Czechoslovak Mathematical Journal, 71 (146) (2021), 1099–1113

A VARIETY OF EULER'S SUM OF POWERS CONJECTURE

Tianxin Cai, Hangzhou, Yong Zhang, Changsha

Received May 22, 2020. Published online May 10, 2021.

Abstract. We consider a variety of Euler's sum of powers conjecture, i.e., whether the Diophantine system

$$\begin{cases} n = a_1 + a_2 + \ldots + a_{s-1}, \\ a_1 a_2 \ldots a_{s-1}(a_1 + a_2 + \ldots + a_{s-1}) = b^s \end{cases}$$

has positive integer or rational solutions n, b, a_i, $i = 1, 2, \ldots, s-1$, $s \geqslant 3$. Using the theory of elliptic curves, we prove that it has no positive integer solution for $s = 3$, but there are infinitely many positive integers n such that it has a positive integer solution for $s \geqslant 4$. As a corollary, for $s \geqslant 4$ and any positive integer n, the above Diophantine system has a positive rational solution. Meanwhile, we give conditions such that it has infinitely many positive rational solutions for $s \geqslant 4$ and a fixed positive integer n.

Keywords: Euler's sum of powers conjecture; elliptic curve; positive integer solution; positive rational solution

MSC 2020: 11D72, 11D41, 11G05

1. Introduction

In 1769, Euler (see [5], page 209) conjectured that the Diophantine equation

$$a_1^s + a_2^s + \ldots + a_{s-1}^s = a_s^s \tag{1.1}$$

Project was supported by the Natural Science Foundation of China (Grants No. 11501052 and No. 12071421), Younger Teacher Development Program of Changsha University of Science and Technology (Grant No. 2019QJCZ051), Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering (Changsha University of Science and Technology) and the Natural Science Foundation of Zhejiang Province (Project No. LY18A010016).

DOI: 10.21136/CMJ.2021.0210-20

has no positive integer solution $(a_1, a_2, \ldots, a_s)$ for $s \geqslant 3$. It is called *Euler's sum of powers conjecture.*

For $s = 3$, equation (1.1) corresponds to the case $n = 3$ of Fermat's Last Theorem: $x^n + y^n = z^n$, $n \geqslant 3$.

For $s = 4$, in 1988, Elkies in [4] disproved Euler's sum of powers conjecture by showing that $a_1^4 + a_2^4 + a_3^4 = a_4^4$ has infinitely many positive integer solutions. In particular, he gave the solution

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

Shortly after, Frye (see [5], page 210) got the smallest counterexample

$$95800^4 + 217519^4 + 414560^4 = 422481^4.$$

For $s = 5$, in 1966, Lander and Parkin in [6] found the first counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

In 2004, Frye (see [9]) obtained the only other known primitive solution for $s = 5$:

$$55^5 + 3183^5 + 28969^5 + 85282^5 = 85359^5.$$

For $s \geqslant 6$, there are no known solutions. More information about Euler's sum of powers conjecture can be found in [5], pages 209–218: *D*1 *Sums of like powers. Euler's conjecture.*

In 2013, Cai and Chen in [1] investigated the question to express a positive integer n as a sum of s positive integers whose product is a kth power, i.e.,

$$n = a_1 + a_2 + \ldots + a_s$$

such that

$$a_1 a_2 \ldots a_s = b^k$$

for positive integers n, a_i, b, k and $k \geqslant 2$. This can be considered a new variant of the Hilbert-Waring problem

$$n = a_1^k + a_2^k + \ldots + a_s^k.$$

In 2015, Cai, Chen and Zhang in [2] studied the Diophantine system

$$A + B = C, \quad ABC = D^n,$$

where A, B, C, D, n are positive integers and $n \geqslant 3$, which is a new generalization of Fermat's Last Theorem.

