

5-DISSECTIONS AND SIGN PATTERNS OF RAMANUJAN'S
PARAMETER AND ITS COMPANION

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Abstract. In 1998, Michael Hirschhorn discovered the 5-dissection formulas of the Rogers-Ramanujan continued fraction $R(q)$ and its reciprocal. We obtain the 5-dissections for functions $R(q)R(q^2)^2$ and $R(q)^2/R(q^2)$, which are essentially Ramanujan's parameter and its companion. Additionally, 5-dissections of the reciprocals of these two functions are derived. These 5-dissection formulas imply that the coefficients in their series expansions have periodic sign patterns with few exceptions.

Keywords: 5-dissection; sign pattern; Ramanujan's parameter

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1. INTRODUCTION

Throughout the paper, we always assume that q is a complex number such that $|q| < 1$. As a q -analog of the golden ratio, the Rogers-Ramanujan continued fraction

$$R(q) = \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}}$$

was discovered by Rogers (see [18]) and later popularized by Ramanujan (see [15]). In some contexts like [2], $R(q)$ has an extra factor of $q^{1/5}$. But in this paper, we will drop this factor so that in its series expansion, all powers are nonnegative integers. The Rogers-Ramanujan continued fraction is closely related to the Rogers-Ramanujan identities, as indicated by their names. Before proceeding with our introduction, let

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us record some fairly standard notation:

$$(A; q)_\infty := \prod_{k=0}^{\infty} (1 - Aq^k),$$

$$(A_1, A_2, \dots, A_n; q)_\infty := (A_1; q)_\infty (A_2; q)_\infty \dots (A_n; q)_\infty,$$

$$\left(\begin{matrix} A_1, A_2, \dots, A_n \\ B_1, B_2, \dots, B_m \end{matrix}; q \right)_\infty := \frac{(A_1; q)_\infty (A_2; q)_\infty \dots (A_n; q)_\infty}{(B_1; q)_\infty (B_2; q)_\infty \dots (B_m; q)_\infty}.$$

The celebrated Rogers-Ramanujan identities (see [12], equations (17.4.2), (17.4.3)) state that

$$(1.1) \quad G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\dots(1-q^n)} = \frac{1}{(q, q^4; q^5)_\infty},$$

$$(1.2) \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\dots(1-q^n)} = \frac{1}{(q^2, q^3; q^5)_\infty}.$$

The functions $G(q)$ and $H(q)$ are known as the Rogers-Ramanujan functions. The relation between $G(q)$ and $H(q)$ along with the continued fraction $R(q)$ was initially found by Rogers, see [18]:

$$(1.3) \quad R(q) = \frac{H(q)}{G(q)} = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5 \right)_\infty.$$

It was also recorded as Entry 38 (iii) in Chapter 16 of Ramanujan's Notebook, see [15], page 204.

Another two important functions in the theory of Rogers-Ramanujan continued fraction are the so-called *Ramanujan's parameter*

$$(1.4) \quad k := qR(q)R(q^2)^2$$

and its companion $R(q)^2/R(q^2)$. These functions are extensively used in the modular equations related to the Rogers-Ramanujan continued fraction. See, for instance, Raghavan and Rangachari in [14], Kang in [13], Gugg in [10] and Cooper in [4]; see also Hirschhorn's book [12], Chapter 40. Also, there is a general theory at level 10 with Ramanujan's parameter k involved, from which various series for $1/\pi$ are discovered, see [5] or [6], Chapter 10 for a detailed discussion.

In 1968 or early 1969, Szekeres observed a surprising phenomenon that in the series expansions of $R(q)$ and its reciprocal $R(q)^{-1}$, the signs of their coefficients are eventually periodic with period 5. This observation was later confirmed in the asymptotic sense by Richmond and Szekeres, see [17]. In 1978, Andrews in [1] found some formulas of Ramanujan in the Lost Notebook, and used these to give a complete proof

of Szekeres' observation. The study of this topic reached the climax in 1998, when Hirschhorn in [11] discovered the following explicit 5-dissections of $R(q)$ and $R(q)^{-1}$:

$$R(q) = \frac{(q^{125}; q^{125})_\infty}{(q^5; q^5)_\infty} \left(\left(\frac{q^{30}, q^{95}}{q^{15}, q^{110}}; q^{125} \right)_\infty - q \left(\frac{q^{20}, q^{105}}{q^{10}, q^{115}}; q^{125} \right)_\infty \right. \\ \left. + q^2 \left(\frac{q^{55}, q^{70}}{q^{35}, q^{90}}; q^{125} \right)_\infty - q^{18} \left(\frac{q^5, q^{120}}{q^{60}, q^{65}}; q^{125} \right)_\infty - q^4 \left(\frac{q^{45}, q^{80}}{q^{40}, q^{85}}; q^{125} \right)_\infty \right)$$

