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# ON THE BALANCED DOMINATION OF GRAPHS

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*Abstract.* Let $G=(V_G,E_G)$ be a graph and let $N_G[v]$ denote the closed neighbourhood of a vertex $v$ in $G$. A function $f\colon V_G\to\{-1,0,1\}$ is said to be a balanced dominating function (BDF) of $G$ if $\sum\limits_{u\in N_G[v]} f(u)=0$ holds for each vertex $v\in V_G$. The balanced domination number of $G$, denoted by $\gamma_b(G)$, is defined as

$$\gamma_b(G)=\max\biggl\{\sum_{v\in V_G} f(v)\colon\ f \text{ is a BDF of } G\biggr\}.$$

A graph $G$ is called $d$-balanced if $\gamma_b(G)=0$. The novel concept of balanced domination for graphs is introduced. Some upper bounds on the balanced domination number are established, in which one is the best possible bound and the rest are sharp, all the corresponding extremal graphs are characterized; several classes of $d$-balanced graphs are determined. Some open problems are proposed.

*Keywords*: domination number; balanced dominating function; balanced domination number; $d$-balanced graph

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## 1. Introduction

We denote $G=(V_G,E_G)$ a finite simple graph with a vertex set $V_G$ and edge set $E_G$, without loops and multiple edges (for more background on graph theory and for the notation not defined here we refer the reader to [1]). The *order* of $G$ is the number $n=|V_G|$ of its vertices and its *size* is the number $|E_G|$ of its edges.

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Given a graph $G$, its *adjacency matrix* $A(G)$ is a 0-1 square matrix of order $n$ whose $(i,j)$-entry is 1 if and only if $ij \in E_G$. For $v \in V_G$, we define the *open neighbourhood* and the *closed neighbourhood* of $v$ as

$$N_G(v) = \{u\colon\ u \in V_G,\, uv \in E_G\} \quad \text{and} \quad N_G[v] = \{v\} \cup N_G(v),$$

respectively. Let $\delta(G)$ be the *minimum degree* of $G$, whereas let $\Delta(G)$ be the *maximum degree* of $G$. A vertex of degree 1 is called a *pendant vertex*. Denote by $c(G)$ the *cyclomatic number* of $G$, that is $c(G) = |E_G| - |V_G| + \omega(G)$, where $\omega(G)$ is the number of connected components of $G$.

For a subset $S \subseteq V_G$, let $G[S]$ denote the subgraph of $G$ induced by $S$. For $A, B \subseteq V_G$, let $E(A,B) = \{uv \in E_G\colon\ u \in A,\, v \in B\}$.

Given a graph $G$, a subset $D \subseteq V_G$ is said to be a *dominating set* of $G$ if $N_G(u) \cap D \neq \emptyset$ for every vertex $u \in V_G \setminus D$. The *domination number* of $G$, written as $\gamma(G)$, is the smallest cardinality of a dominating set of $G$.

In recent years, the study of domination theory has attracted more and more researchers' attention. In 1995, Dunbar et al. in [5] first put forward and studied the signed domination of graphs. Cockayne and Mynhardt in [3] generalized the signed dominating function. Thus, many variations of domination concepts were introduced, such as the minus domination (see [4]), the signed total domination (see [14]) and so on. Some important results of this field were surveyed by Haynes, Hedetniemi and Slater, see [6]. Xu in [9] proposed and studied the signed edge domination in graphs. After that many concepts of the edge domination appeared, which include the signed cycle domination (see [11]), the signed star domination (see [10]) and so on.

In this paper, we propose the concepts of the balanced dominating function and balanced domination number of a graph. Let $f\colon V_G \to \mathbb{R}$ be a real valued function defined on $V_G$. Then, for a subset $S \subseteq V_G$, we put $f(S) := \sum\limits_{v \in S} f(v)$.

