

ON THE DISTRIBUTION OF (k, r) -INTEGERS
IN PIATETSKI-SHAPIRO SEQUENCES

TEERAPAT SRICHAN, Bangkok

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Abstract. A natural number n is said to be a (k, r) -integer if $n = a^k b$, where $k > r > 1$ and b is not divisible by the r th power of any prime. We study the distribution of such (k, r) -integers in the Piatetski-Shapiro sequence $\{[n^c]\}$ with $c > 1$. As a corollary, we also obtain similar results for semi- r -free integers.

Keywords: (k, r) -integer; Piatetski-Shapiro sequence

MSC 2020: 11L07, 11N37

1. INTRODUCTION AND RESULTS

Piatetski-Shapiro sequences (PS-sequences) are defined by

$$\mathbb{N}^c = \{[n^c]\}_{n \in \mathbb{N}} \quad (c > 1, c \notin \mathbb{N}),$$

where $[z]$ is the integer part of a real z . The PS-sequence was introduced first by Piatetski-Shapiro (see [5]) to study prime numbers in a sequence of the form $[f(n)]$, where $f(n)$ is a polynomial. A positive integer is called *r-free* if it is not divisible by the r th power of any prime. In 1978, Rieger in [6] remarked on a paper of Stux (see [7]) concerning square-free integers of the form $[n^c]$, and showed that for real $x > 1$,

$$\sum_{\substack{n \leq x \\ [n^c] \text{ is square-free}}} 1 = \frac{6}{\pi^2} x + O(x^{(2c+1)/4+\varepsilon}) \quad \text{for } 1 < c < \frac{3}{2}.$$

In [1], Cao and Zhai improved Rieger's range by the method of exponential sums and proved that for $\varepsilon > 0$,

$$\sum_{\substack{n \leq x \\ [n^c] \text{ is square-free}}} 1 = \frac{6}{\pi^2} x + O(x^{36(c+1)/97+\varepsilon}) \quad \text{for } 1 < c < \frac{61}{36},$$

and in [2] they improved their range to $c < \frac{149}{87}$. In 2017, Zhang and Li in [9] studied in case of cube-free integers and showed that for $\varepsilon > 0$

$$(1.1) \quad \sum_{\substack{n \leq x \\ [n^c] \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + O(x^{1-\varepsilon}) \quad \text{for } 1 < c < \frac{11}{6}.$$

Very recently, Deshouillers in [3] used the approach of Rieger to improve the error term in (1.1) and showed that

$$\sum_{\substack{n \leq x \\ [n^c] \text{ is cube-free}}} 1 = \frac{x}{\zeta(3)} + O(x^{(c+1)/3} \log x) \quad \text{for } 1 < c < 2.$$

It is natural to investigate this problem for other special integers. In this paper, we study the distribution of (k, r) -integers in PS-sequences. Let k and r be fixed integers such that $1 < r < k$. Any positive integer n is called a (k, r) -integer if n can be written in the form $a^k b$, where b is an r -free integer. When k tends to infinity, a (∞, r) -integer is the same as an r -free integer, one might consider (k, r) -integers as generalized r -free integers. A positive integer n is called *semi- r -free* if in its canonical factorization no exponent is equal to r . The (k, r) -integers include also the semi- r -free integers when $k = r + 1$. The semi- r -free integers in PS-sequences are also obtained in this paper. Let us first recall the notion of exponent pair taken from [4], Chapter 2.

Definition 1.1. Let $A \geq 1$, $B \geq 1$, and suppose that for all C in $[B, 2B]$,

$$\sum_{B \leq n \leq C \leq 2B} e^{2\pi i f(n)} = O(A^\kappa B^\lambda)$$

for some pair (κ, λ) of real numbers satisfying $0 \leq \kappa \leq 1/2 \leq \lambda \leq 1$, and for any real function $f \in C^\infty[B, 2B]$ satisfying for all $r \geq 1$ and for $x \in [B, 2B]$, $AB^{1-r} \ll |f^{(r)}(x)| \ll AB^{1-r}$, where the constants implied by \ll depend only on r . Then we call (κ, λ) an *exponent pair*.

Using the simple one-dimensional exponent pair, we obtain the following results.