and

$$\frac{1}{R(q)} = \frac{(q^{125}; q^{125})_\infty}{(q^5; q^5)_\infty} \left(\left(\frac{q^{40}, q^{85}}{q^{20}, q^{105}}; q^{125} \right)_\infty + q \left(\frac{q^{60}, q^{65}}{q^{30}, q^{95}}; q^{125} \right)_\infty \right. \\ \left. - q^7 \left(\frac{q^{35}, q^{90}}{q^{45}, q^{80}}; q^{125} \right)_\infty - q^3 \left(\frac{q^{10}, q^{115}}{q^5, q^{120}}; q^{125} \right)_\infty - q^{14} \left(\frac{q^{15}, q^{110}}{q^{55}, q^{70}}; q^{125} \right)_\infty \right).$$

It is notable that Ramanujan indeed partially discovered the two 5-dissections in terms of Lambert series in his Lost Notebook, see [16], page 50. But he left the final punch, which can be done by the quintuple product identity, to Hirschhorn.

Considering the significance of dissection formulas in the study of sign patterns in q -series expansions, a natural question is about the 5-dissections of Ramanujan's parameter, its companion and their reciprocals, since these functions also play an important role in the theory of Rogers-Ramanujan continued fraction. In this paper, we obtain the following 5-dissections.

Theorem 1.1. *We have*

$$(1.5) \quad R(q)R(q^2)^2 = \left(\frac{q^5, q^5, q^{25}, q^{25}, q^{25}, q^{25}, q^{45}, q^{45}}{q^{10}, q^{10}, q^{10}, q^{20}, q^{30}, q^{40}, q^{40}, q^{40}}; q^{50} \right)_\infty \\ - q \left(\frac{q^5, q^5, q^{15}, q^{25}, q^{25}, q^{35}, q^{45}, q^{45}}{q^{10}, q^{10}, q^{10}, q^{20}, q^{30}, q^{40}, q^{40}, q^{40}}; q^{50} \right)_\infty \\ - q^2 \left(\frac{q^5, q^{15}, q^{15}, q^{25}, q^{25}, q^{35}, q^{35}, q^{45}}{q^{10}, q^{20}, q^{20}, q^{20}, q^{30}, q^{30}, q^{30}, q^{40}}; q^{50} \right)_\infty \\ + 2q^3 \left(\frac{q^5, q^5, q^{15}, q^{25}, q^{25}, q^{35}, q^{45}, q^{45}}{q^{10}, q^{10}, q^{20}, q^{20}, q^{30}, q^{30}, q^{40}, q^{40}}; q^{50} \right)_\infty \\ + q^9 \left(\frac{q^5, q^5, q^5, q^{15}, q^{35}, q^{45}, q^{45}, q^{45}}{q^{10}, q^{20}, q^{20}, q^{20}, q^{30}, q^{30}, q^{30}, q^{40}}; q^{50} \right)_\infty$$

and

$$(1.6) \quad \frac{1}{R(q)R(q^2)^2} = \left(\begin{matrix} q^{15}, q^{15}, q^{25}, q^{25}, q^{25}, q^{25}, q^{35}, q^{35} \\ q^{10}, q^{20}, q^{20}, q^{20}, q^{30}, q^{30}, q^{30}, q^{40} \end{matrix} ; q^{50} \right)_{\infty} \\ + q \left(\begin{matrix} q^5, q^{15}, q^{15}, q^{15}, q^{35}, q^{35}, q^{35}, q^{45} \\ q^{10}, q^{10}, q^{10}, q^{20}, q^{30}, q^{40}, q^{40}, q^{40} \end{matrix} ; q^{50} \right)_{\infty} \\ + 2q^2 \left(\begin{matrix} q^5, q^{15}, q^{15}, q^{25}, q^{25}, q^{35}, q^{35}, q^{45} \\ q^{10}, q^{10}, q^{20}, q^{20}, q^{30}, q^{30}, q^{40}, q^{40} \end{matrix} ; q^{50} \right)_{\infty} \\ + q^3 \left(\begin{matrix} q^5, q^5, q^{15}, q^{25}, q^{25}, q^{35}, q^{45}, q^{45} \\ q^{10}, q^{10}, q^{10}, q^{20}, q^{30}, q^{40}, q^{40}, q^{40} \end{matrix} ; q^{50} \right)_{\infty} \\ + q^4 \left(\begin{matrix} q^5, q^{15}, q^{15}, q^{25}, q^{25}, q^{35}, q^{35}, q^{45} \\ q^{10}, q^{20}, q^{20}, q^{20}, q^{30}, q^{30}, q^{30}, q^{40} \end{matrix} ; q^{50} \right)_{\infty}.$$