Now we expand this idea to Euler's sum of powers conjecture and consider whether the Diophantine system

$$(1.2) \qquad \begin{cases} n = a_1 + a_2 + \ldots + a_{s-1}, \\ a_1 a_2 \ldots a_{s-1}(a_1 + a_2 + \ldots + a_{s-1}) = b^s \end{cases}$$

has positive integer or rational solutions n, a_i, $i = 1, 2, \ldots, s-1$, b, and $s \geqslant 3$.

Obviously, the solutions of (1.1) are a subset of the solutions of (1.2). If a_i, $i = 1, 2, \ldots, s-1$, and $a_1 + a_2 + \ldots + a_{s-1}$ are relatively prime in pairs, (1.2) reduces to Euler's sum of powers conjecture. The motivation for studying (1.2) is that we try to find a counterexample to Euler's sum of powers conjecture for $s = 6$. Although we cannot obtain any counterexamples, we find some interesting results by the theory of elliptic curves.

For positive integer solutions of (1.2), we have:

Theorem 1.1. *For $s = 3$ and any positive integer n, (1.2) has no positive integer solution (a_1, a_2, b).*

This is a special case studied in [1], Theorem 2, but we give a proof in a different way.

Theorem 1.2.

(1) *For $s = 4$, there are infinitely many positive integers n such that (1.2) has a positive integer solution (a_1, a_2, a_3, b).*

(2) *For $s \geqslant 5$, there are infinitely many positive integers n such that (1.2) has a positive integer solution $(a_1, a_2, \ldots, a_{s-1}, b)$, which depends on $s-3$ positive rational parameters.*

For positive rational solutions, let us note that if (1.2) has a positive rational solution for some positive integer n, then it has a positive rational solution for each positive integer N, by the following transformation

$$\begin{cases} N = \dfrac{a_1 N}{n} + \dfrac{a_2 N}{n} + \ldots + \dfrac{a_{s-1} N}{n}, \\ \dfrac{a_1 N}{n} \cdot \dfrac{a_2 N}{n} \cdot \ldots \cdot \dfrac{a_{s-1} N}{n} \cdot \Big(\dfrac{a_1 N}{n} + \dfrac{a_2 N}{n} + \ldots + \dfrac{a_{s-1} N}{n}\Big) = \Big(\dfrac{bN}{n}\Big)^s. \end{cases}$$

Therefore, from Theorem 1.2, we obtain:

Corollary 1.1. *For $s \geqslant 4$ and any positive integer n, (1.2) has a positive rational solution.*

Furthermore, we can ask the question: for which positive integer n does (1.2) have infinitely many positive rational solutions? We give a partial answer to it under some conditions in the following theorem.

Theorem 1.3. *For $s \geqslant 4$ and a fixed positive integer n, if (1.2) has a positive rational solution $(a'_1, a'_2, \ldots, a'_{s-1}, b')$ and the elliptic curve*

$$\mathcal{E}_3\colon\ Y^2 = X^3 - 27u'^3v'(u'v'^3 - 24)X + 54u'^4(u'^2v'^6 - 36u'v'^3 + 216)$$

has positive rank, then (1.2) has infinitely many positive rational solutions.

2. Proofs of the theorems

P r o o f of Theorem 1.1. For $s = 3$, (1.2) reduces to

$$\begin{cases} \dfrac{n}{b} = b_1 + b_2, \\ b_1b_2(b_1 + b_2) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, 2.$$

Let us consider the cubic curve

$$b_1b_2(b_1 + b_2) = 1,$$

which equals

$$\Big(\frac{b_1}{b_2}\Big)^2 + \frac{b_1}{b_2} = \frac{1}{b_2^3}.$$

Let

$$u = \frac{b_1}{b_2}, \quad v = \frac{1}{b_2},$$

we have

$$u^2 + u = v^3.$$

Taking $y = 16u + 8$, $x = 4v$, we get the elliptic curve

$$y^2 = x^3 + 64.$$

Using the Magma package (see [7]), the rank of it is 0, and the only rational points are

$$(x, \pm y) = (8, 24),\ (0, 8),\ (-4, 0).$$

Tracing back, there is no positive integer solution of (1.2). □

P r o o f of Theorem 1.2. (1) For $s = 4$, (1.2) leads to

$$\begin{cases} \dfrac{n}{b} = b_1 + b_2 + b_3, \\ b_1 b_2 b_3 (b_1 + b_2 + b_3) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, 2, 3.$$