Theorem 1.1. *Let k and r be fixed integers such that $1 < r < k$. For any two exponent pairs (κ_1, λ_1) and (κ_2, λ_2) satisfying $r\lambda_1 - \kappa_1 < 1$, $r\lambda_2 - \kappa_2 > 1$, $k\lambda_1 - \kappa_1 > 1$ and $(r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2)) > 1$, we have*

$$\sum_{\substack{n \leq N \\ \lfloor n^c \rfloor \text{ is } (k,r)\text{-integer}}} 1 = \frac{\zeta(k)}{\zeta(r)} N + O(N^{(c/r) + \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N)$$

for $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))$, where

$$\delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2) = \frac{(\lambda_2 \kappa_1 - \lambda_1 \kappa_2) + r^{-1}(\kappa_2 - \kappa_1)}{\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2)}.$$

For example, if $(\kappa_1, \lambda_1) = (\frac{2}{7}, \frac{4}{7})$ and $(\kappa_2, \lambda_2) = (\frac{1}{6}, \frac{2}{3})$, we obtain the result in Theorem 1.1 for $(k, 2)$ -integers as

$$\sum_{\substack{n \leq N \\ \lfloor n^c \rfloor \text{ is square-free}}} 1 = \frac{\zeta(k)}{\zeta(2)} N + O(N^{(c/2) + (3/16)} \log N) \quad \text{for } 1 < c < \frac{13}{8}.$$

Since the (k, r) -integers include the semi- r -free integers when $k = r + 1$, we obtain the following corollary.

Corollary 1.1. *With the same assumptions as in Theorem 1.1, we have*

$$\sum_{\substack{n \leq N \\ \lfloor n^c \rfloor \text{ is semi-}r\text{-free}}} 1 = \frac{\zeta(r+1)}{\zeta(r)} N + O(N^{(c/r) + \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N)$$

for $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))$.

Notation. Throughout this paper, ε denotes a fixed positive constant, not necessarily the same in all occurrences. As usual, let $\mu(n)$ denote the Möbius function, and $\psi(x) = x - [x] - \frac{1}{2}$.

2. LEMMAS

In this section we state the lemmas which are needed in our proof.

Lemma 2.1 ([8], Lemma 2.6). *If $q_{k,r}$ denotes the characteristic function of the set of (k,r) -integers, then*

$$q_{k,r}(n) = \sum_{a^k b^r c = n} \mu(b).$$

Lemma 2.2 ([8], Theorem 3.1). *For $x \geq 3$, we have*

$$Q_{k,r}(x) := \sum_{n \leq x} q_{k,r}(n) = \frac{x\zeta(k)}{\zeta(r)} + O(x^{1/r} \exp(-B \log^{3/5}(x)(\log \log x)^{-1/5})),$$

where B is a positive constant depending on r and the O -estimate is uniform in k .

Lemma 2.3 ([1], Lemma). *Let $y > 0$, $X > 1$, $0 \leq \sigma < 1$, $g(n) = (n + \sigma)^\gamma$. Then, for any exponent pair (κ, λ) ,*

$$\sum_{n \sim X} \psi(yg(n)) \ll y^{\kappa/(1+\kappa)} X^{(\lambda+\gamma\kappa)/(1+\kappa)} + y^{-1} X^{1-\lambda}.$$

3. PROOF OF THEOREM 1.1

Let k and r be fixed integers such that $1 < r < k$. For any two exponent pairs (κ_1, λ_1) and (κ_2, λ_2) satisfying $r\lambda_1 - \kappa_1 < 1$, $r\lambda_2 - \kappa_2 > 1$, $k\lambda_1 - \kappa_1 > 1$ and $1 < c < (r(\lambda_2 - \lambda_1) - (\kappa_2 - \kappa_1))/(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))$, let $\gamma = 1/c$ and

$$T_c(N) = \sum_{\substack{n \leq N \\ \lfloor n^c \rfloor \text{ is } (k,r)\text{-integer}}} 1.$$

We note that, $\lfloor n^c \rfloor$ is a (k,r) -integer, if and only if $m^\gamma \leq n < (m+1)^\gamma$, where m is a (k,r) -integer. Therefore

$$T_c(N) = \sum_{\substack{m \leq N^c \\ m \text{ is } (k,r)\text{-integer}}} (\lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor) + O(1).$$

In view of Lemma 2.1, we have

$$T_c(N) = \sum_{a^k b^r m \leq N^c} \mu(b) (\lfloor -a^{k\gamma} b^{r\gamma} m^\gamma \rfloor - \lfloor -(a^k b^r m + 1)^\gamma \rfloor) + O(1).$$