Theorem 1.2. We have

$$(1.7) \quad \frac{R(q)^2}{R(q^2)} = \left(\begin{matrix} -q^5, -q^{10}, -q^{10}, -q^{10}, -q^{15}, -q^{15}, -q^{15}, -q^{20} \\ q^5, q^{10}, q^{10}, q^{10}, q^{15}, q^{15}, q^{15}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ - 2q \left(\begin{matrix} -q^5, -q^{10}, -q^{10}, -q^{15}, -q^{15}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^5, q^5, q^{10}, q^{15}, q^{20}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ + 4q^2 \left(\begin{matrix} -q^5, -q^{10}, -q^{10}, -q^{15}, -q^{15}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^5, q^{10}, q^{10}, q^{15}, q^{15}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ - 4q^3 \left(\begin{matrix} -q^{10}, -q^{10}, -q^{15}, -q^{15}, -q^{25}, -q^{25}, -q^{25}, -q^{25} \\ q^5, q^5, q^5, q^{10}, q^{15}, q^{20}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ + 2q^4 \left(\begin{matrix} -q^5, -q^5, -q^{10}, -q^{15}, -q^{20}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^{10}, q^{10}, q^{10}, q^{15}, q^{15}, q^{15}, q^{20} \end{matrix} ; q^{25} \right)_{\infty}$$

and

$$(1.8) \quad \frac{R(q^2)}{R(q)^2} = \left(\begin{matrix} -q^5, -q^5, -q^5, -q^{10}, -q^{15}, -q^{20}, -q^{20}, -q^{20} \\ q^5, q^5, q^5, q^{10}, q^{15}, q^{20}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ + 2q \left(\begin{matrix} -q^5, -q^{10}, -q^{10}, -q^{15}, -q^{15}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^5, q^5, q^{10}, q^{15}, q^{20}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ - 4q^7 \left(\begin{matrix} -q^5, -q^5, -q^{20}, -q^{20}, -q^{25}, -q^{25}, -q^{25}, -q^{25} \\ q^5, q^{10}, q^{10}, q^{10}, q^{15}, q^{15}, q^{15}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ - 4q^3 \left(\begin{matrix} -q^5, -q^5, -q^{10}, -q^{15}, -q^{20}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^5, q^{10}, q^{10}, q^{15}, q^{15}, q^{20}, q^{20} \end{matrix} ; q^{25} \right)_{\infty} \\ - 2q^4 \left(\begin{matrix} -q^5, -q^5, -q^{10}, -q^{15}, -q^{20}, -q^{20}, -q^{25}, -q^{25} \\ q^5, q^{10}, q^{10}, q^{10}, q^{15}, q^{15}, q^{15}, q^{20} \end{matrix} ; q^{25} \right)_{\infty}.$$

As direct consequences of the 5-dissection formulas (1.5)–(1.8), the following sign patterns hold.

Corollary 1.1. *Define the sequences $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$ and $\{\delta(n)\}$ by*

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha(n)q^n &= R(q)R(q^2)^2, & \sum_{n=0}^{\infty} \beta(n)q^n &= \frac{1}{R(q)R(q^2)^2}, \\ \sum_{n=0}^{\infty} \gamma(n)q^n &= \frac{R(q)^2}{R(q^2)}, & \sum_{n=0}^{\infty} \delta(n)q^n &= \frac{R(q^2)}{R(q)^2}. \end{aligned}$$

For any $n \geq 0$, we have

$$(1.9) \quad \alpha(n) \begin{cases} > 0 & \text{if } n \equiv 0, 3, 6, 7, 9 \pmod{10}, \\ < 0 & \text{if } n \equiv 1, 2, 4, 5, 8 \pmod{10}, \end{cases}$$

$$(1.10) \quad \beta(n) \begin{cases} > 0 & \text{if } n \equiv 0, 1, 2, 3, 4 \pmod{10}, \\ < 0 & \text{if } n \equiv 5, 6, 7, 8, 9 \pmod{10}, \end{cases}$$

$$(1.11) \quad \gamma(n) \begin{cases} > 0 & \text{if } n \equiv 0, 2, 4 \pmod{5}, \\ < 0 & \text{if } n \equiv 1, 3 \pmod{5}, \end{cases}$$

$$(1.12) \quad \delta(n) \begin{cases} > 0 & \text{if } n \equiv 0, 1 \pmod{5}, \\ < 0 & \text{if } n \equiv 2, 3, 4 \pmod{5}, \end{cases}$$

except for $\alpha(4) = \beta(5) = \delta(2) = 0$.