It's easy to see that $(a_1, a_2, a_3) = (1, 2, 24)$ satisfies (1.2), then

$$(b_1, b_2, b_3) = \Big(\frac{1}{6}, \frac{1}{3}, 4\Big).$$

Let us consider the positive rational solutions of the Diophantine system

$$\begin{cases} b_1 b_2 b_3 = \dfrac{2}{9}, \\ b_1 + b_2 + b_3 = \dfrac{9}{2}. \end{cases}$$

Eliminating b_3, we get

$$18 b_1^2 b_2 + 18 b_1 b_2^2 - 81 b_1 b_2 + 4 = 0,$$

then

$$18 \frac{b_2}{b_1} + 18 \Big(\frac{b_2}{b_1}\Big)^2 - 81 \frac{b_2}{b_1} \frac{1}{b_1} + 4 \Big(\frac{1}{b_1}\Big)^3 = 0.$$

Taking

$$u = \frac{b_2}{b_1}, \quad v = \frac{1}{b_1},$$

we have the curve

$$\mathcal{C}_1\colon\ 18u^2 + 18u - 81uv + 4v^3 = 0.$$

By the map $\varphi_1\colon\ \mathcal{C}_1 \mapsto \mathcal{E}_1$,

$$y = 384u - 864v + 192, \quad x = -32v + 243$$

and we obtain the elliptic curve

$$\mathcal{E}_1\colon\ y^2 = x^3 - 166779x + 26215254.$$

Using the Magma package (see [7]), the rank of $\mathcal{E}_1$ is 1 and then there are infinitely many rational points on it.

From the above transformations, we have

$$\begin{cases} b_1 = \dfrac{32}{243-x}, \\ b_2 = \dfrac{y-27x+6369}{12(243-x)}, \\ b_3 = \dfrac{-y-27x+6369}{12(243-x)}. \end{cases}$$

Then

$$\begin{cases} a_1 = & \dfrac{32}{243-x}b, \\ a_2 = & \dfrac{y-27x+6369}{12(243-x)}b, \\ a_3 = & \dfrac{-y-27x+6369}{12(243-x)}b \end{cases}$$

is the solution of (1.2).

In view of $b_i > 0$, we get the condition

$$x < \frac{2123}{9}, \quad |y| < -27x + 6369.$$

It is easy to see that the point $P = (235, 8)$ satisfies this condition. By the theorem of Poincaré and Hurwitz (see [10], Chapter V, page 78, Satz 11) about the density of rational points, then there are infinitely many rational points on $\mathcal{E}_1$ satisfying

$$x < \frac{2123}{9}, \quad |y| < -27x + 6369.$$

Therefore, we can find infinitely many positive rational solutions b_i, $i = 1, 2, 3$, which leads to infinitely many positive integers a_i, $i = 1, 2, 3$, by multiplying the least common denominator of b_i. This proves that for $s = 4$ there exist infinitely many positive integers n such that (1.2) has a positive integer solution.

(2) For $s \geqslant 5$, (1.2) reduces to

$$\begin{cases} \dfrac{n}{b} = b_1 + b_2 + \ldots + b_{s-1}, \\ b_1 b_2 \ldots b_{s-1}(b_1 + b_2 + \ldots + b_{s-1}) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, \ldots, s-1.$$

Take

$$x = b_1, \quad y = b_2, \quad z = b_3, \quad u = b_4 \ldots b_{s-1}, \quad v = b_4 + \ldots + b_{s-1},$$

then

$$\begin{cases} \dfrac{n}{b} = x + y + z + v, \\ xyzu(x + y + z + v) = 1. \end{cases}$$

