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# ON THE POWERS OF QUASIHOMOGENEOUS TOEPLITZ OPERATORS

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*Abstract.* We present sufficient conditions for the existence of $p$th powers of a quasihomogeneous Toeplitz operator $T_{\mathrm{e}^{\mathrm{i}s\theta}\psi}$, where $\psi$ is a radial polynomial function and $p$, $s$ are natural numbers. A large class of examples is provided to illustrate our results. To our best knowledge those examples are not covered by the current literature. The main tools in the proof of our results are the Mellin transform and some classical theorems of complex analysis.

*Keywords*: quasihomogeneous Toeplitz operator; Mellin transform

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## 1. Introduction

Let $\mathrm{d}A(r\mathrm{e}^{\mathrm{i}\theta}) = r\,\mathrm{d}r\,\mathrm{d}\theta/\pi$ be the normalized Lebesgue area measure in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. The Bergman space $L^2_a(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D}, \mathrm{d}A)$ consisting of all holomorphic functions on $\mathbb{D}$ and it has the set $\{z^n\}_{n\in\mathbb{N}}$ as an orthogonal basis.

Let $P$ denote the Bergman projection which is the orthogonal projection from $L^2(\mathbb{D}, \mathrm{d}A)$ onto $L^2_a(\mathbb{D})$. For $f \in L^2(\mathbb{D}, \mathrm{d}A)$, the Toeplitz operator $T_f$, with symbol $f$, acting on $L^2_a(\mathbb{D})$ is defined by

$$T_f g = P(fg)$$

for all $g$ in $L^2_a(\mathbb{D})$ such that the product $fg$ is in $L^2(\mathbb{D}, \mathrm{d}A)$. It is easy to see that any bounded holomorphic function is in the domain of $T_f$. Therefore, $T_f$ is a densely defined operator on $L^2_a(\mathbb{D})$. Moreover, if the symbol $f$ is bounded, then $T_f$ is a bounded operator and $\|T_f\| \leqslant \|f\|_\infty$.

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A Toeplitz operator $T_f$ is called *quasihomogeneous Toeplitz operator* of degree an integer $p$ if its symbol $f$ can be written as $f(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}p\theta}\phi(r)$, where $\phi$ is a radial function in $L^2([0,1], r\,\mathrm{d}r)$. Such class of Toeplitz operators has been extensively studied. The reader can refer to [3], [5], [6], [7], [8], [9].

In [5], Louhichi has introduced the notion of the $p$th root (or power) of a quasihomogeneous Toeplitz operator which turns out to be very useful in investigating the question of commutativity of Toeplitz operators. In fact, Louhichi proved the existence of $p$th roots for the case $\phi(r) = r^n$, $n \in \mathbb{N}$ and for any $p \in \mathbb{N}$. Later with Rao in [6], they extended this result to a more general class of $\phi(r)$.

The aim of this work is to study the powers of quasihomogeneous Toeplitz operators when the radial part of the symbol is a linear combination of $r^\alpha$ and $r^\beta \log^\gamma(r)$, where $\alpha$, $\beta$, $\gamma$ are nonnegative integers. Under certain conditions, we show the existence of $p$th powers for any $p \in \mathbb{N}$.

## 2. Preliminaries

For a function $\phi \in L^1([0,1], r\,\mathrm{d}r)$ we define the Mellin transform of $\phi$, denoted $\widehat{\phi}$ by

$$\widehat{\phi}(z) = \int_0^1 \phi(r) r^{z-1}\,\mathrm{d}r.$$

It is clear that for $\phi \in L^1([0,1], r\,\mathrm{d}r)$, $\widehat{\phi}$ is a bounded holomorphic function on the half-plane $\Pi = \{z\colon \Re z > 2\}$. Moreover, the Mellin transform $\widehat{\phi}$ is uniquely determined by its values on any arithmetic sequence of integers. In fact, we have the following classical theorem, see [10], page 102.

**Theorem 2.1.** *Suppose $f$ is a bounded holomorphic function on $\{z\colon \Re z > 0\}$ that vanishes at the pairwise distinct points $z_1, z_2, \ldots$, where*

(1) $\inf\{|z_n|\} > 0$,

(2) $\sum\limits_{n\geqslant 1} \Re(1/z_n) = \infty$.

*Then $f$ vanishes identically on $\{z\colon \Re z > 0\}$.*

The inversion formula of the Mellin transform is given by

$$\phi(r) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \widehat{\phi}(z) r^{-z}\,\mathrm{d}r, \tag{2.1}$$

where the integration is along a vertical line through $\Re(z) = c$ in $\Pi$.

For the sake of completeness we choose to state the following classical lemma of complex analysis (see [2], Lemma 2.2, page 29), which we will use later to prove our results.