2. LEMMAS

Let us record some necessary identities for our proofs. The first several are related to the Rogers-Ramanujan functions $G(q)$ and $H(q)$.

Lemma 2.1. *We have*

$$(2.1) \quad G(q)^2 H(q^2) + G(q^2) H(q)^2 = \frac{2G(q)G(q^2)^2}{(q^5; q^{10})_{\infty}^2},$$

$$(2.2) \quad G(q)^2 H(q^2) - G(q^2) H(q)^2 = \frac{2qH(q)H(q^2)^2}{(q^5; q^{10})_{\infty}^2},$$

$$(2.3) \quad \frac{G(q)^2 H(q^2) - G(q^2) H(q)^2}{G(q)^2 H(q^2) + G(q^2) H(q)^2} = \frac{qH(q)H(q^2)^2}{G(q)G(q^2)^2}.$$

Proof. Identities (2.1)–(2.3) are (17.4.10)–(17.4.12) in [12], respectively. □

The next lemma gives three modular equations related to Ramanujan's parameter k as in (1.4).

Lemma 2.2. *We have*

$$(2.4) \quad \frac{R(q)^2}{R(q^2)} = \frac{1-k}{1+k},$$

$$(2.5) \quad \frac{\varphi(-q)^2}{\varphi(-q^5)^2} = \frac{1-4k-k^2}{1-k^2},$$

$$(2.6) \quad \frac{\psi(q)^2}{q\psi(q^5)^2} = \frac{1+k-k^2}{k}.$$

Here $\varphi(q)$ and $\psi(q)$ are two of Ramanujan's classical theta functions defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}, \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}.$$

Proof. The identity (2.4) appears in [2], equation (1.9.29). For (2.5) and (2.6), see [2], Entry 1.8.2 (i) and (ii), respectively. See also [12], Chapters 40 and 41. \square

Further, we need the following identity involving $\varphi(-q)$.

Lemma 2.3. *We have*

$$(2.7) \quad \varphi(-q)^2 - \varphi(-q^5)^2 = -\frac{4q(q; q)_{\infty} (q^{10}; q^{10})_{\infty}^3}{(q^2; q^2)_{\infty} (q^5; q^5)_{\infty}}.$$

Proof. The identity (2.7) follows from [12], equation (34.1.21). \square

Finally, we require the following 5-dissections.

Lemma 2.4. *We have*

$$(2.8) \quad \frac{G(q^2)^2}{G(q)H(q)^2} = G(q^5)G(q^{10})^2 - qH(q^5)G(q^{10})^2 \\ - 2q^3H(q^5)G(q^{10})H(q^{10}) + q^4G(q^5)H(q^{10})^2,$$

$$(2.9) \quad \frac{H(q^2)^2}{G(q)^2H(q)} = H(q^5)G(q^{10})^2 - 2qG(q^5)G(q^{10})H(q^{10}) \\ + q^3G(q^5)H(q^{10})^2 + q^4H(q^5)H(q^{10})^2,$$

$$(2.10) \quad \frac{G(q)^2}{G(q^2)^2H(q^2)} = G(q^5)^2G(q^{10}) + 2qG(q^5)H(q^5)G(q^{10}) \\ + q^2G(q^5)^2H(q^{10}) + q^4H(q^5)^2H(q^{10}),$$

$$(2.11) \quad \frac{H(q)^2}{G(q^2)H(q^2)^2} = G(q^5)^2G(q^{10}) + q^2H(q^5)^2G(q^{10}) \\ + 2q^3G(q^5)H(q^5)H(q^{10}) - q^4H(q^5)^2H(q^{10}).$$

Proof. The identities (2.8) and (2.9) follow from Theorems 1.1 and 1.2 in [7] together with (1.1) and (1.2), respectively. For (2.10) and (2.11), we make use of Theorems 1.1 and 1.2 in [20]. \square

3. AUXILIARY IDENTITIES

We shall establish two auxiliary identities that are also of independent interest.