We need to study the rational points on the cubic curve

$$xyzu(x + y + z + v) = 1.$$

Let $z = ut^2y$, then we get

$$t^2u^2y^2x^2 + t^2u^2y^2(t^2uy + v + y)x - 1 = 0. \tag{2.1}$$

Consider the above equation as a quadratic equation in x. If it has rational solutions, then the discriminant

$$\Delta(y) = u^2t^2y^2(u^2t^2(ut^2 + 1)^2y^4 + 2vu^2t^2(ut^2 + 1)y^3 + u^2v^2t^2y^2 + 4)$$

should be a perfect square. It follows from the study of the rational parametric solutions in $\mathbb{Q}(t)$ on the quartic curve

$$\mathcal{C}_2\colon\ w^2 = u^2t^2(ut^2 + 1)^2y^4 + 2vu^2t^2(ut^2 + 1)y^3 + u^2v^2t^2y^2 + 4.$$

The discriminant of $\mathcal{C}_2$ is

$$\Delta(t) = 256u^6t^6(ut^2 + 1)^4(64u^2t^4 + u(uv^4 + 128)t^2 + 64)$$

and is nonzero as an element of $\mathbb{Q}(t)$ since $u, v \in \mathbb{Q}^+$. Then $\mathcal{C}_2$ is smooth.

By the method described in [8], page 77 (or [3], page 476, Proposition 7.2.1), we can transform $\mathcal{C}_2$ into the family of elliptic curves

$$\mathcal{E}_2\colon\ Y^2 = X(X^2 + u^2v^2t^2X - 16u^2(ut^2 + 1)^2t^2)$$

by the inverse birational map $\varphi_2\colon\ \mathcal{C}_2 \mapsto \mathcal{E}_2$ with

$$y = \frac{Y - uvtX}{ut(ut^2 + 1)X}, \quad w = \frac{Y^2 - u^2v^2t^2X^2 - 2X^3}{4ut(ut^2 + 1)X^2}$$

and

$$\begin{aligned} X &= 2ut(ut^2 + 1)(t^3u^2y^2 + tuvy + tuy^2 - w), \\ Y &= 2u^2t^2(ut^2 + 1)(2t^2uy + v + 2y)(t^3u^2y^2 + tuvy + tuy^2 - w). \end{aligned}$$

In view of $u, v \in \mathbb{Q}^+$, we can take

$$u = \frac{p}{q}, \quad v = \frac{c}{d},$$

where p, q, c, d are positive integers. Let

$$U = q^4 d^2 X, \quad V = q^6 d^3 Y,$$

then $\mathcal{E}_2$ reduces to

$$\mathcal{E}_2' \colon\ V^2 = U(U^2 + c^2p^2q^2t^2U - 16d^4p^2q^4t^2(pt^2+q)^2).$$

Note that the point

$$P = (4pd^2q^2(pt^2+q)t, 4cd^2p^2q^3(pt^2+q)t^2)$$

lies on $\mathcal{E}_2'$. Using the group law on the elliptic curve, we obtain the point

$$2P = \Big(\frac{16q^2d^4(pt^2+q)^2}{c^2}, -\frac{64q^3d^6(pt^2+q)^3}{c^3}\Big).$$

Next, we determine the positive integer solutions of (1.2). From the birational map φ_2 and the point $-2P$, i.e., the reflected point of $2P$, we get

$$x = \frac{uv^3t}{2(4ut^2 - uv^2t + 4)}, \quad y = \frac{4ut^2 - uv^2t + 4}{2uvt(ut^2+1)}, \quad z = \frac{(4ut^2 - uv^2t + 4)t}{2v(ut^2+1)}.$$

