exist a positive radius $\sigma > 0$ and a σ-lattice $\{a_n\}$ in D such that if the sequence $\{\widehat{\mu}_\sigma(a_n)\}_{n=1}^{\infty}$ belongs to l^p, then $T_\mu^{(j)} \in S_p$ on A^2.*

Proof. Since for a σ-lattice $\{a_n\}_{n=1}^{\infty}$, the sequence $\{\widehat{\mu}_\sigma(a_n)\}_{n=1}^{\infty}$ belongs to l^p and must be bounded, then the Toeplitz operator T_μ is bounded on A^2 and μ is a Carleson measure, see [9]. Theorem 4.2 of [8] implies that $T_\mu^{(j)}$ is bounded on A^2. By Lemma 3.1, for any $b > \frac{1}{2}(3+p^{-1})$ there exist a positive radius σ' and a σ'-lattice $\{z_n\}$ in the Bergman metric such that the space A^2 consists exactly of functions of the form

$$f(z) = \sum_{n=1}^{\infty} c_n \frac{(1 - |z_n|^2)^{b-1}}{(1 - \overline{z_n} z)^b},$$

where $\{c_n\} \in l^2$, the above series converges in A^2, and

$$\int_D |f(z)|^2 \, dA(z) \leqslant C \sum_{n=1}^{\infty} |c_n|^2 \tag{3.3}$$

for some constant C independent of $\{c_n\}$.

Let $\{e_n\}$ be an orthonormal basis on A^2 and define the operator T on A^2 by

$$T\Big(\sum_{n=1}^{\infty} c_n e_n\Big) = \sum_{n=1}^{\infty} c_n \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^b},$$

then T is a bounded surjective linear operator on A^2. According to Proposition 1.30 of [11], $T_\mu^{(j)} \in S_p$ is equivalent to $T^*T_\mu^{(j)}T \in S_p$. Since $T^*T_\mu^{(j)}T$ is positive, in order to complete the proof, we need to check that $M = \sum\limits_{n=1}^{\infty} \langle T^*T_\mu^{(j)}Te_n, e_n\rangle^p < \infty$. In fact,

$$M = \sum_{n=1}^{\infty} \Big\langle T_\mu^{(j)} \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^b}, \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^b} \Big\rangle^p = \sum_{n=1}^{\infty} I_n^p,$$

where

$$\begin{aligned} I_n &= \Big\langle T_\mu^{(j)} \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^b}, \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^b} \Big\rangle \\ &= \int_D \Big|\Big\langle U_z \frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}w)^b}, e_j \Big\rangle\Big|^2 K(z,z)\,\mathrm{d}\mu(z). \end{aligned} \tag{3.4}$$

Since $\{a_n\}$ is a σ-lattice in the Bergman metric, by Lemma 4.30 of [11] and the proof of (2.8), we get

$$\begin{aligned} I_n &\leqslant (j+1)^2 \sum_{k=0}^{j} \int_D |h_n^{(k)}(z)|^2 (1-|z|^2)^{2k}\,\mathrm{d}\mu(z) \\ &\leqslant (j+1)^2 \sum_{k=0}^{j} \int_D \Big|\frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^{b+k}}\Big|^2 \\ &\qquad \times [b(b+1)\dots(b+k)]^2 (1-|z|^2)^{2k}\,\mathrm{d}\mu(z) \\ &\leqslant (j+1)^2 \sum_{k=0}^{j}\sum_{l=1}^{\infty} \int_{D(a_l,\sigma)} \Big|\frac{(1-|z_n|^2)^{b-1}}{(1-\overline{z_n}z)^{b+k}}\Big|^2 \\ &\qquad \times [b(b+1)\dots(b+k)]^2 (1-|z|^2)^{2k}\,\mathrm{d}\mu(z) \\ &\leqslant C(j+1)^2 \sum_{k=0}^{j}\sum_{l=1}^{\infty} \int_{D(a_l,\sigma)} |h_n(a_l)|^2 \\ &\qquad \times [b(b+1)\dots(b+k)]^2 \frac{(1-|a_l|^2)^{2k}}{|1-\overline{z_n}a_l|^{2k}}\,\mathrm{d}\mu(z) \\ &\leqslant C(j+1)^2 \sum_{k=0}^{j} [b(b+1)\dots(b+k)2^k]^2 \\ &\qquad \times \sum_{l=1}^{\infty} \frac{1}{(1-|a_l|^2)^2} |h_n(a_l)|^2 \widehat{\mu}_\sigma(a_l), \end{aligned} \tag{3.5}$$

where $h_n(z) = (1-|z_n|^2)^{b-1}/(1-\overline{z_n}z)^b$, and C depends on σ, b and j. Since $0 < p < 1$, there is a constant $C_1 > 0$ such that

$$I_n^p \leqslant C_1 \sum_{l=1}^{\infty} \frac{1}{(1-|a_l|^2)^{2p}} |h_n(a_l)|^{2p} \widehat{\mu}_\sigma^p(a_l). \tag{3.6}$$