**Lemma 2.2.** *Let $\overline{f}(s)$ be a holomorphic function in the right half-plane $\Re s > \gamma$. If $|\overline{f}(r\mathrm{e}^{\mathrm{i}\theta})| < Cr^{-\nu}$ with $-\pi \leqslant \theta \leqslant \pi$ and $r > R_0$ for some constants $R_0$, $C$ and $\nu$ $(> 0)$, then for all $t > 0$ we have*

$$\lim_{r\to\infty}\int_{\Gamma_1} \mathrm{e}^{st}\overline{f}(s)\,\mathrm{d}s = 0 \quad \textit{and} \quad \lim_{r\to\infty}\int_{\Gamma_2} \mathrm{e}^{st}\overline{f}(s)\,\mathrm{d}s = 0,$$

*where $\Gamma_1$ and $\Gamma_2$ are, respectively, the arcs BCD and DEA of $\Gamma$ as shown in Figure 1.*

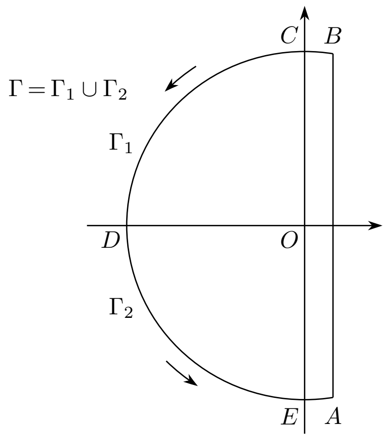


Figure 1. The standard Bromwich contour

We will also need the following easy lemma.

**Lemma 2.3.** *Let $f$ be a linear combination of $r^{\alpha}$ and $r^{\beta}\log^{\gamma}(r)$, where $\alpha$, $\beta$, and $\gamma$ are in $\mathbb{Z}$. If $\alpha \geqslant -1$ and $\beta, \gamma \in \mathbb{N}$, then $f \in L^2([0,1], r\,\mathrm{d}r)$.*

The following lemma determines the values of powers of a bounded quasihomogeneous Toeplitz operator evaluated at any element of the orthogonal basis of $L^2_a(\mathbb{D})$. In fact, quasihomogeneous Toeplitz operator and its powers map the space of polynomials in $z$ into itself.

**Lemma 2.4.** *Let $p, s \in \mathbb{N}$ and let $\psi$ be a radial function in $L^1([0,1], r\,\mathrm{d}r)$. Then for all $n \in \mathbb{N}$ we have*

$$\begin{aligned}(T_{\mathrm{e}^{\mathrm{i}s\theta}\psi})^p(\xi^n)(z) &= \left[\prod_{j=0}^{p-1} 2(n+js+s+1)\widehat{\psi}(2n+2js+s+2)\right] z^{n+ps}\\ &= \frac{\prod_{j=0}^{p-1}\widehat{\psi}(2n+2js+s+2)}{\prod_{j=0}^{p-1}\widehat{\mathbb{1}}(2n+2js+2s+2)} z^{n+ps},\end{aligned}$$

*where $\mathbb{1}$ denotes the constant function with value one.*

## 3. Main results

With respect to the definition of the $p$th root in [5], we analogously say that a quasihomogeneous Toeplitz operator $T_{\mathrm{e}^{\mathrm{i}s\theta}\psi}$ has a $p$th power if and only if there exists a radial function $\phi$ such that

$$(T_{\mathrm{e}^{\mathrm{i}s\theta}\psi})^p = T_{\mathrm{e}^{\mathrm{i}ps\theta}\phi}.$$

In particular, $(T_{\mathrm{e}^{\mathrm{i}p\theta}\psi})^0 = I$, where $I$ is the identity operator in $L^2_a(\mathbb{D})$.

We are now ready to state our main result which can be seen as an extension of [5], Corollary 18, page 1474.

**Theorem 3.1.** *Let $\psi(r) = \sum\limits_{i=1}^{m} a_i r^{k_i}$ be a nonzero radial polynomial function and let $P_{\widehat{\psi}} = \{-k_i\colon\ i = 1, \ldots, m\}$ and $Z_{\widehat{\psi}}$ be the sets of poles and zeros of its Mellin transform $\widehat{\psi}$, respectively. Assume that:*

(1) *For $i = 1, \ldots, m$, at least one $k_i$ is an odd number. We denote by $k_{i_0}$ the biggest odd number.*

(2) *There exists a set of integers $\{\alpha_i\}_{i=1, i\neq i_0}^{i=m}$ such that:*
  - (i) *$\{\alpha_i\}_{i=1, i\neq i_0}^{i=m} \subseteq Z_{\widehat{\psi}}$, and*
  - (ii) *for all $i \in \{1, \ldots, m\} \setminus \{i_0\}$ we have $-k_i < \alpha_i \leqslant k_i + 1$ and $k_i$ and $\alpha_i$ have the same parity.*