To get $x > 0$, $y > 0$, $z > 0$ from $u > 0$, $v > 0$, we need

$$4ut^2 - uv^2t + 4 > 0,$$

the discriminant of it is $\delta = u(uv^4 - 64)$. If $\delta < 0$, then for any $t \in \mathbb{Q}$, we have

$$4ut^2 - uv^2t + 4 > 0.$$

If $\delta > 0$, then for any

$$t \in \left(0, \frac{uv^2 - \sqrt{u(uv^4 - 64)}}{8u}\right) \cup \left(\frac{uv^2 + \sqrt{u(uv^4 - 64)}}{8u}, \infty\right),$$

we have

$$4ut^2 - uv^2t + 4 > 0.$$

Hence, by the density of rational numbers, for any $u > 0, v > 0$, there are infinitely many positive rational numbers t such that $x > 0$, $y > 0$, $z > 0$. Then for any given positive rational numbers $b_4, \dots, b_{s-1}$, there are infinitely many positive rational numbers t such that $b_i > 0$, $i = 1, 2, 3$. Multiplying the least common multiple of the denominators of t, b_i, $i = 1, \dots, s-1$, we can get $a_i \in \mathbb{Z}^+$, $i = 1, \dots, s-1$.

Therefore, for $s \geqslant 5$ there are infinitely many positive integers n such that (1.2) has a positive integer solution $(a_1, a_2, \dots, a_{s-1}, b)$, which depends on $s - 3$ positive rational parameters t, b_i, $i = 4, \dots, s-1$. $\square$

Example 2.1.

(1) For $s=4$, the points

$$(x,y)=(235,8),\ \left(\frac{60266587}{257049},\frac{3852230624}{130323843}\right)$$

on $\mathcal{E}_1$ lead to

$$(a_1,a_2,a_3)=(1,2,24),\ (781943058,138991832,18609625).$$

Then

$$\begin{cases}27=1+2+24,\\ 1\cdot 2\cdot 24\cdot(1+2+24)=6^4\end{cases}$$

and

$$\begin{cases}939544515=781943058+138991832+18609625,\\ 781943058\cdot 138991832\cdot 18609625\\ \qquad\qquad\times(781943058+138991832+18609625)=208787670^4.\end{cases}$$

(2) For $s=5$, we have

$$x=\frac{uv^3t}{2(4ut^2-uv^2t+4)},\ y=\frac{4ut^2-uv^2t+4}{2uvt(ut^2+1)},\ z=\frac{(4ut^2-uv^2t+4)t}{2v(ut^2+1)},\ u=v=b_4.$$

Take

$$b=2uvt(ut^2+1)(4ut^2-uv^2t+4),$$

then

$$\begin{aligned}a_1&=t^2u^6(ut^2+1), & a_2&=(4ut^2-tu^3+4)^2,\\ a_3&=ut^2(4ut^2-tu^3+4)^2, & a_4&=2tu^3(ut^2+1)(4ut^2-tu^3+4).\end{aligned}$$

If $t=1$, $u=1$, we have

$$\begin{cases}128=2+49+49+28,\\ 2\cdot 49\cdot 49\cdot 28\cdot(2+49+49+28)=28^5.\end{cases}$$

When $t=2$, $u=1$, we obtain

$$\begin{cases}2000=20+324+1296+360,\\ 20\cdot 324\cdot 1296\cdot 360\cdot(20+324+1296+360)=360^5,\end{cases}$$

which reduces to

$$\begin{cases}500=5+81+324+90,\\ 5\cdot 81\cdot 324\cdot 90\cdot(5+81+324+90)=90^5.\end{cases}$$

P r o o f of Theorem 1.3. For $s \geqslant 4$, (1.2) is equivalent to

$$\begin{cases} \frac{n}{b} = b_1 + b_2 + \ldots + b_{s-1}, \\ b_1 b_2 \ldots b_{s-1}(b_1 + b_2 + \ldots + b_{s-1}) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \ i = 1, \ldots, s-1.$$

Put

$$x = b_1, \quad y = b_2, \quad u = b_3 \ldots b_{s-1}, \quad v = b_3 + \ldots + b_{s-1},$$

then

$$\begin{cases} \frac{n}{b} = x + y + v, \\ xyu(x + y + v) = 1. \end{cases}$$

For $s \geqslant 4$ and a fixed positive integer n, if (1.2) has a positive rational solution $(a'_1, a'_2, \ldots, a'_{s-1}, b')$, then we have

$$u' = b'_3 \ldots b'_{s-1}, \quad v' = b'_3 + \ldots + b'_{s-1},$$

where

$$b'_i = \frac{a'_i}{b'} \in \mathbb{Q}^+, \ i = 3, \ldots, s-1.$$

Now we need to study the rational points on the cubic curve

$$\mathcal{C}_3\colon \ xyu'(x + y + v') = 1,$$

which can be transformed into the elliptic curve

$$\mathcal{E}_3\colon \ Y^2 = X^3 - 27u'^3v'(u'v'^3 - 24)X + 54u'^4(u'^2v'^6 - 36u'v'^3 + 216).$$