**Definition 1.1.** Let $G$ be a graph. A function $f\colon V_G \to \{-1, 0, 1\}$ is said to be a *balanced dominating function* (BDF) of $G$ if $f(N_G[v]) = 0$ holds for each vertex $v \in V_G$. The *balanced domination number* of $G$, written as $\gamma_b(G)$, is defined as

$$\gamma_b(G) = \max\{f(V_G)\colon\ f \text{ is a BDF of } G\}.$$

By Definition 1.1, we know that the constant function $f = 0$ is a BDF for any graph $G$. Hence, $\gamma_b(G) \geqslant 0$ holds for any graph $G$. Obviously, if the function $f$ is a BDF of $G$, then $-f$ is also a BDF of $G$. A balanced dominating function $f$ of $G$ is maximum if $f(V_G) = \gamma_b(G)$. A graph $G$ is said to be *d-balanced* if $\gamma_b(G) = 0$. Notice that not all graphs are $d$-balanced. The tree depicted in Figure 1 is a nice example. It is interesting and challenging to determine $\gamma_b(G)$ of a graph $G$.

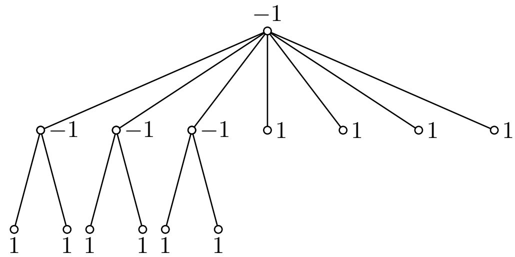

Figure 1. A tree $T$ with $\gamma_b(T) = 6$.

Our paper is organized as follows. In Section 2, we give four sharp upper bounds on the balanced domination number and some of the extremal graphs are characterized. In Section 3, we determine some $d$-balanced graphs. In the last section, some open problems are proposed.

Further on we need the following preliminary results.

**Lemma 1.2** ([7]). *Let $G$ be a graph with $n$ vertices. If $\delta(G) \geqslant 1$, then $\gamma(G) \leqslant \frac{1}{2}n$.*

Let $G$ be a graph. The *corona* of $G$, denoted by $G \circ K_1$, is the graph obtained from $G$ by attaching exactly one pendant vertex to each vertex of $G$.

**Lemma 1.3** ([12], [13]). *Let $G$ be a connected graph of order $n$. If $\delta(G) \geqslant 1$, then $\gamma(G) = \frac{1}{2}n$ if and only if $G \cong C_4$ or $G$ is the corona of some connected graph with $\frac{1}{2}n$ vertices.*

**Lemma 1.4** ([8]). *Let $G$ be a graph. If $H$ is a subgraph of $G$, then $c(H) \leqslant c(G)$.*

In the whole context, for $i \in \{-1, 0, 1\}$, let

$$A_i = \{v \in V_G \colon\ f(v) = i\}$$

and put $r = |A_0|$, $s = |A_1|$ and $t = |A_{-1}|$.

## 2. Some upper bounds on balanced domination numbers

In this section, we establish some sharp upper bounds on balanced domination numbers of graphs. Some corresponding extremal graphs are identified.

**Theorem 2.1.** *Let $G$ be a graph with $n$ vertices. Then*

$$\gamma_b(G) \leqslant n + 1 - \sqrt{1 + 4n}.$$

*The equality holds if and only if $G$ is obtained by attaching exactly $t$ pendant vertices to each vertex in $K_t$.*

P r o o f. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Clearly, $n_1 := s + t \leqslant n$ and $\gamma_b(G) = s - t = 2s - n_1$. Since $\gamma_b(G) \geqslant 0$, one has $s \geqslant t$.

For every vertex $v \in A_1$, by Definition 1.1 we have $f(N_G[v]) = 0$. Hence, $v$ is adjacent to at least one vertex in $A_{-1}$. That is, $|E(A_1, A_{-1})| \geqslant s$. Thus, there exists a vertex $u \in A_{-1}$ such that $|N_G[u] \cap A_1| \geqslant \lceil \frac{s}{t} \rceil$. Notice that $f(N_G[u]) = 0$. Hence, $|N_G[u] \cap A_1| = |N_G[u] \cap A_{-1}|$. Then

$$t = |A_{-1}| \geqslant |N_G[u] \cap A_{-1}| = |N_G[u] \cap A_1| \geqslant \left\lceil \frac{s}{t} \right\rceil \geqslant \frac{s}{t},$$

which implies that $(n_1 - s)^2 = t^2 \geqslant s$. That is, $s \leqslant n_1 - \frac{1}{2}(\sqrt{1 + 4n_1} - 1)$. Hence,

$$\gamma_b(G) = 2s - n_1 \leqslant n_1 + 1 - \sqrt{1 + 4n_1} \leqslant n + 1 - \sqrt{1 + 4n}. \tag{2.1}$$

In what follows, we show the second part of our result.

