Czechoslovak Mathematical Journal, 71 (146) (2021), 1149–1155

A NOTE ON ARITHMETIC DIOPHANTINE SERIES

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Received July 23, 2020. Published online June 30, 2021.

Abstract. We consider an asymptotic analysis for series related to the work of Hardy and Littlewood (1923) on Diophantine approximation, as well as Davenport. In particular, we expand on ideas from some previous work on arithmetic series and the RH. To accomplish this, Mellin inversion is applied to certain infinite series over arithmetic functions to apply Cauchy's residue theorem, and then the remainder of terms is estimated according to the assumption of the RH. In the last section, we use simple properties of the fractional part function and its Fourier series to state some identities involving different arithmetic functions. We then discuss some of their individual properties, such as convergence, as well as implications related to known work.

Keywords: arithmetic series; Riemann zeta function; Möbius function

MSC 2020: 11L20, 11M06

1. Introduction

In 1923 paper by Hardy and Littlewood (see [4]), we find a mention of the series

$$\sum_{n\geqslant 1}\frac{\overline{B}_m(nx)}{n^s}$$

with $\sigma := \Re(s) > 1$, in the setting of analysis on problems of Diophantine approximation. Here $\overline{B}_m(x) := \sum\limits_{j\geqslant 0}^{m}\binom{m}{j}B_{m-j}\{x\}^j$, where B_j is the jth Bernoulli number, and $\{x\} = x - [x]$, $[x]$ being the floor function. Not long after, Davenport's famous work (see [2]) was published showing interesting properties on arithmetic series of the form

$$\sum_{n\geqslant 1}\frac{a_n\overline{B}_1(nx)}{n}, \tag{1.1}$$

DOI: 10.21136/CMJ.2021.0311-20

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where a_n is taken to be a multiplicative arithmetic function $a\colon \mathbb{N} \to \mathbb{C}$. Here and throughout the paper we will take the set of natural numbers $\mathbb{N}$ to exclude 0, and write $\mathbb{N}_0$ to mean non-negative integers. Recall that the Möbius function is denoted by $\mu(n)$. Series (1.1) has been explored to a great extent, see [1], [6], [9]. One of our main results is given in the following, and is reminiscent of criteria for the Riemann hypothesis given in [3].

Theorem 1.1. *Let $k \geqslant 1$ be a natural number and let $\Upsilon_k(x)$ be a polynomial of degree k plus a term of the form $\dot{h}\log(x)x^k$, where $\dot{h}$ is a computable constant. Put $C_k = k!\,(2\pi\mathrm{i})^{-k}(1+(-1)^k)\zeta'(k)/\zeta(k)$, when $k > 1$, and $C_1 = 0$. We have that the Riemann hypothesis is equivalent to*

$$\sum_{n\geqslant 1}\frac{\mu(n)\log(n)}{n^k}\overline{B}_k(nx) = C_k + \Upsilon_{k-1}(x) + O(x^{k-1/2}) \tag{1.2}$$

as $x \to 0^+$.

P r o o f. First, we note from [5], equation (4.16) that for $m \geqslant 1$,

$$\overline{B}_m(x) = -m!\sum_{n\neq 0}(2\pi\mathrm{i}n)^{-m}\mathrm{e}^{2\pi\mathrm{i}nx}. \tag{1.3}$$

Now, it is well-known (see [1], equation (5.12)) that for $0 < c < 1$,

$$\frac{1}{2\pi\mathrm{i}}\int_{(c)}\mathrm{e}^{s(\mathrm{i}\pi/2-\log(2\pi)-\log(nx))}\Gamma(s)\,\mathrm{d}s = \mathrm{e}^{2\pi\mathrm{i}nx}. \tag{1.4}$$

This integral is also noted in [8], pages 91 and 406. We will follow similar lines as [1] in constructing our integral. Combining (1.3) with (1.4), we may sum the desired series, by conditional convergence, to get

$$-\frac{m!}{2\pi\mathrm{i}}\int_{(c)}(\mathrm{e}^{s(\mathrm{i}\pi/2)}+(-1)^m\mathrm{e}^{-s(\mathrm{i}\pi/2)})\mathrm{e}^{-s(\log(2\pi)+\log(x))}\zeta(s+m)\Gamma(s)\,\mathrm{d}s = (2\pi\mathrm{i})^m\overline{B}_m(x). \tag{1.5}$$