The map $\varphi_3\colon \mathcal{C}_3 \mapsto \mathcal{E}_3$ is given by

$$X = \frac{3u'(u'v'^2x + 12)}{x}, \quad Y = -\frac{108u'^2(x + 2y + v')}{x}$$

and its inverse map φ_3^{-1} is

$$x = \frac{36u'}{X - 3u'^2v'^2}, \quad y = -\frac{Y + 3u'(v'X - 3u'^2v'^3 + 36u')}{6u'(X - 3u'^2v'^2)}.$$

To get positive rational solutions x and y from the inverse map φ_3^{-1}, we have the condition

$$X > 3u'^2v'^2, \quad Y + 3u'(v'X - 3u'^2v'^3 + 36u') < 0.$$

From the first assumption of Theorem 1.3, $\mathcal{C}_3$ has a rational point

$$Q = (x, y) = \Big(\frac{a_1'}{b'}, \frac{a_2'}{b'}\Big),$$

so the rational point

$$R = (X, Y) = \varphi_3(Q) = \Big(\frac{3u'(a_1'u'v'^2 + 12b')}{a_1'}, -\frac{108u'^2(b'v' + a_1' + 2a_2')}{a_1'}\Big)$$

on $\mathcal{E}_3$ satisfies the above condition. If $\mathcal{E}_3$ has positive rank, by the theorem of Poincaré and Hurwitz (see [10], Chapter V, page 78, Satz 11) about the density of rational points, $\mathcal{E}_3$ has infinitely many rational points satisfying

$$X > 3u'^2v'^2, \quad Y + 3u'(v'X - 3u'^2v'^3 + 36u') < 0.$$

Therefore, there are infinitely many positive rational solutions (x, y) satisfying

$$\begin{cases} \dfrac{n}{b'} = x + y + v', \\ xyu'(x + y + v') = 1, \end{cases}$$

i.e.,

$$\begin{cases} n = xb' + yb' + a_3' + \ldots + a_{s-1}', \\ (xb')(yb')(a_3') \ldots (a_{s-1}')(xb' + yb' + a_3' + \ldots + a_{s-1}') = b'^s. \end{cases}$$

Then for $s \geqslant 4$ and a fixed positive integer n, (1.2) has infinitely many positive rational solutions $(a_1, a_2, a_3, \ldots, a_{s-1}, b) = (xb', yb', a_3', \ldots, a_{s-1}', b')$. □

Example 2.2. For $s = 4$ and $n = 27$, (1.2) has a positive integer solution

$$\begin{cases} 27 = 1 + 2 + 24, \\ 1 \cdot 2 \cdot 24 \cdot (1 + 2 + 24) = 6^4. \end{cases}$$

Then

$$u' = v' = 4$$

and $\mathcal{E}_3$ reduces to

$$\mathcal{E}_3(4, 4)\colon\ Y^2 = X^3 - 1603584X + 781553664.$$

Using the Magma package (see [7]), the rank of $\mathcal{E}_3(4, 4)$ is 1, then there are infinitely many rational points on it. From the positive rational solution

$$(x, y) = \Big(\frac{1}{6}, \frac{1}{3}\Big)$$

on $\mathcal{C}_3(4,4)$: $4xy(x+y+4)=1$, we can get the point

$$(X,Y)=(1632,-50112)$$

on $\mathcal{E}_3(4,4)$, which satisfies the condition

$$X>768, \quad Y<-48X+35136.$$

By the theorem of Poincaré and Hurwitz, there are infinitely many rational points satisfying this condition. Thus, for $s=4$ and $n=27$, (1.2) has infinitely many positive rational solutions $(a_1,a_2,a_3,b)=(6x,6y,24,6)$. From the points