*Then $(T_{\mathrm{e}^{\mathrm{i}\theta}\psi})^p$ is always a Toeplitz operator for all $p \in \mathbb{N}$.*

Proof. Since $\psi(r) = \sum\limits_{i=1}^{m} a_i r^{k_i}$, we have

$$\widehat{\psi}(z) = \int_0^1 \psi(r) r^{z-1}\,\mathrm{d}r = \int_0^1 \sum_{i=1}^{m} a_i r^{k_i} r^{z-1}\,\mathrm{d}r = \sum_{i=1}^{m} \frac{a_i}{z + k_i},$$

and hence $P_{\widehat{\psi}} = \{-k_i\colon\ i = 1, \ldots, m\}$. Clearly (see [4], pages 105–106) the function $\widehat{\psi}$ can be extended to a meromorphic function in $\mathbb{C}$. This implies, together with the hypothesis of the theorem, that $\widehat{\psi}$ can be written as

$$\widehat{\psi}(z) = \frac{\prod_{i=1, i\neq i_0}^{m} (z - \alpha_i)}{\prod_{i=1}^{m} (z + k_i)} f(z),$$

where $f$ is holomorphic and nonzero in a neighborhood of each pole $-k_i$, with $i = 1, \ldots, m$. Next, for any integer $p \geqslant 1$ we show the existence of $\phi$ in $L^2([0,1], r\,\mathrm{d}r)$ so that $(T_{\mathrm{e}^{\mathrm{i}\theta}\psi})^p = T_{\mathrm{e}^{\mathrm{i}p\theta}\phi}$. Indeed, Lemma 2.4 implies that for all integers $n \geqslant 0$ we have

$$\prod_{j=0}^{p-1} (2n + 2j + 4)\widehat{\psi}(2n + 2j + 3) = (2n + 2p + 2)\widehat{\phi}(2n + p + 2).$$

Using Theorem 2.1, it is easy to see that if $p=1$, then $\phi\equiv\psi$. So we let $p\geqslant 2$, we complexify the previous equality by letting $z=2n+p+2$, and we obtain

$$\begin{aligned}\widehat{\phi}(z) &= \prod_{j=0}^{p-2}(z-p+2j+2)\prod_{j=0}^{p-1}\widehat{\psi}(z-p+2j+1)\\ &= \frac{\prod_{l=0}^{p-2}(z-p+2l+2)\prod_{(i,s)=(1,0),i\neq i_0}^{(m,p-1)}(z-\alpha_i-p+2s+1)}{\prod_{(d,j)=(1,0)}^{(m,p-1)}(z+k_d-p+2j+1)}h(z),\end{aligned}$$

where $h(z)=\prod\limits_{j=0}^{p-1}f(z-p+2j+1)$. Thus, $\widehat{\phi}$ is a meromorphic function in $\mathbb{C}$ and has simple poles at the integers $-k_d+p-2j-1$, with $(d,j)=(1,0),\ldots,(m,p-1)$. Moreover, the line of integration in the inversion formula (2.1) is shifted to the left while taking residues into the Bromwich contour (see Figure 1). Using Lemma 2.2 and the residue theorem, we conclude that $\phi$ is determined by the sum of the residues at all poles to the left of $\Re z=c$ and we have

$$\phi(r)=\sum_{(d,j)=(1,0)}^{(m,p-1)}\operatorname{Res}\widehat{\phi}(z)\big|_{z=-k_d+p-2j-1}r^{k_d-p+2j+1}. \tag{3.1}$$

**Claim 3.2.** *$\phi$ belongs to $L^2([0,1],r\,\mathrm{d}r)$.*

Proof. To prove this, it is sufficient, using Lemma 2.3, to show that $P_{\widehat{\phi}}\subseteq\mathbb{Z}_-$. Without loss of generality, we may assume that $k_1<k_2<\ldots<k_m$. Then we have the following cases:

*Case 1*: $p\leqslant k_1$. We have

$$p\leqslant k_1<k_2<\ldots<k_m$$

and so

$$k_d-p\geqslant 0\quad\forall\, d\in\{1,\ldots,m\}.$$

Therefore

$$k_d-p+2j+1\geqslant 1\quad\forall\, d\in\{1,\ldots,m\}\ \forall\, j\in\{0,\ldots,p-1\}.$$

*Case 2*: $k_n<p\leqslant k_{n+1}$ for $n\in\{1,\ldots,m-1\}$. Here we consider two subcases.