Therefore,

$$M = \sum_{n=1}^{\infty} I_n^p \leqslant C_1 \sum_{l=1}^{\infty} \frac{1}{(1-|a_l|^2)^{2p}} \widehat{\mu}_\sigma^p(a_l) \sum_{n=1}^{\infty} |h_n(a_l)|^{2p}.$$

For any positive integer l, we consider the series

$$S_l = \sum_{n=1}^{\infty} |h_n(a_l)|^{2p} = \sum_{n=1}^{\infty} \frac{(1-|z_n|^2)^{p(2b-2)}}{|1-\overline{a_l}z_n|^{2pb}}.$$

Since $\{z_n\}$ is a σ'-lattice in the Bergman metric, then the Bergman disks $D(z_n, \frac{1}{8}\sigma')$ are mutually disjoint. Let

$$f(z) = \frac{(1-\overline{z_n}z)^{2b-2}}{(1-\overline{a_l}z)^{2b}},$$

by Lemma 3.2, then there exists a positive constant C (depending only on σ') such that

$$\begin{aligned} |f(z_n)|^p = \frac{(1-|z_n|^2)^{p(2b-2)}}{|1-\overline{a_l}z_n|^{2pb}} &\leqslant \frac{C}{(1-|z_n|^2)^2} \int_{D(z_n,\sigma'/8)} \frac{|1-\overline{z_n}z|^{p(2b-2)}}{|1-\overline{a_l}z|^{2pb}} \,\mathrm{d}A(z) \\ &\leqslant C \int_{D(z_n,\sigma'/8)} \frac{(1-|z|^2)^{p(2b-2)-2}}{|1-\overline{a_l}z|^{2pb}} \,\mathrm{d}A(z). \end{aligned}$$

Hence

$$S_l \leqslant C \sum_{n=1}^{\infty} \int_{D(z_n,\sigma'/8)} \frac{(1-|z|^2)^{p(2b-2)-2}}{|1-\overline{a_l}z|^{2pb}} \,\mathrm{d}A(z) \leqslant C \int_D \frac{(1-|z|^2)^{p(2b-2)-2}}{|1-\overline{a_l}z|^{2pb}} \,\mathrm{d}A(z).$$

Since $p(2b-2)-2 > -1$, by Lemma 3.10 of [11], there is a constant $C_2 > 0$ such that

$$S_l \leqslant \frac{C_2}{(1-|a_l|^2)^{2p}}.$$

Therefore,

$$M \leqslant C_1 C_2 \sum_{l=1}^{\infty} \widehat{\mu}_\sigma^p(a_l) < \infty.$$

□

4. The generalized Toeplitz operators on the Bergman spaces A^p $(1 < p < \infty)$

In this section, we assume $1 < p < \infty$. For any fixed $z \in D$, define the operator $U_z\colon A^p \to A^p$ such that

$$U_z f = (f \circ \varphi_z)\varphi_z' \quad \forall\, f \in A^p.$$

Then U_z is bounded. It's easy to check that

$$U_z^* g = (g \circ \varphi_z)\varphi_z' \quad \forall\, g \in A^q, \text{ where } 1/p + 1/q = 1.$$

Let S be a bounded operator on A^p and let $S_z = U_z S U_z$. The Berezin transform of S is the function $\widetilde{S}$ defined on D such that

$$\widetilde{S}(z) = \langle S k_z, k_z\rangle, \quad \text{where } \langle f, g\rangle = \int_D f\overline{g}\,\mathrm{d}A.$$

Let $E_j := e_j \otimes e_j$ be the rank one operator defined on A^p such that

$$E_j f = \langle f, e_j\rangle e_j, \quad f \in A^p.$$

Let $\varphi \in L^\infty(D, \mathrm{d}A)$ and $j \in \mathbb{N}$. The generalized Toeplitz operator $T_\varphi^{(j)}$ on A^p is defined as

$$T_\varphi^{(j)} := \int_D U_z E_j U_z \varphi(z)\,\mathrm{d}\tilde{A}(z), \tag{4.1}$$

where the integral converges in the weak operator topology.

Lemma 4.1. *Suppose that $\varphi \in L^\infty(D, \mathrm{d}A)$ and $j \in \mathbb{N}$, then $T_\varphi^{(j)}$ is bounded on A^p.*

Proof. For any $f \in A^p$, $g \in A^q$,

$$\begin{aligned}|\langle T_\varphi^{(j)} f, g\rangle| &\leqslant \int_D |\langle U_z f, e_j\rangle||\langle e_j, U_z^* g\rangle||\varphi(z)|\,\mathrm{d}\tilde{A}(z)\\ &\leqslant \|\varphi\|_\infty \biggl(\int_D |\langle U_z f, e_j\rangle|^p \frac{1}{(1-|z|^2)^p}\,\mathrm{d}A(z)\biggr)^{1/p}\\ &\quad\times \biggl(\int_D |\langle U_z^* g, e_j\rangle|^q \frac{1}{(1-|z|^2)^q}\,\mathrm{d}A(z)\biggr)^{1/q}.\end{aligned}$$