Lemma 3.1. *We have*

$$(3.1) \quad G(q)^4 H(q^2)^2 - H(q)^4 G(q^2)^2 + 4G(q)^2 H(q)^2 G(q^2) H(q^2) \\ = \frac{4(q^{10}; q^{10})_{\infty}^4}{(q^5; q^5)_{\infty}^4} \cdot \frac{G(q)^3 H(q)^3}{G(q^2)^2 H(q^2)^2}.$$

Proof. We first prove

$$(3.2) \quad \frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} + 4 = \frac{4(q^2; q^2)_{\infty}^3 (q^{10}; q^{10})_{\infty}}{(q; q)_{\infty} (q^5; q^5)_{\infty}^3}.$$

To see this, we obtain from (2.4) that

$$(3.3) \quad \frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} + 4 = \frac{1+k}{1-k} - \frac{1-k}{1+k} + 4 = \frac{4k}{1-k^2} \cdot \frac{1+k-k^2}{k}.$$

Further, from (2.5) we find that

$$\frac{\varphi(-q)^2}{\varphi(-q^5)^2} = \frac{1-4k-k^2}{1-k^2} = 1 - \frac{4k}{1-k^2}.$$

Hence,

$$(3.4) \quad \frac{4k}{1-k^2} = -\frac{\varphi(-q)^2 - \varphi(-q^5)^2}{\varphi(-q^5)^2} = \frac{4q(q; q)_{\infty} (q^{10}; q^{10})_{\infty}^5}{(q^2; q^2)_{\infty} (q^5; q^5)_{\infty}^5},$$

where (2.7) is utilized. We arrive at (3.2) by substituting (2.6) and (3.4) into (3.3).

Finally, we notice that

$$G(q)^4 H(q^2)^2 - H(q)^4 G(q^2)^2 + 4G(q)^2 H(q)^2 G(q^2) H(q^2) \\ = G(q)^2 H(q)^2 G(q^2) H(q^2) \left(\frac{G(q)^2 H(q^2)}{G(q^2) H(q)^2} - \frac{G(q^2) H(q)^2}{G(q)^2 H(q^2)} + 4 \right) \\ = G(q)^2 H(q)^2 G(q^2) H(q^2) \left(\frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} + 4 \right).$$

Substituting (3.2) into the above yields the desired identity. \square

Lemma 3.2. *We have*

$$(3.5) \quad H(q)^4 G(q^2)^2 - G(q)^4 H(q^2)^2 + G(q)^2 H(q)^2 G(q^2) H(q^2) = \frac{G(q^2)^3 H(q^2)^3}{G(q)^2 H(q)^2}.$$

Proof. We first prove

$$(3.6) \quad \frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} + 1 = \frac{\varphi(-q)^2}{\varphi(-q^5)^2}.$$

It follows from (2.4) that

$$\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} + 1 = \frac{1-k}{1+k} - \frac{1+k}{1-k} + 1 = \frac{1-4k-k^2}{1-k^2}.$$

Therefore, (3.6) is a direct consequence of (2.5). Finally,

$$\begin{aligned} & H(q)^4 G(q^2)^2 - G(q)^4 H(q^2)^2 + G(q)^2 H(q)^2 G(q^2) H(q^2) \\ &= G(q)^2 H(q)^2 G(q^2) H(q^2) \left(\frac{G(q^2) H(q)^2}{G(q)^2 H(q^2)} - \frac{G(q)^2 H(q^2)}{G(q^2) H(q)^2} + 1 \right) \\ &= G(q)^2 H(q)^2 G(q^2) H(q^2) \left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2} + 1 \right). \end{aligned}$$

We arrive at the desired identity by substituting (3.6) into the above. \square

4. PROOFS

4.1. Idea behind the proofs. Before presenting our proofs, it appears necessary to explain the idea behind them.

▷ *Guessing the 5-dissection.* Given a formal power series $\sum_{n=0}^{\infty} a_n q^n$, we may start by computing, for example, the first 1000 terms of each m -dissection slice $\sum_{n=0}^{\infty} a_{mn+l} q^n$ (with $0 \leq l \leq m-1$). If the slice has a nice product form, then such a q -product could be induced through the `prodmake` command of Garvan's *Maple* package `qseries`, see [9]. In other words, we can take advantage of the package `qseries` to conjecture the 5-dissections of $R(q)R(q^2)^2$, $1/R(q)R(q^2)^2$, $R(q)^2/R(q^2)$ and $R(q^2)/R(q)^2$.

In a recent work, Frye and Garvan in [8] further implemented another two *Maple* packages `thetaid` and `ramarobinsids` to provide automatic proofs of theta-function identities with the help of modular forms. It should be admitted

that the above conjectural 5-dissections could be shown automatically through the package `thet aids`. The interested reader may consult [21] for a recent application of this computer-assisted approach.

On the other hand, elementary proof of dissection formulas often has its own research interest. In this section, we will present such proofs of (1.5)–(1.8) in which only identities involving the Rogers-Ramanujan functions $G(q)$ and $H(q)$ will be utilized. In particular, the trick of *substitution of one* which will be discussed below makes our proof less routine.