*Subcase 2.1*: $n\in\{1,\ldots,m-1\}\setminus\{i_0\}$. From Case 1 we know that

$$k_d-p+2j+1\geqslant 1\quad\forall\, d\in\{n+1,\ldots,m\}\ \forall\, j\in\{0,\ldots,p-1\}.$$

Now for $d\in\{1,\ldots,n\}$ we have

$$k_d-p+2j+1<0,\quad\text{i.e.,}\ j<\frac{-k_d+p-1}{2}.$$

If we let $j_0 = \lfloor \frac{1}{2}(-k_d + p - 1) \rfloor$ to be the greatest integer function of $\frac{1}{2}(-k_d + p - 1)$, then for all $d \in \{1, \dots, n\}$ and all $j \in \{0, \dots, j_0\}$ we have

$$k_d - p + 2j + 1 < 0. \tag{3.2}$$

Next, we shall prove that the poles of $\widehat{\phi}$ in (3.2) are canceled by zeros of $\widehat{\phi}$. In other words,

$$\forall\, (j, d) \in \{(0,1), \dots, (j_0, n)\},\ \exists\, (i, s) \in \{(1,0), \dots, (m, p-1)\},\ \text{and } i \neq i_0$$

such that

$$-\alpha_i + 2s = k_d + 2j.$$

To do so, we take $i = d$ and we let $s = \frac{1}{2}(k_d + \alpha_d + 2j)$. Then by hypothesis (2) (ii), $s \in \mathbb{N}$. Moreover,

$$2s = k_d + \alpha_d + 2j \leqslant k_d + \alpha_d + 2j_0 \leqslant \alpha_d + p - 1 < 2p,$$

which implies $s \leqslant p - 1$.

*Subcase 2.2*: $n = i_0$. Then for all $j \in \{0, \dots, j_0 = \lfloor \frac{1}{2}(-k_{i_0} + p - 1) \rfloor\}$ we have

$$k_{i_0} - p + 2j + 1 < 0. \tag{3.3}$$

Similarly to the previous subcase, those poles of $\widehat{\phi}$ are canceled by zeros of $\widehat{\phi}$. In fact, by taking $l = \frac{1}{2}(k_{i_0} + 2j - 1)$, it is easy to see that $l \in \{0, \dots, p-2\}$ and also that $2l + 2 = k_{i_0} + 2j + 1$ for all $j \in \{0, \dots, j_0\}$.

*Case 3*: $p > k_m$. We follow the same argument as in Case 2. □

This completes the proof of Theorem 3.1. □

**Remark 3.3.** If $\widehat{\phi}$ has poles of multiplicity greater than 1, the expression (3.1) becomes

$$\phi(r) = \sum_{(d,j)=(1,0)}^{(m,p-1)} \alpha_{d,j} (\log r)^n r^{k_d - p + 2j + 1},$$

where $n \in \mathbb{N}$ is the multiplicity of the pole $z = -k_d + p - 2j - 1$; and the same argument in the proof remains true.

**Example 3.4.** Let $\psi(r) = 3r - 12r^2 + 10r^3$. Then

$$\widehat{\psi}(z) = \frac{3}{z+1} - \frac{12}{z+2} + \frac{10}{z+3} = \frac{(z-1)(z-2)}{(z+1)(z+2)(z+3)}.$$

Using Lemma 2.4, we obtain for all $n \geqslant 0$

$$\begin{aligned}(T_{e^{i\theta}\psi})^p(\xi^n)(z) &= \left[\prod_{j=0}^{p-1} 2(n+j+2)\widehat{\psi}(2n+2j+3)\right] z^{n+p} \\ &= \frac{\prod_{j=0}^{p-1}(2n+2j+4)(2n+2j+2)(2n+2j+1)}{\prod_{j=0}^{p-1}(2n+2j+4)(2n+2j+5)(2n+2j+6)} z^{n+p} \\ &= \frac{\prod_{j=0}^{p-1}(2n+2j+2)(2n+2j+1)}{\prod_{j=2}^{p+1}(2n+2j+1)(2n+2j+2)} z^{n+p} \\ &= \frac{(2n+2)(2n+1)(2n+4)(2n+3)}{(2n+2p+2)(2n+2p+1)(2n+2p+4)(2n+2p+3)} z^{n+p}.\end{aligned}$$

Now we want to find a radial function $\phi$ such that

$$(T_{e^{i\theta}\psi})^p(\xi^n)(z) = T_{e^{ip\theta}\phi}(\xi^n)(z)$$

for every integer $p \geqslant 1$. This is equivalent to finding $\phi$ for which

$$\begin{aligned}T_{e^{ip\theta}\phi}(\xi^n)(z) &= (2n+2p+2)\widehat{\phi}(2n+p+2)z^{n+p} \\ &= \frac{(2n+2)(2n+1)(2n+4)(2n+3)}{(2n+2p+2)(2n+2p+1)(2n+2p+4)(2n+2p+3)} z^{n+p},\end{aligned}$$

and so for all $n \geqslant 0$, we must have

$$\widehat{\phi}(2n+p+2) = \frac{(2n+2)(2n+1)(2n+4)(2n+3)}{(2n+2p+2)^2(2n+2p+1)(2n+2p+4)(2n+2p+3)}.$$