Here the gamma factor $\Gamma(s)$ is estimated by Stirling's formula, see [5], page 151, equation (5.113)

$$\Gamma(\sigma+\mathrm{i}t) = \sqrt{2\pi}t^{\sigma-1/2}\mathrm{e}^{(-\pi/2)|t|+(\mathrm{i}\pi/2)(\sigma-1/2)}\Big(\frac{|t|}{\mathrm{e}}\Big)^{\mathrm{i}t}\Big(1+O\Big(\frac{1}{|t|}\Big)\Big)$$

when $s = \sigma+\mathrm{i}t$, $t \neq 0$, and $\zeta(s+m)$ is also bounded on the line $s = \sigma + \mathrm{i}t$ when $m \geqslant 1$, since [10], page 95

$$\zeta(s) = O(|t|^k)$$

for any fixed $\sigma_0 > 0$ and $\Re(s) > \sigma_0$. Note that the m even case corresponds to the Mellin transform of $\cos(t)$ and the m odd case of this integral corresponds to the Mellin transform of $\sin(t)$, both of which are valid when $0 < \Re(s) < 1$.

Now using the formula $-\sum_{n\geqslant 1} \mu(n)\log(n)/n^s = \zeta'(s)/\zeta^2(s)$, $\Re(s) \geqslant 1$, we may again invert to get

$$(1.6)\quad \frac{m!}{2\pi\mathrm{i}}\int_{(c)} (\mathrm{e}^{s(\mathrm{i}\pi/2)} + (-1)^m\mathrm{e}^{-s(\mathrm{i}\pi/2)})\mathrm{e}^{-s(\log(2\pi)+\log(x))}\frac{\zeta'(s+m)}{\zeta(s+m)}\Gamma(s)\,\mathrm{d}s = (2\pi\mathrm{i})^m \sum_{n\geqslant 1} \frac{\mu(n)\log(n)\overline{B}_m(nx)}{n^m}.$$

Define $K(s) := (\mathrm{e}^{s(\mathrm{i}\pi/2)} + (-1)^m\mathrm{e}^{-s(\mathrm{i}\pi/2)})$ and note that $|K(s)| \ll \mathrm{e}^{|t|\pi/2}$. In particular, we see that

$$\begin{aligned}
&\Big|\int_{c-\mathrm{i}(T+1/\log^2(T))}^{c-\mathrm{i}T} K(s)\mathrm{e}^{-s(\log(2\pi)+\log(x))}\frac{\zeta'(s+m)}{\zeta(s+m)}\Gamma(s)\,\mathrm{d}s\Big|\\
&= \mathrm{e}^{m(\log(2\pi)+\log(x))}\Big|\int_{c+m-\mathrm{i}(T+1/\log^2(T))}^{c+m-\mathrm{i}T} K(s-m)\mathrm{e}^{-s(\log(2\pi)+\log(x))}\frac{\zeta'(s)}{\zeta(s)}\Gamma(s-m)\,\mathrm{d}s\Big|\\
&\gg \mathrm{e}^{m(\log(2\pi)+\log(x))}\int_{-(T+1/\log^2(T))}^{-T}\Big|K(s-m)\mathrm{e}^{-s(\log(2\pi)+\log(x))}\frac{\zeta'(s)}{\zeta(s)}\Gamma(s-m)\Big|\,\mathrm{d}t\\
&\gg \mathrm{e}^{(m-\sigma)(\log(2\pi)+\log(x))}\Big(\frac{\log(2)}{\cosh(\sigma\log(2))} + \frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)T^{\sigma-1/2-m}\int_{-(T+1/\log^2(T))}^{-T}\mathrm{d}t\\
&= \mathrm{e}^{(m-\sigma)(\log(2\pi)+\log(x))}\Big(\frac{\log(2)}{\cosh(\sigma\log(2))} + \frac{\zeta'(\sigma)}{\zeta(\sigma)}\Big)\frac{T^{\sigma-1/2-m}}{\log^2(T)}.
\end{aligned}$$