▷ *Constructing an auxiliary product.* To start the proofs, we first need to reformulate $R(q)R(q^2)^2$, $1/R(q)R(q^2)^2$, $R(q)^2/R(q^2)$ and $R(q^2)/R(q)^2$ as

$$\begin{aligned} R(q)R(q^2)^2 &= \frac{(q^2, q^3; q^5)_\infty (q, q^9; q^{10})_\infty^2}{(q, q^4; q^5)_\infty (q^3, q^7; q^{10})_\infty^2} = \frac{H(q^2)^2}{G(q)^2 H(q)} \Big/ \frac{G(q^2)^2}{G(q) H(q)^2}, \\ \frac{1}{R(q)R(q^2)^2} &= \frac{(q, q^4; q^5)_\infty (q^3, q^7; q^{10})_\infty^2}{(q^2, q^3; q^5)_\infty (q, q^9; q^{10})_\infty^2} = \frac{G(q^2)^2}{G(q) H(q)^2} \Big/ \frac{H(q^2)^2}{G(q)^2 H(q)}, \\ \frac{R(q)^2}{R(q^2)} &= \frac{(q, q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty}{(q^2, q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty} = \frac{H(q)^2}{G(q^2) H(q^2)^2} \Big/ \frac{G(q)^2}{G(q^2)^2 H(q^2)}, \\ \frac{R(q^2)}{R(q)^2} &= \frac{(q^2, q^3; q^5)_\infty (q^2, q^8; q^{10})_\infty}{(q, q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty} = \frac{G(q)^2}{G(q^2)^2 H(q^2)} \Big/ \frac{H(q)^2}{G(q^2) H(q^2)^2}. \end{aligned}$$

As long as we have conjectured the 5-dissections through Garvan’s *Maple* package `qseries`, then in light of the above, it suffices to prove

$$\text{Dissection} = \text{Numerator} / \text{Denominator},$$

or

$$\text{Denominator} \times \text{Dissection} = \text{Numerator},$$

where the “Numerator” and “Denominator” come from the right-hand sides of the above reformulations. Now we multiply by a simple extra term on both sides of the above,

$$\text{Denominator} \times \text{Dissection} \times [\text{extra term}] = \text{Numerator} \times [\text{extra term}].$$

Our auxiliary products then come from here. More precisely, these auxiliary products have the form

$$\Pi_* = \Pi_{*1} \times \Pi_{*2}$$

such that $\Pi_{*1} = \text{Denominator}$ by Lemma 2.4 and $\Pi_{*2} = \text{Dissection} \times [\text{extra term}]$ by (2.1) and (2.2). Therefore, it suffices to verify

$$\Pi_* \stackrel{?}{=} \text{Numerator} \times [\text{extra term}].$$

▷ *Substitution of one.* To complete the proofs, we want to highlight a trick which we call the *substitution of one*. We first expand the auxiliary product $\Pi_* = \Pi_{*1} \times \Pi_{*2}$ and collect terms according to the power of q ,

$$\Pi_* = \Sigma_{*1} + q^5 \Sigma_{*2}$$

such that in both Σ_{*1} and Σ_{*2} , the powers of q are up to q^4 . We then see from (2.3) that

$$(4.1) \quad \frac{G(q^5)G(q^{10})^2}{q^5 H(q^5)H(q^{10})^2} \cdot \frac{G(q^5)^2 H(q^{10}) - H(q^5)^2 G(q^{10})}{G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})} = 1.$$

Hence, we may rewrite Π_* as

$$\Pi_* = \Sigma_{*1} + q^5 \Sigma_{*2} \times \left(\frac{G(q^5)G(q^{10})^2}{q^5 H(q^5)H(q^{10})^2} \cdot \frac{G(q^5)^2 H(q^{10}) - H(q^5)^2 G(q^{10})}{G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})} \right).$$

Expanding and then factoring the above, we arrive at an expression, where either Lemma 3.1 or Lemma 3.2 could be applied. Finally, we deduce $\Pi_* = \text{Numerator} \times [\text{extra term}]$ through Lemma 2.4.

Remark 4.1. It is notable that the trick of *substitution of one* has also been used in [3], [19]. However, in those work, 1 is substituted by different expressions involving the Rogers-Ramanujan functions $G(q)$ and $H(q)$ other than (4.1).

We will use (1.5) to present a detailed proof while the rest could be shown analogously.