Using Theorem 2.1 and letting $z = 2n+p+2$, we obtain

$$\widehat{\phi}(z) = \frac{(z-p)(z-p-1)(z-p+2)(z-p+1)}{(z+p)^2(z+p-1)(z+p+2)(z+p+1)}.$$

Clearly $\widehat{\phi}$ is holomorphic on $\{z\colon \Re z > 0\}$ and has simple poles at $1-p$, $-p-2$, $-p-1$ and a double pole at $-p$. Finally, to find the function $\phi$, we use the inverse Mellin transform and the residue theorem and we obtain

$$\begin{aligned}\phi(r) &= \operatorname{Res} \widehat{\phi}(z)\big|_{z=1-p} r^{p-1} + \operatorname{Res} \widehat{\phi}(z)\big|_{z=-p-1} r^{p+1} \\ &\quad + \operatorname{Res} \widehat{\phi}(z)\big|_{z=-p-2} r^{p+2} + \operatorname{Res} \widehat{\phi}(z) r^{-z}\big|_{z=-p} \\ &= a_1 r^{p-1} + a_2 r^{p+1} + a_3 r^{p+2} + (a_4 + a_5 \log r) r^p,\end{aligned}$$

where $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ are real constants. It is worth mentioning here that for all $p \geqslant 1$, the function $\phi$ is “nearly bounded” (see [1], page 204) and hence $T_{e^{ip\theta}\phi}$ is a bounded Toeplitz operator.

In the following proposition, we prove the existence of nonpolynomial radial functions $\psi$ for which $(T_{e^{i\theta}\psi})^p$ is always a Toeplitz operator for all $p \in \mathbb{N}$.

**Proposition 3.5.** *There exist nonpolynomial functions $\psi$ such that the power $(T_{e^{i\theta}\psi})^p$ is always a Toeplitz operator for all integers $p \geqslant 1$.*

Proof. Let $\psi(r) = r + 4r^2 \log r$. Then

$$\widehat{\psi}(z) = \frac{-4}{(z+2)^2} + \frac{1}{z+1} = \frac{z^2}{(z+1)(z+2)^2}.$$

Using Lemma 2.4, we obtain that for all $n \geqslant 0$ and all $p \geqslant 1$

$$\begin{aligned}(T_{e^{i\theta}\psi})^p(\xi^n)(z) &= \left[\prod_{j=0}^{p-1} 2(n+j+2)\widehat{\psi}(2n+2j+3)\right] z^{n+p} \\ &= \frac{\prod_{j=0}^{p-1}(2n+2j+4)(2n+2j+3)^2}{\prod_{j=0}^{p-1}(2n+2j+4)(2n+2j+5)^2} z^{n+p} = \frac{\prod_{j=0}^{p-1}(2n+2j+3)^2}{\prod_{j=0}^{p-1}(2n+2j+5)^2} z^{n+p} \\ &= \frac{\prod_{j=0}^{p-1}(2n+2j+3)^2}{\prod_{j=1}^{p}(2n+2j+3)^2} z^{n+p} = \frac{(2n+3)^2}{(2n+2p+3)^2} z^{n+p}.\end{aligned}$$

We want to find a radial function $\phi$ such that

$$(T_{e^{i\theta}\psi})^p(\xi^n)(z) = T_{e^{ip\theta}\phi}(\xi^n)(z)$$

for every integer $p \geqslant 1$ and all $n \geqslant 0$. This is equivalent to finding $\phi$ for which

$$T_{e^{ip\theta}\phi}(\xi^n)(z) = (2n+2p+2)\widehat{\phi}(2n+p+2)z^{n+p} = \frac{(2n+3)^2}{(2n+2p+3)^2} z^{n+p}.$$

So for all $n \geqslant 0$ we must have

$$\widehat{\phi}(2n+p+2) = \frac{(2n+3)^2}{(2n+2p+2)(2n+2p+3)^2}.$$

Using Theorem 2.1 and letting $z = 2n+p+2$, we obtain

$$\widehat{\phi}(z) = \frac{(z-p+1)^2}{(z+p)(z+p+1)^2}.$$

Clearly $\widehat{\phi}$ is holomorphic on $\{z : \Re z > 0\}$ and has a simple pole at $-p$ and a double pole at $-p-1$. Finally, to recover the function $\phi$, we use the inverse Mellin transform and the residue theorem and we obtain

$$\phi(r) = \operatorname{Res}\widehat{\phi}(z)\big|_{z=-p} r^p + \operatorname{Res}\widehat{\phi}(z)\big|_{z=-p-1} r^{p+1} = (1-2p)^2 r^p + 4pr^{p+1}((1-p)+p\log r).$$