$$(X,Y)=(26752,-4370752),\ \Big(\frac{1595044704}{616225},-\frac{57181013356608}{483736625}\Big),$$
$$\Big(\frac{1119732063118503868848}{216521728021016209},-\frac{36437253415550579162358351801216}{100751663784490024367909177}\Big)$$

on $\mathcal{E}_3(4,4)$, we obtain the positive rational solutions

$$(x,y)=\Big(\frac{9}{1624},\frac{841}{168}\Big),\ \Big(\frac{616225}{7790166},\frac{2550409}{3829230}\Big),$$
$$\Big(\frac{216521728021016209}{662113455554419041 9},\frac{4217686043540528329}{3000378379947935118}\Big).$$

Remark 2.1. For $s=4$ and $n=1$, from the smallest counterexample of Euler's sum of powers conjecture found by Frye, see [5], page 210:

$$95800^4+217519^4+414560^4=422481^4,$$

we see that (1.2) has a positive rational solution

$$(a_1,a_2,a_3,b)=\Big(\frac{95800^4}{422481^4},\frac{217519^4}{422481^4},\frac{414560^4}{422481^4},\frac{95800\cdot 217519\cdot 414560}{422481^3}\Big),$$

i.e.,

$$\begin{cases} 1=\dfrac{95800^4}{422481^4}+\dfrac{217519^4}{422481^4}+\dfrac{414560^4}{422481^4}, \\ \dfrac{95800^4}{422481^4}\cdot\dfrac{217519^4}{422481^4}\cdot\dfrac{414560^4}{422481^4}\cdot 1=\Big(\dfrac{95800\cdot 217519\cdot 414560}{422481^3}\Big)^4. \end{cases}$$

Then

$$u'=v'=\frac{414560^3}{95800\cdot 217519\cdot 422481}.$$

By the same method, we can show that for $s = 4$ and $n = 1$, (1.2) has infinitely many positive rational solutions

$$(a_1, a_2, a_3, b) = \Big(\frac{95800 \cdot 217519 \cdot 414560}{422481^3}x, \frac{95800 \cdot 217519 \cdot 414560}{422481^3}y, \frac{95800^4}{422481^4}, \frac{95800 \cdot 217519 \cdot 414560}{422481^3}\Big).$$

Because the solutions are huge, we just list an example:

$$(x, y) = \Big(\frac{63888017184631726926167265740411010804865370118657598000000}{16386283355822543519946496843630803411504474368317131012609}, \frac{35001612452233282367362385354102826683854709981313862387928 1}{94090754282869340379266492338266378799875157305180543251840000}\Big).$$

3. Some related questions

From

$$n = a_1 + a_2 + \ldots + a_{s-1}, \quad a_i \in \mathbb{Z}^+,$$

we can see that (1.2) has finitely many or no positive integer solutions for $s \geqslant 4$ and a fixed positive integer n. So we can ask the following question.

Question 3.1. *For $s \geqslant 4$, which is the least positive integer n such that (1.2) has positive integer solutions $(a_1, \ldots, a_{s-1}, b)$?*

When $s = 4, 5, 6$, it is easy to get the least positive integers $n = 18, 27, 8$, where

$$\begin{cases} 18 = 1 + 8 + 9, \\ 1 \cdot 8 \cdot 9 \cdot (1 + 8 + 9) = 6^4, \end{cases} \quad \begin{cases} 27 = 1 + 2 + 12 + 12, \\ 1 \cdot 2 \cdot 12 \cdot 12 \cdot (1 + 2 + 12 + 12) = 6^5, \end{cases}$$

and

$$\begin{cases} 8 = 1 + 1 + 2 + 2 + 2, \\ 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot (1 + 1 + 2 + 2 + 2) = 2^6. \end{cases}$$

For $s \geqslant 4$, let $N(n, s)$ denote the number of the positive integer solutions of (1.2), we raise the following question.