*Necessity.* Let $G$ be a graph with $n$ vertices and $\gamma_b(G) = n + 1 - \sqrt{1 + 4n}$. In view of the proof of (2.1), we may see that $\gamma_b(G) = n + 1 - \sqrt{1 + 4n}$ if and only if $n = n_1$ and $s = t^2$. That is, $A_0 = \emptyset$ and $t = \frac{1}{2}(\sqrt{4n + 1} - 1)$. Let $u$ be an arbitrary vertex in $A_{-1}$. Notice that $|N_G[u] \cap A_1| = |N_G[u] \cap A_{-1}| \leqslant |A_{-1}| = t$. Hence,

$$|E(A_1, A_{-1})| = \sum_{u \in A_{-1}} |N_G[u] \cap A_1| = \sum_{u \in A_{-1}} |N_G[u] \cap A_{-1}| \leqslant t^2.$$

Together with $|E(A_1, A_{-1})| \geqslant |A_1| = s = t^2$, we have

$$|E(A_1, A_{-1})| = |A_1| = t^2 \quad \text{and} \quad |N_G[u] \cap A_1| = |N_G[u] \cap A_{-1}| = |A_{-1}| = t.$$

Hence, each vertex in $A_1$ has exactly one neighbour in $A_{-1}$ and any two vertices in $A_{-1}$ have no common neighbours in $A_1$. Furthermore, $G[A_{-1}] = K_t$. That is, $G$ is a graph obtained by attaching exactly $t$ pendant vertices at each vertex of $K_t$.

*Sufficiency.* Let $G$ be a graph obtained by attaching exactly $t$ pendant vertices at each vertex of $K_t$. Clearly, $n = |V_G| = t + t^2$. Define a function $f'$ on $V_G$ as

$$f'(v) = \begin{cases} -1 & \text{if } v \in K_t, \\ 1 & \text{otherwise.} \end{cases}$$

It is routine to check that $f'$ is a BDF of $G$. Hence, $\gamma_b(G) \geqslant f'(V_G) = t^2 - t = n + 1 - \sqrt{1 + 4n}$. Together with (2.1), we have $\gamma_b(G) = n + 1 - \sqrt{1 + 4n}$. □

**Theorem 2.2.** *Let $G$ be an $n$-vertex graph with maximum degree $\Delta$ and minimum degree $\delta$. Then,*

$$\gamma_b(G) \leqslant \frac{(\Delta - \delta)n}{2(\delta + 1)}. \tag{2.2}$$

*The equality holds if and only if $G$ is $\Delta$-regular.*

Proof. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G)=\gamma_b(G)$. Then it is easy to see that $s+t\leqslant n$ and $\gamma_b(G)=s-t$. Since $\gamma_b(G)\geqslant 0$, one has $s\geqslant t$ and thus $t\leqslant\frac{1}{2}n$. By Definition 1.1, we have $f(N_G[v])=0$ for each vertex $v\in A_1$. Hence,

$$\begin{aligned}0&=\sum_{v\in A_1}f(N_G[v])=\sum_{v\in A_1}(|E(\{v\},A_1)|+1-|E(\{v\},A_{-1})|)\\&=2|E_{G[A_1]}|+s-|E(A_1,A_{-1})|.\end{aligned}$$

That is, $s=|E(A_1,A_{-1})|-2|E_{G[A_1]}|$. Similarly, we have $t=|E(A_1,A_{-1})|-2|E_{G[A_{-1}]}|$. Therefore,

$$\gamma_b(G)=s-t=2(|E_{G[A_{-1}]}|-|E_{G[A_1]}|). \tag{2.3}$$

Note, also, that

$$\begin{aligned}\Delta t&\geqslant\sum_{v\in A_{-1}}d_G(v)=|E(A_0,A_{-1})|+|E(A_1,A_{-1})|+2|E_{G[A_{-1}]}|\\&=|E(A_0,A_{-1})|+4|E_{G[A_{-1}]}|+t\end{aligned}$$

and

$$\begin{aligned}\delta s&\leqslant\sum_{v\in A_1}d_G(v)=|E(A_0,A_1)|+|E(A_1,A_{-1})|+2|E_{G[A_1]}|\\&=|E(A_0,A_1)|+4|E_{G[A_1]}|+s.\end{aligned}$$