Since for all $p \geqslant 1$, the function $\phi$ is nearly bounded, $T_{e^{ip\theta}\phi}$ is a genuine Toeplitz operator. $\square$

**Remark 3.6.** Note that in Example 3.4 (or Proposition 3.5), instead of using the inverse Mellin transform and the residue theorem to obtain the function $\phi$, one can recover $\phi$ from its Mellin transform by writing the partial fraction decomposition of $\widehat{\phi}(z)$ and then by using the identities $\widehat{r^m}(z) = 1/(z+m)$ and $\widehat{r^m \log^n}(r)(z) = (-1)^n n!/(z+m)^{n+1}$ for all nonnegative integers $m$ and $n$.

Using similar arguments and notation as in the proof of Theorem 3.1, we obtain the following corollary. The proof is omitted.

**Corollary 3.7.** *Let $\psi(r) = \sum_{i=1}^{m} a_i r^{k_i}$ be a nonzero polynomial function and $s \in \mathbb{N}^*$. Assume that:*

(1) *For $i = 1, \ldots, m$ there exists at least one $k_i$ such that $k_i - s$ is a nonnegative integer and is divisible by $2s$. Let $k_{i_0}$ be the biggest of such numbers.*
(2) *There exists a set of integers $\{\alpha_i\}_{i=1, i\neq i_0}^{i=m}$ such that:*
   (i) $\{\alpha_i\}_{i=1, i\neq i_0}^{i=m} \subseteq Z_{\widehat{\psi}}$;
   (ii) *for all $i \in \{1, \ldots, m\} \setminus \{i_0\}$ we have $-k_i < \alpha_i \leqslant k_i + s$ and $\alpha_i + k_i$ is divisible by $2s$.*

*Then $(T_{e^{is\theta}\psi})^p$ is always a Toeplitz operator for all $p \in \mathbb{N}$.*

**Example 3.8.** Let $m$, $n$ be in $\mathbb{N}$.

(1) There exist $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{N}$ such that $T_{e^{is\theta}(\alpha r^n + \beta r^m)}$ has always a $p$th power for all $p \geqslant 1$. For example, $(T_{e^{3i\theta}(-6r^2/7 + 13r^9/7)})^p$ is always a Toeplitz operator.
(2) For all $p, s \in \mathbb{N}$, the product $(T_{e^{is\theta}r^m})^p$ is a Toeplitz operator if and only if $m \geqslant s$ and $m - s$ is divisible by $2s$.

**Remark 3.9.**

(i) In [5] Theorem 13, page 1472, Louhichi showed that if $T_{e^{i\theta}\psi}$ has $p$th powers and if $T_f$ is a bounded Toeplitz operator such that $T_f T_{e^{i\theta}\psi} = T_{e^{i\theta}\psi} T_f$, then $T_f$ must be the sum of powers of $T_{e^{i\theta}\psi}$. In the same spirit and under the hypothesis of Theorem 3.1 (or Corollary 3.7), if $T_f$ commutes with $T_{e^{i\theta}\psi}$ (or $T_{e^{is\theta}\psi}$), then $T_f$ is the sum of powers of $T_{e^{i\theta}\psi}$ (or $T_{e^{is\theta}\psi}$) as well.
(ii) Let $\psi$ be a nonzero polynomial function and let $s$ be a natural number. If $T_{e^{is\theta}\psi}$ has $p$th powers for all $p \in \mathbb{N}$, then by Theorem 3.1 and Corollary 3.7 it is easy to see that, there exists a positive integer $n$ such that $(T_{e^{in\theta}\psi})^p$ is a Toeplitz operator for all $p$.
(iii) We recall that $T_f^* = T_{\overline{f}}$, where $\overline{f}$ is the complex conjugate of $f$. So by taking the adjoint, $T_{e^{-i\theta}\psi}$ has a $p$th power if and only if $T_{e^{i\theta}\psi}$ has it as well. Therefore, the previous results remain true for quasihomogeneous Toeplitz operator of negative degrees.

In what follows, we discuss the case of radial Toeplitz operators.