Let $1 < b < \infty$, $h \in A^b$. Note that for any $g \in A^b$ and g_n being the nth Taylor polynomial of g we have $\|g_n - g\|_{L^b} \to 0$ as $n \to \infty$. Repeating the course of the proof of (2.8), we get

$$\int_D |\langle U_z h, e_j\rangle|^b \frac{1}{(1-|z|^2)^b}\,\mathrm{d}A(z) \leqslant C_j (j+1)^{b/2} \|h\|_b^b, \tag{4.2}$$

where C_j is a constant depending on j. Hence

$$|\langle T^{(j)}_{\varphi} f, g\rangle| \leqslant C_j(j+1)\|\varphi\|_\infty \|f\|_p\|g\|_q. \tag{4.3}$$

□

The following lemma is Lemma 4.2 of [7].

Lemma 4.2. *For any fixed $z, w \in D$, if $t = (w\overline{z}-1)/(1-\overline{w}z)$, then $U_zU_w = U_{\varphi_z(w)}V_t$, where $(V_tf)(u) = tf(tu)$ for $f \in A^p$.*

Lemma 4.3. *Suppose that $\varphi \in L^\infty(D, \mathrm{d}A)$ and $w \in D$, then $U_wT^{(j)}_{\varphi}U_w = T^{(j)}_{\varphi\circ\varphi_w}$.*

P r o o f. For any $f \in A^p$, $g \in A^q$, we get

$$\begin{aligned}\langle U_wT^{(j)}_{\varphi}U_wf, g\rangle &= \int_D \langle U_zU_wf, e_j\rangle\langle e_j, U_z^*U_w^*g\rangle\varphi(z)\,\mathrm{d}\tilde{A}(z) \\ &= \int_D \langle f, U_w^*U_z^*e_j\rangle\langle U_wU_ze_j, g\rangle\varphi(z)\,\mathrm{d}\tilde{A}(z).\end{aligned} \tag{4.4}$$

By Lemma 4.2, we have $U_wU_z = U_{\varphi_w(z)}V_\lambda$, where $\lambda = (z\overline{w}-1)/(1-w\overline{z})$. Hence,

$$\langle U_wT^{(j)}_{\varphi}U_wf, g\rangle = \int_D \langle f, U_u^*e_j\rangle\langle U_ue_j, g\rangle\varphi\circ\varphi_w(u)d\tilde{A}(u) = \langle T^{(j)}_{\varphi\circ\varphi_w}f, g\rangle.$$

□

Lemma 4.4. *If S is a finite sum of operators of the form $T^{(j)}_{\varphi_1}\dots T^{(j)}_{\varphi_n}$, where $\varphi_i \in L^\infty(D, \mathrm{d}A)$ and $j \in \mathbb{N}$, then*

$$\sup_{z\in D}\|S_z1\|_p < \infty, \quad \sup_{z\in D}\|S_z^*1\|_p < \infty \tag{4.5}$$

for every $p \in (1, \infty)$.

P r o o f. Without loss of generality, we may assume that $S = T^{(j)}_{\varphi_1}\dots T^{(j)}_{\varphi_n}$. For $p \in (1, \infty)$, by Lemmas 4.1 and 4.3, we have

$$\|S_z1\|_p = \|T^{(j)}_{\varphi_1\circ\varphi_z}\dots T^{(j)}_{\varphi_n\circ\varphi_z}1\|_p \leqslant C_j^n(j+1)^n\|\varphi_1\|_\infty\dots\|\varphi_n\|_\infty. \tag{4.6}$$

It is easy to check that $(T^{(j)}_{\varphi_i})^* = T^{(j)}_{\overline{\varphi_i}}$ and then

$$\|S_z^*1\|_p = \|T^{(j)}_{\overline{\varphi_n}\circ\varphi_z}\dots T^{(j)}_{\overline{\varphi_1}\circ\varphi_z}1\|_p \leqslant C_j^n(j+1)^n\|\varphi_n\|_\infty\dots\|\varphi_1\|_\infty. \tag{4.7}$$

□

The following theorem can be found in [4].

Theorem 4.5. *Suppose that S is a bounded operator on A^p such that*

$$\sup_{z\in D}\|S_z 1\|_m < \infty \quad \text{and} \quad \sup_{z\in D}\|S_z^* 1\|_m < \infty \tag{4.8}$$

for some $m > 3/(p_1 - 1)$, where $p_1 = \min\{p, q\}$, then S is compact if and only if $\widetilde{S} \to 0$ as $z \to \partial D$.

Theorem 4.6. *Suppose that S is a finite sum of operators of the form $T_{\varphi_1}^{(j)} \dots T_{\varphi_n}^{(j)}$ on A^p, where each $\varphi_i \in L^\infty(D, \mathrm{d}A)$, $j \in \mathbb{N}$, then S is compact on A^p if and only if $\widetilde{S}(z) \to 0$ as $z \to \partial D$.*

Proof. By Lemma 4.4 and Theorem 4.5, it is easy to get the result desired. $\square$

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