Then

$$|E_{G[A_{-1}]}|\leqslant\frac{(\Delta-1)t-|E(A_0,A_{-1})|}{4}\quad\text{and}\quad|E_{G[A_1]}|\geqslant\frac{(\delta-1)s-|E(A_0,A_1)|}{4}.$$

Recall that $f(N_G[v])=0$ for each vertex $v\in A_0$. Hence, $|E(A_0,A_1)|=|E(A_0,A_{-1})|$. In view of (2.3), we have

$$\gamma_b(G)=s-t=2(|E_{G[A_{-1}]}|-|E_{G[A_1]}|)\leqslant\frac{(\Delta-1)t-(\delta-1)s}{2}.$$

That is, $s\leqslant(\Delta+1)t/(\delta+1)$. Together with $t\leqslant\frac{1}{2}n$, one has

$$\gamma_b(G)=s-t\leqslant\frac{(\Delta-\delta)t}{\delta+1}\leqslant\frac{(\Delta-\delta)n}{2(\delta+1)}.$$

In what follows, we show the second part of our result.

*Necessity.* Let $G$ be a graph with $n$ vertices and $\gamma_b(G) = (\Delta - \delta)n/2(\delta + 1)$. In view of the proof of (2.2), we may derive that $\gamma_b(G) = (\Delta - \delta)n/2(\delta + 1)$ if and only if at least one of the following conditions holds:

(a) $\Delta = \delta$,

(b) $s = t = \frac{1}{2}n$, each vertex in $A_1$ is of degree $\delta$ and each vertex in $A_{-1}$ is of degree $\Delta$.

If (a) holds, then we are done. Now, we assume $\delta \neq \Delta$. Hence, (b) holds and thus $A_0 = \emptyset$. Let $v$ be an arbitrary vertex of $A_1$. Since each vertex in $A_1$ has degree $\delta$, we have $|E(\{v\}, A_1)| + |E(\{v\}, A_{-1})| = \delta$. Together with $f(N_G[v]) = |E(\{v\}, A_1)| + 1 - |E(\{v\}, A_{-1})| = 0$, one has $|E(\{v\}, A_{-1})| = \frac{1}{2}(\delta + 1)$. By a similar reasoning, we may show that $|E(\{u\}, A_1)| = \frac{1}{2}(\Delta + 1)$ for each vertex $u \in A_{-1}$. Therefore,

$$|E(A_1, A_{-1})| = \sum_{v \in A_1} |E(\{v\}, A_{-1})| = s\frac{\delta + 1}{2}$$

and

$$|E(A_{-1}, A_1)| = \sum_{u \in A_{-1}} |E(\{u\}, A_1)| = t\frac{\Delta + 1}{2}.$$

Notice that $s = t$. Then $\delta = \Delta$, a contradiction.

*Sufficiency.* It suffices to show the following claim.

**Claim 2.3.** *Each $k$-regular graph $G$ is $d$-balanced.*

Proof of Claim 2.3. Let $f$ be the maximum BDF of $G$, that is, $f(V_G) = \gamma_b(G)$. Note that $f(N_G[v]) = 0$ holds for every vertex $v \in V_G$. Therefore,

$$(k + 1)f(V_G) = \sum_{v \in V_G} f(N_G[v]) = 0.$$

Hence, $\gamma_b(G) = f(V_G) = 0$, i.e., $G$ is $d$-balanced. □

In view of Claim 2.3, the sufficiency is obviously true. Theorem 2.2 is proved. □

**Theorem 2.4.** *Let $G$ be an $n$-vertex graph with the minimum degree $\delta \geqslant 1$. Then*

$$\gamma_b(G) \leqslant n - 2\gamma(G) \tag{2.4}$$

*and the bound is tight.*

P r o o f. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Let $G_1 = G[A_0]$ and $I = \{v \in A_0\colon\ d_{G_1}(v) = 0\}$. Notice that $\delta \geqslant 1$ and $f(N_G[v]) = 0$ for each vertex $v \in V_G$. Hence, each vertex in $I$ is adjacent to at least one vertex in $A_{-1}$. Let $G_2 = G_1 - I$. Then $\delta(G_2) \geqslant 1$. By Lemma 1.2, one has $\gamma(G_2) \leqslant \frac{1}{2}|V_{G_2}| \leqslant \frac{1}{2}|A_0|$. Let $D_2$ be the minimum dominating set of $G_2$, i.e., $|D_2| = \gamma(G_2) \leqslant \frac{1}{2}|A_0|$.