4.2. Proof of (1.5). We define the auxiliary product by $\Pi_1 := \Pi_{11} \times \Pi_{12}$, where

$$\Pi_{11} = G(q^5)G(q^{10})^2 - qH(q^5)G(q^{10})^2 - 2q^3H(q^5)G(q^{10})H(q^{10}) + q^4G(q^5)H(q^{10})^2$$

and

$$\begin{aligned} \Pi_{12} = & 4G(q^5)^2 H(q^5)^3 G(q^{10})^3 H(q^{10})^3 \\ & - 2qG(q^5)H(q^5)^2 G(q^{10})^2 H(q^{10})^3 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & - 2q^2 G(q^5)^2 H(q^5)G(q^{10})H(q^{10})^4 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & + 4q^3 G(q^5)H(q^5)^2 G(q^{10})H(q^{10})^4 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & + q^4 H(q^5)H(q^{10})^4 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & \times (G(q^5)^2 H(q^{10}) - H(q^5)^2 G(q^{10})). \end{aligned}$$

Then by (2.8), we have

$$(4.2) \quad \Pi_{11} = \frac{G(q^2)^2}{G(q)H(q)^2}.$$

Also, in light of (2.1) and (2.2),

$$(4.3) \quad \begin{aligned} \Pi_{12} &= 4G(q^5)^4 H(q^5)^3 \\ &\times \left(\frac{G(q^{10})^3 H(q^{10})^3}{G(q^5)^2} - \frac{qG(q^{10})^4 H(q^{10})^3}{G(q^5)^2 H(q^5)(q^{25}; q^{50})_\infty^2} - \frac{q^2 G(q^{10})^3 H(q^{10})^4}{G(q^5) H(q^5)^2 (q^{25}; q^{50})_\infty^2} \right. \\ &\left. + \frac{2q^3 G(q^{10})^3 H(q^{10})^4}{G(q^5)^2 H(q^5)(q^{25}; q^{50})_\infty^2} + \frac{q^9 G(q^{10})^2 H(q^{10})^6}{G(q^5)^3 H(q^5)(q^{25}; q^{50})_\infty^4} \right). \end{aligned}$$

On the other hand, we expand the product in Π_1 and rearrange terms so that $\Pi_1 = \Sigma_{11} + q^5 \Sigma_{12}$, where

$$\begin{aligned} \Sigma_{11} &= 4G(q^5)^3 H(q^5)^3 G(q^{10})^5 H(q^{10})^3 \\ &\quad - 2q(3G(q^5)^2 H(q^5)^4 G(q^{10})^5 H(q^{10})^3 + G(q^5)^4 H(q^5)^2 G(q^{10})^4 H(q^{10})^4) \\ &\quad + 2q^2(G(q^5)H(q^5)^5 G(q^{10})^5 H(q^{10})^3 - G(q^5)^5 H(q^5)G(q^{10})^3 H(q^{10})^5) \\ &\quad - 2q^3(G(q^5)^2 H(q^5)^4 G(q^{10})^4 H(q^{10})^4 - 3G(q^5)^4 H(q^5)^2 G(q^{10})^3 H(q^{10})^5) \\ &\quad - q^4(G(q^5)H(q^5)^5 G(q^{10})^4 H(q^{10})^4 - 4G(q^5)^3 H(q^5)^3 G(q^{10})^3 H(q^{10})^5) \\ &\quad - G(q^5)^5 H(q^5)G(q^{10})^2 H(q^{10})^6) \end{aligned}$$

and

$$\begin{aligned} \Sigma_{12} &= (H(q^5)^6 G(q^{10})^4 H(q^{10})^4 + 2G(q^5)^2 H(q^5)^4 G(q^{10})^3 H(q^{10})^5 \\ &\quad + G(q^5)^4 H(q^5)^2 G(q^{10})^2 H(q^{10})^6) \\ &\quad - 2q(4G(q^5)H(q^5)^5 G(q^{10})^3 H(q^{10})^5 + 5G(q^5)^3 H(q^5)^3 G(q^{10})^2 H(q^{10})^6 \\ &\quad + G(q^5)^5 H(q^5)G(q^{10})H(q^{10})^7) \\ &\quad + 2q^2(H(q^5)^6 G(q^{10})^3 H(q^{10})^5 + 2G(q^5)^2 H(q^5)^4 G(q^{10})^2 H(q^{10})^6 \\ &\quad + G(q^5)^4 H(q^5)^2 G(q^{10})H(q^{10})^7) \\ &\quad - q^3(G(q^5)H(q^5)^5 G(q^{10})^2 H(q^{10})^6 - G(q^5)^5 H(q^5)H(q^{10})^8). \end{aligned}$$