Question 3.2. *For $s \geqslant 4$ and a fixed positive integer $n \geqslant 1$, can we give the formula for $N(n, s)$?*

In Theorem 1.3, we give conditions such that (1.2) has infinitely many positive rational solutions for $s \geqslant 4$ and a fixed positive integer n. However, the conditions are not easy to check. Meanwhile, we have the following question.

Question 3.3. *For positive rational numbers u and v, are there infinitely many positive rational solutions $b_3, \ldots, b_{s-1}$ of the Diophantine system*

$$\begin{cases} u = b_3 \ldots b_{s-1}, \\ v = b_3 + \ldots + b_{s-1} \end{cases} \quad \text{for } s \geqslant 5?$$

When $s = 5$, we have

$$\begin{cases} u = b_3 b_4, \\ v = b_3 + b_4. \end{cases}$$

It is easy to get the condition, which is

$$u = \frac{v^2 - w^2}{4}, \quad v > w,$$

then

$$b_3 = \frac{v + w}{2}, \quad b_4 = \frac{v - w}{2}.$$

In 2014, Ulas in [11] proved that for each $k \geqslant 4$ and rational numbers A, B with $AB \neq 0$, the Diophantine system

$$\begin{cases} A = x_1 + x_2 + \ldots + x_k, \\ B = x_1 x_2 \ldots x_k \end{cases}$$

has infinitely many solutions depending on $k - 3$ free parameters. In his proof, $x_3 = -4Bt^2 x_1$, so if $B > 0$, then x_1 and x_3 cannot be positive simultaneously. Therefore, we cannot get a positive answer to Question 3.3 by Ulas' result.

References

[1] *T. Cai, D. Chen*: A new variant of the Hilbert-Waring problem. Math. Comput. *82* (2013), 2333–2341. zbl MR doi

[2] *T. Cai, D. Chen, Y. Zhang*: A new generalization of Fermat's last theorem. J. Number Theory *149* (2015), 33–45. zbl MR doi

[3] *H. Cohen*: Number Theory. Vol. I: Tools and Diophantine Equations. Graduate Texts in Mathematics 239. Springer, New York, 2007. zbl MR doi

[4] *N. D. Elkies*: On $A^4 + B^4 + C^4 = D^4$. Math. Comput. *184* (1988), 825–835. zbl MR doi

[5] *R. K. Guy*: Unsolved Problems in Number Theory. Problem Books in Mathematics. Springer, New York, 2004. zbl MR doi

[6] *L. J. Lander, T. R. Parkin*: Counterexample to Euler's conjecture on sums of like powers. Bull. Am. Math. Soc. *72* (1966), 1079. zbl MR doi

[7] Magma computational algebra system for algebra, number theory and geometry. Available at `http://magma.maths.usyd.edu.au/magma/`. sw

[8] *L. J. Mordell*: Diophantine Equations. Pure and Applied Mathematics 30. Academic Press, London, 1969. zbl MR

[9] *E. S. Rowland*: Elliptic curve and integral solutions to $A^4+B^4+C^4=D^4$. Available at `https://ericrowland.github.io/papers/Elliptic_curves_and_integral_solutions_to_A^4+B^4+C^4=D^4.pdf` (2004), 7 pages.
[10] *T. Skolem*: Diophantische Gleichungen. Ergebnisse der Mathematik und ihrer Grenzgebiete 5. Springer, Berlin, 1938. (In German.) zbl
[11] *M. Ulas*: On some Diophantine systems involving symmetric polynomials. Math. Comput. *83* (2014), 1915–1930. zbl MR doi

Authors' addresses: Tianxin Cai, School of Mathematical Sciences, Zhejiang University, 38 Zheda Rd, Hangzhou, 310027, P. R. China, e-mail: `txcai@zju.edu.cn`; Yong Zhang (corresponding author), School of Mathematics and Statistics, Changsha University of Science and Technology, Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, 960, 2nd Section, Wanjiali RD (S), Changsha 410114, P. R. China, e-mail: `zhangyongzju@163.com`.