Recall that $f(N_G[v]) = 0$ for each vertex $v \in A_1$. Then each vertex in $A_1$ is adjacent to at least one vertex in $A_{-1}$. Hence, $D_2 \cup A_{-1}$ is a dominating set of $G$, which implies that $\gamma(G) \leqslant |A_{-1}| + |D_2| \leqslant |A_{-1}| + \frac{1}{2}|A_0|$. Therefore,

$$\gamma_b(G) = |A_1| - |A_{-1}| = n - 2|A_{-1}| - |A_0| \leqslant n - 2\gamma(G).$$

In order to show that the upper bound in (2.4) is the best possible, we construct a class of graphs $G$ such that $\gamma_b(G) = |V_G| - 2\gamma(G)$.

Let $H$ be a graph and let $G$ be a graph obtained from $H$ by attaching exactly $1 + d_H(v)$ pendant vertices to each vertex $v \in V_H$. Clearly, $|V_G| = 2|V_H| + 2|E_H|$ and $\gamma(G) = |V_H|$. By (2.4), we have $\gamma_b(G) \leqslant |V_G| - 2\gamma(G) = 2|E_H|$. On the other hand, we define a function $f'$ on $V_G$ as

$$f'(v) = \begin{cases} -1 & \text{if } v \in V_H, \\ 1 & \text{otherwise.} \end{cases}$$

It is routine to check that $f'$ is a BDF of $G$. Hence, $\gamma_b(G) \geqslant f'(V_G) = |V_G| - 2|V_H| = 2|E_H|$. Thus, $\gamma_b(G) = |V_G| - 2\gamma(G)$. □

The following result is an immediate consequence of (2.4) and Lemma 1.3.

**Corollary 2.5.** *Let $G$ be a graph. Then the corona of $G$ is d-balanced.*

**Theorem 2.6.** *Let $G$ be a connected graph with $|V_G| \geqslant 2$. Then*

$$\gamma_b(G) \leqslant \frac{2|E_G| - |V_G|}{2}. \tag{2.5}$$

*The equality holds if and only if $G$ is obtained from a connected graph $H$ by attaching exactly $d_H(v) + 1$ pendant vertices to each vertex $v$ in $V_H$.*

P r o o f. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$, and let $H = G[A_{-1}]$. Recall that $r = |A_0|$. Obviously, $s + t + r = |V_G|$ and $\gamma_b(G) = s - t$. Notice that $f(N_G[v]) = 0$ for each vertex $v \in V_G$. Hence, every vertex in $A_1$ has at least one neighbour in $A_{-1}$, which implies that $|E(A_1, A_{-1})| \geqslant |A_1|$. Moreover,

$$0 = \sum_{v \in A_{-1}} f(N_G[v]) = |E(A_1, A_{-1})| - |A_{-1}| - 2|E_H|.$$

Therefore, $|E(A_1, A_{-1})| = |A_{-1}| + 2|E_H|$. By Lemma 1.3, one has $c(H) \leqslant c(G)$. Notice that the cyclomatic number of $G$ is $c(G) = |E_G| - |V_G| + 1$. Then

$$|E_H| = c(H) + |V_H| - \omega(H) \leqslant c(G) + t - \omega(H) \leqslant |E_G| - |V_G| + t.$$

Therefore,

$$s = |A_1| \leqslant |E(A_1, A_{-1})| = |A_{-1}| + 2|E_H| \leqslant 3t + 2|E_G| - 2|V_G|.$$

Together with $s = |V_G| - r - t \leqslant |V_G| - t$, we have

$$2s \leqslant 3t + 2|E_G| - 2|V_G| + |V_G| - t = 2t - |V_G| + 2|E_G|.$$

Hence, $\gamma_b(G) = s - t \leqslant \frac{1}{2}(2|E_G| - |V_G|)$.

In what follows, we show the second part of our result.