**Theorem 3.10.** *Let* $\psi(r)=\sum\limits_{i=1}^{m}a_ir^{k_i}$ *be a nonzero polynomial symbol. Then for all* $p\in\mathbb{N}$ *there exists a radial symbol* $\phi\in L^2([0,1],r\,\mathrm{d}r)$*, such that*

$$(T_\psi)^p=T_\phi.$$

*Moreover, when* $p\geqslant 2$*, we have*

$$\phi(r)=\sum_{i,j}\alpha_{i,j}r^{\beta_i}(\log r)^{\gamma_j},\quad \text{where } \beta_i,\gamma_j\in\mathbb{N}\quad\text{and}\quad \alpha_{i,j}\in\mathbb{R}.$$

P r o o f. As shown at the beginning of the proof of Theorem 3.1, $\widehat{\psi}$ can be written as

$$\widehat{\psi}(z)=\frac{1}{\prod_{i=1}^{m}(z+k_i)}f(z),$$

where $f$ is holomorphic and nonzero in a neighborhood of every pole $-k_i$, $i=1,\dots,m$. Now, we prove the existence of $\phi$ in $L^2([0,1],r\,\mathrm{d}r)$ for which $(T_{\mathrm{e}^{\mathrm{i}\theta}\psi})^p=T_{\mathrm{e}^{\mathrm{i}p\theta}\phi}$ for any integer $p\geqslant 1$. If such $\phi$ exists, Lemma 2.4 implies that we must have

$$(2n+2)^{p-1}[\widehat{\psi}(2n+2)]^p=\widehat{\phi}(2n+2)\quad\forall\,n\geqslant 0.$$

Note that $p$ is a positive integer and that our discussion is trivial for $p=1$ since in this case $\phi\equiv\psi$. So we assume $p\geqslant 2$. By setting $z=2n+2$, we obtain

$$\widehat{\phi}(z)=z^{p-1}[\widehat{\psi}(z)]^p=\frac{z^{p-1}}{\prod_{i=1}^{m}(z+k_i)^p}h(z),$$

where $h(z)=(f(z))^p$. In a similar way as in the proof of Theorem 3.1 and using Leibniz formula, we have that

$$\begin{aligned}\phi(r)&=\sum_{i=1}^{m}\operatorname{Res}\widehat{\phi}(z)\cdot r^{-z}\big|_{z=-k_i}\\&=\sum_{i=1}^{m}\Big(\frac{1}{(p-1)!}\lim_{z\to-k_i}\frac{\partial^{p-1}}{\partial z^{p-1}}\frac{z^{p-1}}{\prod_{l=1,l\neq i}^{m}(z+k_l)^p}h(z)\cdot r^{-z}\Big)\\&=\sum_{i=1}^{m}\Big(\frac{1}{(p-1)!}\lim_{z\to-k_i}\sum_{j=0}^{p-1}\frac{(p-1)!}{j!(p-1-j)!}g^{(j)}(z)\cdot(r^{-z})^{(p-1-j)}\Big)\\&=\sum_{i=1}^{m}\Big(\sum_{j=0}^{p-1}\frac{1}{j!(p-1-j)!}g^{(j)}(-k_i)\cdot(-1)^{p-1-j}(\log r)^{p-1-j}(r^{k_i})\Big),\end{aligned}$$

where $g^{(j)}$ is the $j$th derivative of the function

$$g(z)=\frac{z^{p-1}}{\prod_{l=1,l\neq i}^{m}(z+k_l)^p}h(z).$$

Finally, by letting

$$\alpha_{i,j} = \frac{(-1)^{p-1-j}}{j!\,(p-1-j)!} g^{(j)}(-k_i), \quad \beta_i = k_i \geqslant 0 \quad \text{and} \quad \gamma_i = p-1-j \geqslant 0,$$

we obtain the desired result. □

**Remark 3.11.** Theorem 3.9 remains true in the case where $\psi$ is a linear combination of functions of the form $r^\beta \log^\gamma(r)$, where $\beta$, $\gamma$ are nonnegative integers.

**Example 3.12.** Let $\psi(r) = r^m$ with $m \in \mathbb{N}$. Then $\widehat{\psi}(z) = 1/(z+m)$. Again, Lemma 2.4 implies that for all $p \geqslant 1$ and all $n \geqslant 0$ we have

$$(T_\psi)^p(\xi^n)(z) = \left[\prod_{j=0}^{p-1}(2n+2)\widehat{\psi}(2n+2)\right] z^n = \frac{(2n+2)^p}{(2n+2+m)^p} z^n.$$