By (4.1), we rewrite Π_1 as

$$\begin{aligned} \Pi_1 &= \Sigma_{11} + q^5 \Sigma_{12} \left(\frac{G(q^5)G(q^{10})^2}{q^5 H(q^5)H(q^{10})^2} \cdot \frac{G(q^5)^2 H(q^{10}) - H(q^5)^2 G(q^{10})}{G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})} \right) \\ &= G(q^5)G(q^{10})^2 H(q^{10})^2 \\ &\quad \times (G(q^5)^4 H(q^{10})^2 - H(q^5)^4 G(q^{10})^2 + 4G(q^5)^2 H(q^5)^2 G(q^{10})H(q^{10})) \\ &\quad \times (G(q^{10})^2 H(q^5) - 2qG(q^5)G(q^{10})H(q^{10})) \\ &\quad + q^3 G(q^5)H(q^{10})^2 + q^4 H(q^5)H(q^{10})^2). \end{aligned}$$

Applying (3.1) to the second factor and (2.9) to the third factor, we find that

$$(4.4) \quad \Pi_1 = \frac{4(q^{50}; q^{50})_\infty^4}{(q^{25}; q^{25})_\infty^4} \cdot G(q^5)^4 H(q^5)^3 \cdot \frac{H(q^2)^2}{G(q)^2 H(q)}.$$

Therefore, (1.5) follows (after simplification) from (4.2), (4.3) and (4.4) together with the fact that

$$R(q)R(q^2)^2 = \frac{H(q^2)^2}{G(q)^2 H(q)} \Big/ \frac{G(q^2)^2}{G(q)H(q)^2}.$$

4.3. The remaining dissections. We list the required auxiliary products in the proofs of (1.6), (1.7) and (1.8). The calculations are similar to those for (1.5) and therefore the details are omitted.

▷ *Dissection (1.6).* We require the auxiliary product $\Pi_2 = \Pi_{21} \times \Pi_{22}$, where

$$\Pi_{21} = H(q^5)G(q^{10})^2 - 2qG(q^5)G(q^{10})H(q^{10}) + q^3G(q^5)H(q^{10})^2 + q^4H(q^5)H(q^{10})^2$$

and

$$\begin{aligned} \Pi_{22} = & 4G(q^5)^4 H(q^5)G(q^{10})^3 H(q^{10})^3 \\ & + qG(q^5)G(q^{10})^2 H(q^{10})^2 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10}))^2 \\ & + 4q^2 G(q^5)^2 H(q^5)G(q^{10})^2 H(q^{10})^3 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & + 2q^3 G(q^5)H(q^5)^2 G(q^{10})^2 H(q^{10})^3 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & + 2q^4 G(q^5)^2 H(q^5)G(q^{10})H(q^{10})^4 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})). \end{aligned}$$

▷ *Dissection (1.7).* We require the auxiliary product $\Pi_3 = \Pi_{31} \times \Pi_{32}$, where

$$\Pi_{31} = G(q^5)^2 G(q^{10}) + 2qG(q^5)H(q^5)G(q^{10}) + q^2 G(q^5)^2 H(q^{10}) + q^4 H(q^5)^2 H(q^{10})$$

and

$$\begin{aligned} \Pi_{32} = & G(q^5)^2 H(q^5)^6 G(q^{10})^3 \\ & - qG(q^5)^3 H(q^5)^3 G(q^{10})H(q^{10}) \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & + 2q^2 G(q^5)^2 H(q^5)^4 G(q^{10})H(q^{10}) \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\ & - q^3 G(q^5)H(q^5)^3 H(q^{10}) \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10}))^2 \\ & + q^4 G(q^5)^2 H(q^5)^4 H(q^{10})^2 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})). \end{aligned}$$

▷ *Dissection (1.8).* We require the auxiliary product $\Pi_4 = \Pi_{41} \times \Pi_{42}$, where

$$\Pi_{41} = G(q^5)^2 G(q^{10}) + q^2 H(q^5)^2 G(q^{10}) + 2q^3 G(q^5)H(q^5)H(q^{10}) - q^4 H(q^5)^2 H(q^{10})$$

and

$$\begin{aligned}
 \Pi_{42} = & G(q^5)^6 H(q^5)^2 G(q^{10}) H(q^{10})^2 \\
 & + qG(q^5)^3 H(q^5)^3 G(q^{10}) H(q^{10}) \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\
 & - q^2 G(q^5)^2 H(q^5)^2 H(q^{10}) \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\
 & \times (G(q^5)^2 H(q^{10}) - H(q^5)^2 G(q^{10})) \\
 & - 2q^3 G(q^5)^3 H(q^5)^3 H(q^{10})^2 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})) \\
 & - q^4 G(q^5)^2 H(q^5)^4 H(q^{10})^2 \cdot (G(q^5)^2 H(q^{10}) + H(q^5)^2 G(q^{10})).
 \end{aligned}$$

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