Here we have made the change of variable $s \to s - m$ and estimated $|\zeta'(s)/\zeta(s)|$ using arguments from [10], Theorem 11.5 (A). Hence, we require $\sigma - \frac{1}{2} - m < 0$ for convergence. Since we know $0 < \sigma < 1$, our condition $m \geqslant 1$ is sufficient. (For similar examples and arguments related to algebraic decay, see [8], page 127.) We integrate over the positively oriented rectangle with corners $(c, \mathrm{i}T)$, $(-M - \frac{1}{2}, \mathrm{i}T)$, $(-M - \frac{1}{2}, -\mathrm{i}T)$ and $(c, -\mathrm{i}T)$, and sufficiently large $M > 0$. It follows from analytic continuation that our integral is valid when $m \geqslant 1$. We move the line of integration of (1.6) to the left and compute the residues of the poles at the non-trivial zeros $s = -m + \varrho$, the residue at the pole $s = 0$, and poles at the negative integers. We compute the residues when $s = -l$, for $l < m$, giving the polynomial of degree $m - 1$ plus the term $\dot{h}\log(x)x^{m-1}$ arising from the double pole at $s = -m + 1$ (the polynomial $\Upsilon_{m-1}(x)$). This term is included since $\log(x)x^{m-1} \leqslant x^{m-1/2}$ implies

$\log(x) \leqslant x^{1/2}$, which is valid when $x \in (0,\infty)$. We find

$$(2\pi i)^m \sum_{n\geqslant 1} \frac{\mu(n)\log(n)\overline{B}_m(nx)}{n^m}$$
$$= C_m + \Upsilon_{m-1}(x) + m! \sum_{\varrho} K(\varrho - m) e^{-(\varrho-m)(\log(2\pi)+\log(x))}\Gamma(\varrho - m)$$
$$+ \frac{m!}{2\pi i}\int_{(d)} K(s) e^{-s(\log(2\pi)+\log(x))}\frac{\zeta'(s+m)}{\zeta(s+m)}\Gamma(s)\,ds$$

with $C_m = m!(1+(-1)^m)\zeta'(m)/\zeta(m)$ when $m>1$, $C_1 = 0$, where $-m < d < -m+\frac{1}{2}$. Next we consider $l \geqslant m$. Computing the residues at the double poles $s = -m-2l$, $l \in \mathbb{N}_0$, gives rise to a series over l of the form $\sum_{l\geqslant 0}(p_l + r_l\log(x))x^{m+2l}$. The poles at $s = -m-2l-1$ give rise to a series of the form $\sum_{l\geqslant 0} q_l x^{m+2l+1}$. (Here p_l, r_l and q_l are computable constants.) Combining these observations we find that

$$(2\pi i)^m \sum_{n\geqslant 1} \frac{\mu(n)\log(n)\overline{B}_m(nx)}{n^m}$$
$$= C_m + m! \sum_{\varrho} K(\varrho - m) e^{-(\varrho-m)(\log(2\pi)+\log(x))}\Gamma(\varrho - m) + \Upsilon_{m-1}(x)$$
$$+ \sum_{l\geqslant 0}((p_l + r_l\log(x))x^{m+2l} + q_l x^{m+2l+1}).$$

If we let x become increasingly small, we find the desired result upon inspecting the term $e^{-(\varrho-m)\log(x)}$ in the sum over ϱ and then replacing m with k. That is, the equivalence of the Riemann hypothesis follows from the condition that the non-trivial zeros must have $\Re(\varrho) = \frac{1}{2}$, and hence we estimate the sum by $O(e^{-(1/2-m)\log(x)})$ and negate terms from the last series we computed. □

In [10], page 198, equation (8.9.10), we find the arithmetic function $b_r(n)$, $r \in \mathbb{N}$, and its Dirichlet generating function

$$\frac{1}{\zeta^r(s)} = \sum_{n\geqslant 1}\frac{b_r(n)}{n^s} \tag{1.7}$$

for $\Re(s) > 1$. We offer an analogue of Theorem 1.1 for $b_r(n)$.

Theorem 1.2. *Let $k \geqslant 1$ be a natural number, $r > 1$, and let $\overline{\Upsilon}_k(x)$ be a polynomial of degree k. Put $C_k = k!\,(2\pi i)^{-k}(1+(-1)^k)1/\zeta^{r-1}(k)$ when $k>1$, and $C_1 = 0$. We have that the Riemann hypothesis is equivalent to*

$$\sum_{n\geqslant 1}\frac{b_r(n)}{n^k}\overline{B}_k(nx) = C_k + \overline{\Upsilon}_{k-1}(x) + O(x^{k-1/2}) \quad \text{as } x \to 0^+. \tag{1.8}$$