Thus,

$$(3.1) \quad T_c(N) = \sum_{a^k b^r m \leq N^c} \mu(b)((a^k b^r m + 1)^\gamma - a^{k\gamma} b^{r\gamma} m^\gamma) + E_c(N),$$

where

$$(3.2) \quad E_c(N) = \sum_{a^k b^r m \leq N^c} \mu(b)(\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) + O(1).$$

The first sum on the right-hand side of (3.1) is

$$\begin{aligned} & \sum_{a^k b^r m \leq N^c} \mu(b) a^{k\gamma} b^{r\gamma} m^\gamma \left(\frac{\gamma}{a^k b^r m} + O(a^{-2k} b^{-2r} m^{-2}) \right) \\ &= \gamma \sum_{a^k b^r m \leq N^c} \mu(b) a^{k\gamma-k} b^{r\gamma-r} m^{\gamma-1} + O\left(\left| \sum_{a^k b^r m \leq N^c} a^{k\gamma-2k} b^{r\gamma-2r} m^{\gamma-2} \right| \right) \\ &= \gamma \sum_{a^k b^r m \leq N^c} \mu(b) a^{k\gamma-k} b^{r\gamma-r} m^{\gamma-1} + O(1) = \gamma \sum_{\substack{n \leq N^c \\ n \text{ is } (k,r)\text{-integer}}} n^{\gamma-1} + O(1). \end{aligned}$$

We apply Abel's identity and Lemma 2.2, then we have

$$(3.3) \quad \gamma \sum_{\substack{n \leq N^c \\ n \text{ is } (k,r)\text{-integer}}} n^{\gamma-1} = \frac{\zeta(k)}{\zeta(r)} N + O(N^{(r-cr+c)/r} \exp(-C \log^{3/5}(x) (\log \log x)^{-1/5})),$$

where C is a positive constant depending on r and the O -estimate is uniform in k .

Now it remains to bound (3.2). For this we use the simple one dimensional exponent pair in Lemma 2.3. We write the sum in (3.2) as

$$\begin{aligned} & \sum_{a^k b^r m \leq N^c} \mu(b)(\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\ &= \sum_{a \leq N^{c/k}} \sum_{b \leq N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)). \end{aligned}$$

Taking $B = N^{c/r + (\kappa_2 - \kappa_1)/(r(\lambda_2(\kappa_1 + 1) - \lambda_1(\kappa_2 + 1)))}$, we write

$$(3.4) \quad \begin{aligned} & \sum_{a \leq N^{c/k}} \sum_{b \leq N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\ &= \sum_{a \leq N^{c/k}} \sum_{b \leq B} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\ & \quad + \sum_{a \leq N^{c/k}} \sum_{B < b \leq N^{c/r}/a^{k/r}} \mu(b) \\ & \quad \times \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)). \end{aligned}$$

In view of Lemma 2.3 with the exponent pair $(\kappa, \lambda) = (\kappa_1, \lambda_1)$, we have

$$\begin{aligned}
(3.5) \quad & \sum_{b \leq B} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
& \ll \log N \sum_{b \leq B} \left((a^{k\gamma} b^{r\gamma})^{\kappa_1/(1+\kappa_1)} \left(\frac{N^c}{a^k b^r} \right)^{(\lambda_1 + \gamma \kappa_1)/(1+\kappa_1)} \right. \\
& \quad \left. + (a^{k\gamma} b^{r\gamma})^{-1} \left(\frac{N^c}{a^k b^r} \right)^{1-\gamma} \right) \\
& \ll a^{-k\lambda_1/(1+\kappa_1)} N^{(c\lambda_1 + \kappa_1)/(1+\kappa_1)} \log N \sum_{b \leq B} b^{-r\lambda_1/(1+\kappa_1)} + a^{-k} N^{c-1} \log N \\
& \ll a^{-k\lambda_1/(1+\kappa_1)} B^{1-r\lambda_1/(1+\kappa_1)} N^{(c\lambda_1 + \kappa_1)/(1+\kappa_1)} \log N + a^{-k} N^{c-1} \log N.
\end{aligned}$$

