NOTES ON THE AVERAGE NUMBER OF SYLOW SUBGROUPS OF FINITE GROUPS

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Abstract. We show that if the average number of (nonnormal) Sylow subgroups of a finite group is less than $\frac{29}{4}$ then G is solvable or $G/F(G) \cong A_5$. This generalizes an earlier result by the third author.

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1. INTRODUCTION

All groups considered in this paper are finite. Given a group G, we define the average class size of G to be $\operatorname{acs}(G) = |G|/k(G)$, where k(G) is the number of conjugacy classes of G. Using this notation, Theorem 11 of [2] (which also follows from the earlier results of Lescot, as mentioned in the addendum), asserts that if $\operatorname{acs}(G) < \frac{40}{3}$ then either G is solvable or $G \cong A_5 \times T$. In particular, since $\operatorname{acs}(A_5) = 12$ this implies that if $\operatorname{acs}(G) < 12$ then G is solvable.

An analog of the final part of this result for Sylow numbers was considered in [5]. Let us introduce some notation from [5] to be used in this note. Given a prime p, $\nu_p(G)$ stands for the number of Sylow *p*-subgroups of *G*. Let S =

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{p prime: $\nu_p(G) > 1$ } and put $\operatorname{asn}(G) = \left(\sum_{p \in S} \nu_p(G)\right) / |S|$. Theorem A of [5] asserts that if $\operatorname{asn}(G) < 7$ then G is solvable. Our goal here is to extend this result to include also the analog of the first part of the result of Lescot-Guralnick-Robinson. Our main result is the following.

Theorem 1.1. Let G be a finite nonsolvable group. Assume that $\operatorname{asn}(G) < \frac{29}{4}$. Then $G/F(G) \cong A_5$. Furthermore, if Z(G) = 1 then $G \cong A_5$.

In this case, we cannot get a factorization as a direct product of A_5 and a nilpotent group because $\operatorname{asn}(\operatorname{SL}(2,5)) = \operatorname{asn}(A_7) = 7$. We expect that it should be possible to find $C > \frac{29}{4}$ such that if G is nonsolvable and $\operatorname{asn}(G) < C$ then we still get that $G/F(G) \cong A_5$. Our main aim in the proof, rather than trying to find the best possible value of C, was to keep the proof as elementary as possible. In fact, as in [5], Burnside's $p^a q^b$ -theorem is the most advanced result that we are using.

We also prove the following result, which can be compared with Theorem B of [4].

Theorem 1.2. Let G be a finite group. Assume that $\operatorname{asn}(G) < \frac{7}{2}$. Then G is supersolvable. Furthermore, if Z(G) = 1, then $G \cong S_3$.

2. Proofs

We start with two elementary lemmas.

Lemma 2.1. If N is a normal subgroup of a finite group G, then $\nu_p(N)\nu_p(G/N)$ divides $\nu_p(G)$. In particular, if S_1, \ldots, S_t are the composition factors of G including repetitions, then $\nu_p(S_1) \cdot \ldots \cdot \nu_p(S_t) | \nu_p(G)$.

Proof. See [3] for the first part. The second part is an immediate consequence. \Box

Lemma 2.2. Let S be a simple subgroup of A_7 . Then $S \cong A_5, A_6, A_7$ or PSL(2,7). Furthermore, if A_6, A_7 or PSL(2,7) is a composition factor of a finite group G, then $\operatorname{asn}(G) \geq \frac{89}{6}$.

Proof. The first part is a group theory exercise, or it can be checked with GAP, see [1]. For the second part, assume first that $S = A_6$. As another exercise or using GAP, one can check that $\nu_2(A_6) = 45$, $\nu_3(A_6) = 10$ and $\nu_5(A_6) = 36$. The average of these three integers is $\frac{91}{3}$. Using Lemma 2.1, we get $\nu_2(G) \ge 45$, $\nu_3(G) \ge 10$ and $\nu_5(G) \ge 36$. If we want to find a group G with $\operatorname{asn}(G)$ as low as possible among the groups that satisfy these conditions, it is an arithmetic exercise (using Sylow's

Theorem) to check that we cannot do better than having $\nu_7(G) = 8$, $\nu_{11}(G) = 12$, $\nu_{13}(G) = 14$, $\nu_{17}(G) = 18$ and $\nu_{19}(G) = 20$. In this case, $asn(G) \ge \frac{164}{8} = \frac{41}{2}$.

If $S = A_7$ we can argue as in the previous case to see that $\operatorname{asn}(G) > \frac{41}{2}$. (In fact, it is much bigger.)

If S = PSL(2,7), then $\nu_2(S) = 21$, $\nu_3(S) = 28$ and $\nu_7(S) = 8$. Arguing as in the case when $G = A_6$, we see that $asn(G) \ge \frac{89}{6}$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let S be a nonabelian composition factor of G. Assume first that $\nu_p(S) \ge 8$ for every prime divisor p of |S|. Assume that 5 divides |S|. Then $\nu_5(G) \ge \nu_5(S) \ge 11$ using Lemma 2.1 and Sylow's Theorem. If 2 divides |S|, then $\nu_2(G) \ge \nu_2(S) \ge 9$. Even if $\nu_3(G) = 4$ we get that the average of $\nu_2(G), \nu_3(G)$ and $\nu_5(G)$ is at least 8, so $\operatorname{asn}(G) \ge 8$, a contradiction. Therefore, 2 does not divide |S|. If 3 divides |S|, then $\nu_3(G) \ge \nu_3(S) \ge 10$. As before, we get that the average of $\nu_2(G), \nu_3(G)$ and $\nu_5(G)$ is at least 8, so $\operatorname{asn}(G) \ge 8$, another contradiction. But then all Sylow numbers that are bigger than 1 are at least 8, so $\operatorname{asn}(G) \ge 8$. It follows that 5 does not divide |S|.