*Necessity.* Let $G$ be a connected graph with $|V_G| \geqslant 2$ and $\gamma_b(G) = \frac{1}{2}(2|E_G| - |V_G|)$. In view of the proof of (2.5), we know that $\gamma_b(G) = \frac{1}{2}(2|E_G| - |V_G|)$ if and only if all the following conditions hold:

(i) $s = |V_G| - t$,

(ii) $|A_1| = |E(A_1, A_{-1})|$,

(iii) $c(H) = c(G)$ and $\omega(H) = 1$.

Let $u$ be an arbitrary vertex of $A_1$. By item (ii), we have $|N_G[u] \cap A_1| = |N_G[u] \cap A_{-1}| = 1$. The item (i) implies that $A_0 = \emptyset$. Therefore, $u$ is a pendant vertex of $G$.

Let $w$ be an arbitrary vertex of $A_{-1}$. Notice that $f(N_G[w]) = 0$. Hence, $w$ is adjacent to exactly $d_H(w) + 1$ vertices in $A_1$. On the other hand, the item (iii) implies that $H$ is connected. Hence, $G$ is the graph obtained from a connected graph $H$ by attaching exactly $d_H(w) + 1$ pendant vertices to each vertex $w \in V_H$.

*Sufficiency.* Assume that $G$ is the graph obtained from a connected graph $H$ by attaching exactly $d_H(w) + 1$ pendant vertices to each vertex $w \in V_H$. Then, $|V_G| = 2|V_H| + 2|E_H|$ and $|E_G| = 3|E_H| + |V_H|$. According to the proof of the second part of Theorem 2.4, we have $\gamma_b(G) = 2|E_H| = \frac{1}{2}(2|E_G| - |V_G|)$. □

## 3. $d$-balanced graphs

In this section, we determine some classes of $d$-balanced graphs.

**Proposition 3.1.** *Let $G$ be a graph. Then $\gamma_b(G) \equiv 0 \pmod 2$.*

P r o o f. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Let

$$G_1 = G[A_1 \cup A_{-1}].$$

For every vertex $v \in V_{G_1}$, by Definition 1.1 we have $f(N_G[v]) = 0$. It is routine to check that

$$0 = \sum_{v \in V_{G_1}} f(N_G[v]) = \sum_{v \in V_{G_1}} f(v) + \sum_{uv \in E_{G_1}} (f(u) + f(v)).$$

Notice that 0 is even and $f(u) + f(v)$ is even for each edge $uv \in E_{G_1}$. Thus, $\gamma_b(G) = \sum\limits_{v \in V_{G_1}} f(v)$ is even. This completes the proof. $\square$

**Proposition 3.2.** *Let $G$ be a graph. If $\Delta(G) \leqslant 2$ or $\Delta(G) = |V_G| - 1$, then $G$ is $d$-balanced.*

P r o o f. Notice that a graph is $d$-balanced if and only if all of its components are $d$-balanced. Hence, we assume that $G$ is connected. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$.

If $\Delta(G) = 0$, then $G \cong K_1$. If $\Delta(G) = 1$, then $G \cong K_2$. It is straightforward to check that $\gamma_b(K_1) = \gamma_b(K_2) = 0$, as desired.

If $\Delta(G) = 2$ and $\delta(G) = 1$, then $G \cong P_n$ with $n \geqslant 3$. Notice that $f(N_G[v]) = 0$ for each vertex $v \in V_G$. Therefore, each vertex in $A_1$ (or $A_{-1}$) is adjacent to exactly one vertex in $A_{-1}$ (or $A_1$, respectively). Hence, $|A_1| = |A_{-1}|$, which implies that $\gamma_b(G) = |A_1| - |A_{-1}| = 0$, i.e., $G$ is $d$-balanced. If $\Delta(G) = \delta(G) = 2$, then $G \cong C_n$ and $G$ is 2-regular. By Claim 2.3 (in the proof of Theorem 2.2), we have that $G$ is $d$-balanced.

If $\Delta(G) = |V_G| - 1$, then assume without loss of generality that $w$ is such a maximum degree vertex of $G$. Thus, $\gamma_b(G) = f(V_G) = f(N_G[w]) = 0$, i.e., $G$ is $d$-balanced. $\square$

A tree $T$ is said to be a *double star* if $T$ can be obtained from the disjoint union $K_{1,p} \cup K_{1,q}$ by adding one edge to join the maximum degree vertices of $K_{1,p}$ and $K_{1,q}$. We denote $T$ by $S(p,q)$.