We want to find a radial symbol $\phi$ such that

$$(T_\psi)^p(\xi^n)(z) = T_\phi(\xi^n)(z)$$

for all $n \geqslant 0$. This is equivalent to finding $\phi$ such that

$$\widehat{\phi}(2n+2) = \frac{(2n+2)^{p-1}}{(2n+2+m)^p}.$$

Using Theorem 2.1 and letting $z = 2n+2$, we obtain

$$\widehat{\phi}(z) = \frac{z^{p-1}}{(z+m)^p}.$$

Clearly $\widehat{\phi}$ has a pole of order $p$ at $z = -m$. In order to obtain $\phi$, we choose to proceed as follows (but one can also use the partial fraction decomposition of $\widehat{\phi}(z)$ as mentioned in Remark 3.6):

$$\begin{aligned}
\phi(r) &= \operatorname{Res} \widehat{\phi}(z) \cdot r^{-z}\big|_{z=-m} = \frac{1}{(p-1)!} \lim_{z\to -m} \frac{\partial^{p-1}}{\partial z^{p-1}}[z^{p-1} r^{-z}] \\
&= \frac{1}{(p-1)!} \lim_{z\to -m} \sum_{j=0}^{p-1} \Big[\frac{(p-1)!}{j!\,(p-1-j)!} (z^{p-1})^{(p-1-j)} (r^{-z})^{(j)}\Big] \\
&= \frac{1}{(p-1)!} \lim_{z\to -m} \sum_{j=0}^{p-1} \Big[\frac{(p-1)!}{j!\,(p-1-j)!} (p-1)(p-2)\ldots(j+1) z^j (-1)^j (\log r)^j r^{-z}\Big] \\
&= r^m \sum_{j=0}^{p-1} \Big[\frac{(p-1)(p-2)\ldots(j+1)m^j}{j!\,(p-1-j)!} (\log r)^j\Big] = r^m \sum_{j=0}^{p-1} \alpha_{j,m} (\log r)^j,
\end{aligned}$$

where $\alpha_{j,m} = (p-1)(p-2)\ldots(j+1)m^j/j!\,(p-1-j)!$. Finally, it is easy to see that $\phi$ is a nearly bounded function and therefore $T_\phi$ is a genuine Toeplitz operator.

We conclude by a simple but interesting consequence of our main results.

**Corollary 3.13.** *Let $s \in \mathbb{N}^*$ and let $\psi(r) = \sum\limits_{i=1}^{m} a_i r^{k_i}$ be nonzero polynomial function. Then there exists an integer $N \in \mathbb{N}^*$ such that $T_{\mathrm{e}^{\mathrm{i}s\theta}\psi}$ has pth powers for all integers $1 \leqslant p \leqslant N$.*

Proof. Since $\psi(r) = \sum\limits_{i=1}^{m} a_i r^{k_i}$, we can write

$$\widehat{\psi}(z) = \frac{f(z)}{\prod_{i=1}^{m}(z+k_i)},$$

where the numerator $f$ is a polynomial function of degree less or equal to $m-1$. Obviously, if $\psi$ satisfies the conditions of Corollary 3.7, then $N$ can be any integer in $\mathbb{N}$. Now, assume the hypotheses of Corollary 3.7 do not hold. We want to find $N \in \mathbb{N}$ such that for any random integer $p$ between 1 and $N$ there exists a radial function $\varphi$ satisfying $(T_{\mathrm{e}^{\mathrm{i}s\theta}\psi})^p = T_{\mathrm{e}^{\mathrm{i}ps\theta}\varphi}$. If this is the case, then by using Lemma 2.4 and by letting $z = 2n+ps+2$, we must have that for all integers $n \geqslant 0$

$$\begin{aligned}(T_{\mathrm{e}^{\mathrm{i}s\theta}\psi})^p(\xi^n)(z) &= \left[\prod_{j=0}^{p-1} 2(n+js+s+1)\widehat{\psi}(2n+2js+s+2)\right] z^{n+ps}\\ &= \frac{\prod_{j=0}^{p-1}(z-ps+2js+2s)f(z-ps+2js+s)}{\prod_{(i,j)=(1,0)}^{(m,p-1)}(z-ps+k_i+2js+s)} z^{n+ps}\\ &= (z+ps)\widehat{\varphi}(z).\end{aligned}$$

Similarly and as in the proof of Theorem 3.1, we deduce that $\varphi$ must be of the from

$$\varphi(r) = \sum_{(i,j)=(1,0)}^{(m,p-1)} \alpha_{i,j} r^{k_i-ps+2js+s}, \tag{3.4}$$

where $\alpha_{i,j}$ are constants. Furthermore, since $k_1 - ps + s \leqslant k_i - ps + 2js + s$ for all $(i,j) = (1,0)\dots(m,p-1)$, the function $\varphi$ will be in $L^2([0,1], r\,\mathrm{d}r)$ if $k_1 - ps + s \geqslant 0$. Otherwise $T_{\mathrm{e}^{\mathrm{i}ps\theta}\varphi}$ will not be bounded and hence not a genuine Toeplitz operator. Therefore it is sufficient to take $N = \lfloor \frac{s+k_1}{s} \rfloor$. □

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