P r o o f. The proof is identical to Theorem 1.1, but we estimate our integral with $|\zeta(s)| \leqslant \zeta(\sigma)$ for $\sigma > 1$, which implies (since $\Re(s) > 1$ is a zero free region for $\zeta(s)$)

$$\frac{1}{\zeta^{r-1}(\sigma)} \leqslant \frac{1}{|\zeta^{r-1}(s)|}.$$

The computation involving the sum over ϱ includes computing the residues

$$R_{\varrho,m,r}(x) := \lim_{s\to\varrho-m} \frac{1}{(r-2)!}\frac{\mathrm{d}^{r-2}}{\mathrm{d}s^{r-2}}\Big((s-\varrho+m)^{r-1}K(s)(2\pi x)^{-s}\frac{1}{\zeta^{r-1}(s+m)}\Gamma(s)\Big),$$

and further, if $\Re(\varrho) = \frac{1}{2}$,

$$\sum_{\varrho} R_{\varrho,k,r}(x) = O(x^{k-1/2}),$$

since the terms involving $\log(x)^{r-2}x^{k-1/2}$ decay faster as $x \to 0^+$. The polynomial $\overline{\Upsilon}_k(x)$ is defined as in the theorem, since this time the pole at $s = -m+1$ is simple. The remaining details are left to the interested reader. $\square$

2. Some further observations

We mention some corollaries that are related to the $k = 2$ series from (1.5). It can be obtained from (1.3) and the property $\{-x\} = 1 - \{x\}$ that

$$\overline{B}_1(x)^2 - \frac{1}{12} = \frac{1}{2\pi^2}\sum_{n\geqslant 1}\frac{\mathrm{e}^{2\pi\mathrm{i}xn}}{n^2}, \tag{2.1}$$

$$\sum_{n\geqslant 1}\Big(\frac{\overline{B}_1(nx)}{n}\Big)^2 - \frac{\pi^2}{72} = \frac{1}{2\pi^2}\sum_{n\geqslant 1} d(n)\frac{\mathrm{e}^{2\pi\mathrm{i}xn}}{n^2}, \tag{2.2}$$

where $d(n)$ is the number of positive divisors of n. Furthermore, we have

$$\sum_{n\geqslant 1}\frac{\mu(n)}{n^2}\overline{B}_1(nx)^2 = \frac{1}{2\pi^2}(\mathrm{e}^{2\pi\mathrm{i}x} + 1), \tag{2.3}$$

$$\sum_{n\geqslant 1}\Big(\frac{\mu(n)}{n}\overline{B}_1(nx)\Big)^2 = \frac{1}{12}\frac{\zeta^2(2)}{\zeta(4)} + \frac{1}{2\pi^2}\sum_{n\geqslant 1}\frac{2^{v(n)}}{n^2}\mathrm{e}^{2\pi\mathrm{i}xn}, \tag{2.4}$$

$$\sum_{n\geqslant 1}\frac{\lambda(n)}{n^2}\overline{B}_1(nx)^2 = \frac{1}{12}\frac{\zeta(4)}{\zeta(2)} + \frac{1}{2\pi^2}\sum_{n\geqslant 1}\frac{1}{n^4}\mathrm{e}^{2\pi\mathrm{i}xn^2}, \tag{2.5}$$

where $v(n)$ is the number of distinct prime factors of n (see [10], page 5, equation (1.2.8), and $\lambda(n)$ is equal to $(-1)^k$ if n has k prime factors, counted with multiplicity, see [10], page 6, equation (1.2.11). The series considered in [2], [9] may be obtained from (2.1) (or (2.3)) by differentiating and then taking the real part. The series on the right-hand side of (2.5) is related to a function of Riemann in considering a function which is non-differentiable, see [7]. Equivalent results could be obtained as (2.5) regarding twice differentiability. Recall that a periodic function with a continuous first order derivative has a uniformly convergent Fourier series. In [4], page 216, several relevant results are given concerning convergence, one of which is that the series

$$\sum_{n\geqslant 1}\frac{1}{n}\mathrm{e}^{2\pi i n^2 x}$$

is not convergent for all irrational x. We close with a short proof concerning convergence of some of our arithmetic sums.

Corollary 2.1. *The function defined by*

$$f(z)=\sum_{n\geqslant 1}\Big(\frac{\mu(n)}{n}\overline{B}_1(nz)\Big)^2,$$

is 1*-periodic, converges locally uniformly on* $\mathbb{H}:=\{z\in\mathbb{C}\colon\ \Re(z)>0\}$, *and consequently is analytic there. Furthermore, the same can be said about* (2.3).

P r o o f. Since the Dirichlet series $\sum_{n\geqslant 1}a_n/n^s$ with $a_n=2^{v(n)}/n^2$ converges absolutely for $\Re(s)>-1$, we have $a_n=O(n^{-1})$. The result now clearly follows when comparing with the right side of (2.4). The series (2.3) follows from the comparison test. □

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