In view of $k\lambda_1/(1+\kappa_1) > 1$ and (3.5), the first term in (3.4) is bounded by

$$\begin{aligned}
(3.6) \quad & \sum_{a \leq N^{c/k}} \sum_{b \leq B} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
& \ll B^{1-r\lambda_1/(1+\kappa_1)} N^{(c\lambda_1 + \kappa_1)/(1+\kappa_1)} \log N + N^{c-1} \log N \\
& \ll N^{(c/r) + \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N,
\end{aligned}$$

where

$$\delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2) = \frac{(\lambda_2 \kappa_1 - \lambda_1 \kappa_2) + r^{-1}(\kappa_2 - \kappa_1)}{\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2)}.$$

Now we bound the second sum in (3.4). In view of Lemma 2.3 with the exponent pair $(\kappa, \lambda) = (\kappa_2, \lambda_2)$, we have

$$\begin{aligned}
(3.7) \quad & \sum_{B < b \leq N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\
& \ll \log N \sum_{B < b \leq N^{c/r}/a^{k/r}} \left((a^{k\gamma} b^{r\gamma})^{\kappa_2/(1+\kappa_2)} \left(\frac{N^c}{a^k b^r} \right)^{(\lambda_2 + \gamma \kappa_2)/(1+\kappa_2)} \right. \\
& \quad \left. + (a^{k\gamma} b^{r\gamma})^{-1} \left(\frac{N^c}{a^k b^r} \right)^{1-\gamma} \right) \\
& \ll a^{-k\lambda_2/(1+\kappa_2)} N^{(c\lambda_2 + \kappa_2)/(1+\kappa_2)} \\
& \quad \times \log N \sum_{B < b \leq N^{c/r}/a^{k/r}} b^{-r\lambda_2/(1+\kappa_2)} + a^{-k} N^{c-1} \log N \\
& \ll a^{-k\lambda_2/(1+\kappa_2)} B^{1-r\lambda_2/(1+\kappa_2)} N^{(c\lambda_2 + \kappa_2)/(1+\kappa_2)} \log N \\
& \quad + a^{-k} N^{c-1} \log N.
\end{aligned}$$

In view of $k\lambda_2/(1 + \kappa_2) > 1$ and (3.7), the second term in (3.4) is bounded by

$$(3.8) \quad \sum_{a \leq N^{c/k}} \sum_{B < b \leq N^{c/r}/a^{k/r}} \mu(b) \sum_{m \leq N^c/a^k b^r} (\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) \\ \ll B^{1-r\lambda_2/(1+\kappa_2)} N^{(c\lambda_2+\kappa_2)/(1+\kappa_2)} \log N + N^{c-1} \log N \\ \ll N^{(c/r)+\delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N.$$

From $r\lambda_1/(1 + \kappa_1) < 1 < r\lambda_2/(1 + \kappa_2)$, we have

$$0 < \frac{r\lambda_2}{1 + \kappa_2} - \frac{r\lambda_1}{1 + \kappa_1} = \frac{r(\lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2))}{(1 + \kappa_1)(1 + \kappa_2)}.$$

Then

$$(3.9) \quad \lambda_2(1 + \kappa_1) - \lambda_1(1 + \kappa_2) > 0.$$

From $\kappa_1 > r\lambda_1 - 1$ and $\kappa_2 < r\lambda_2 - 1$, we have

$$(3.10) \quad (\lambda_2\kappa_1 - \lambda_1\kappa_2) + r^{-1}(\kappa_2 - \kappa_1) = \frac{1}{r}(r\lambda_2\kappa_1 - r\lambda_1\kappa_2 + \kappa_2 - \kappa_1) \\ = \kappa_1(r\lambda_2 - 1) - \kappa_2(r\lambda_1 - 1) \\ > \kappa_1\kappa_2 - \kappa_2\kappa_1 = 0.$$

From (3.9) and (3.10), we have

$$(3.11) \quad \delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2) > 0.$$

In view of (3.6) and (3.8) we have

$$(3.12) \quad \sum_{a^k b^r m \leq N^c} \mu(b)(\psi(-(a^k b^r m + 1)^\gamma) - \psi(-a^{k\gamma} b^{r\gamma} m^\gamma)) = O(N^{(c/r)+\delta(\kappa_1, \lambda_1, \kappa_2, \lambda_2)} \log N).$$

In view of (3.11), the bound (3.12) dominates the error term in (3.3). Thus, the conclusion follows from (3.1), (3.3) and (3.12).

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Author's address: Teerapat Srichan, Department of Mathematics, Faculty of Science, Kasetsart University, PO Box 50 Phahon Yothin Rd, Chatuchak District, Bangkok, 10900 Thailand, e-mail: fscitrp@ku.ac.th.