If 6 divides |S|, then $\nu_2(G) \ge \nu_2(S) \ge 9$ and $\nu_3(G) \ge \nu_3(S) \ge 10$ (using Sylow's Theorem and the fact that $\nu_p(S) \ge 8$ for every prime divisor p of |S|). As before, one can see that this implies that $\operatorname{asn}(G) \ge 8$. This contradiction and Burnside's Theorem imply that there exist two different primes $u, v \ge 7$ such that $uv \mid |S|$. Arguing as in the last paragraph of the proof of Theorem 2.2 of [4], we get that $\operatorname{asn}(G) \ge \frac{39}{5} > \frac{29}{4}$. This is the final contradiction.

Therefore $\nu_p(S) < 8$ for some prime divisor p of |S|. Then S has a proper subgroup of index ≤ 7 and we deduce that S is isomorphic to a simple subgroup of S_7 . By Lemma 2.2, we deduce that $S = A_5$. Therefore, $\nu_2(S) = 5$, $\nu_3(S) = 10$ and $\nu_5(S) = 6$. By Lemma 2.1, $\nu_p(S) \mid \nu_p(G)$ for every prime p. Since $\operatorname{asn}(G) < \frac{29}{4}$, it follows that $\nu_2(G) = 5$, $\nu_3(G) = 10$ and $\nu_5(G) = 6$.

Assume that there exists a prime $q \ge 7$ that divides |G|. If $\nu_q(G) > 1$ then $\nu_q(G) \ge 8$ and since the average of 5, 10, 6 and 8 is $\frac{29}{4}$, $\operatorname{asn}(G) \ge \frac{29}{4}$. It follows that $\nu_q(G) = 1$ for every $q \ge 7$.

Let N be the largest normal solvable subgroup of G and let M/N be a chief factor of G. We know that M/N is a direct product of copies of A_5 . Using Lemma 2.1, we see that $M/N = A_5$. Let $C/N = C_{G/N}(M/N)$. Notice that $C/N \times M/N \leq G/N$. Using Lemma 2.1 again we see that C/N is solvable, so C = N. It follows that G/Nis isomorphic to a subgroup of $\operatorname{Aut}(A_5) = S_5$. If $G/N = S_5$ then $\nu_2(G/N) = 15$ and $\operatorname{asn}(G) > \frac{29}{4}$, a contradiction. Hence, G = M.

By Lemma 2.1, $\nu_p(N) = 1$ for every prime p, so N is nilpotent. The first part of the statement follows. Now, we assume that Z(G) = 1 and we want to prove that

N = 1. By way of contradiction, assume that N > 1. Let $R \in \text{Syl}_r(N)$ for some prime $r \mid \mid N \mid$ and let P be a Sylow subgroup of G for some prime $p \in \{2, 3, 5\} - \{r\}$. Since $R \leq G$, P normalizes R and $RP \leq G$. On the other hand, $R \leq N \leq N_G(P)$ (otherwise $\nu_p(G) > \nu_p(G/N)$ and we saw in the third paragraph of the proof that this is not the case), so [P, R] = 1. Therefore, R centralizes all the Sylow p-subgroups of G for all primes $p \neq r$. Since G/N is generated by its Sylow p-subgroups for any $p \in \{2, 3, 5\}$ it follows that $Z(R) \leq Z(G)$. This contradicts the hypothesis Z(G) = 1. It follows that $G = A_5$.

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. Notice that in order to have $\operatorname{asn}(G) < \frac{7}{2}$, we must have $\nu_2(G) = 3$, $\nu_p(G) = 1$ for every prime $p \ge 3$. Therefore, G has a normal nilpotent Hall 2'-subgroup N and G = PN, where $P \in \operatorname{Syl}_2(G)$. Since $|G : N_G(P)| = 3$, we have that $|N : C_N(P)| = |N : N \cap N_G(P)| = 3$. Since $C_N(P)$ has its prime index in the nilpotent subgroup N, we deduce that $C_N(P) \le N$. Clearly, P normalizes $C_N(P)$, so $C_N(P) \le G$. Observe that any chief series of N that contains $C_N(P)$ consists of normal subgroups of G. Extending this chief series to a chief series of G, we see that G is supersolvable.

Assume now that Z(G) = 1. If $C_N(P) > 1$, we can take a minimal normal subgroup M of N contained in $C_N(P)$. This subgroup is central in G and this is a contradiction. We conclude that $C_N(P) = 1$ so |N| = 3. Since Z(G) = 1, we deduce that |P| = 2 and $G = S_3$.

References

- [1] GAP Group: GAP Groups, Algorithms, Programming a System for Computational Discrete Algebra. Version 4.4. Available at http://www.gap-system.org (2005).
- [2] R. M. Guralnick, G. R. Robinson: On the commuting probability in finite groups. J. Algebra 300 (2006), 509–528; addendum ibid 319 (2008), 1822.
- [3] M. Hall, Jr.: On the number of Sylow subgroups in a finite group. J. Algebra 7 (1967), 363–371.
- [4] I. M. Isaacs, M. Loukaki, A. Moretó: The average degree of an irreducible character of a finite group. Isr. J. Math. 197 (2013), 55–67.

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zbl MR doi

zbl MR doi

[5] A. Moretó: The average number of Sylow subgroups of a finite group. Math. Nachr. 287 (2014), 1183–1185.

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