**Proposition 3.3.** *Let $S(p,q)$ be a double star. Then $S(p,q)$ is $d$-balanced if and only if $(p,q) \neq (2,2)$. In addition, $\gamma_b(S(2,2)) = 2$.*

P r o o f. Let $G = S(p,q)$. Assume, without loss of generality, that $p \geqslant q$ and $u_1$ (or $u_2$) is the center of $K_{1,p}$ (or $K_{1,q}$, respectively). If $q = 0$ or $(p,q) = (1,1)$, then by Proposition 3.2, we obtain that $G$ is $d$-balanced. So, in what follows, we assume that $q \geqslant 1$ and $p \geqslant 2$.

Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Notice that $f(N_G[v]) = 0$ for every vertex $v \in V_G$. By Definition 1.1, we have $f(v) = -f(u_1)$ for each vertex $v \in V_{K_{1,p}} \setminus \{u_1\}$ and $f(v) = -f(u_2)$ for each vertex $v \in V_{K_{1,q}} \setminus \{u_2\}$. Hence,

$$(3.1)\quad f(N_G[u_1]) = (1-p)f(u_1)+f(u_2) = 0, \quad f(N_G[u_2]) = (1-q)f(u_2)+f(u_1) = 0.$$

If $f(u_1) = 0$, then by (3.1) one has $f(u_2) = 0$. Therefore, $f = 0$ and so $\gamma_b(G) = f(V_G) = 0$. If $f(u_1) \neq 0$, then it follows from (3.1) that $p = q = 2$ and $f(u_1) = f(u_2)$ (based on the fact that $f(u_i) \in \{-1, 0, 1\}$ for $i = 1, 2$). Thus, $S(p, q)$ is $d$-balanced unless $(p, q) = (2, 2)$.

Note that

$$f(V_{S(2,2)}) = f(u_1) - 2f(u_1) + f(u_2) - 2f(u_2) = -f(u_1) - f(u_2) \leqslant 2.$$

We define a function $f'$ on $V_{S(2,2)}$ satisfying that $f'(u_1) = f'(u_2) = -1$ and $f'(v) = 1$ for $v \in V_{S(2,2)} \setminus \{u_1, u_2\}$. It is routine to check that $f'$ is a BDF of $S(2, 2)$ and $f'(V_{V_{S(2,2)}}) = 2$. Hence, $\gamma_b(S(2, 2)) = 2$.

This completes the proof. $\square$

For any two disjoint graphs $G$ and $H$, the *join* of $G$ and $H$, denoted by $G \vee H$, is obtained from the disjoint union $G \cup H$ by adding all edges between $G$ and $H$. The *complete multipartite graph*, denoted by $K_{n_1,n_2,\ldots,n_r}$, is defined to be the join $n_1K_1 \vee n_2K_1 \vee \ldots \vee n_rK_1$.

**Proposition 3.4.** *If $G$ is a complete multipartite graph, then $G$ is $d$-balanced.*

Proof. Let $G = K_{n_1,n_2,\ldots,n_r}$ be a complete $r$-partite graph and let $V_1 \cup V_2 \cup \ldots \cup V_r$ be a partition of $V_G$ with $|V_i| = n_i$ for $1 \leqslant i \leqslant r$. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Notice that $A_{-1} \cup A_0 \neq \emptyset$. Then choose $w \in A_{-1} \cup A_0$, that is, $f(w) \leqslant 0$. Without loss of generality, we may assume that $w \in V_1$. If $|V_1| = 1$, then $d_G(w) = |V_G| - 1$. By Proposition 3.2, $G$ is $d$-balanced. Now, we assume that $|V_1| \geqslant 2$. Let $u$ be an arbitrary vertex in $V_1$ and $u \neq w$. Notice that $f(N_G[u]) = f(N_G[w]) = 0$. Thus,

$$f(u) = f(N_G[u]) - (f(N_G[w]) - f(w)) = f(w) \leqslant 0.$$

Hence,

$$\gamma_b(G) = f(V_G) = f(N_G[w]) + f(V_1 \setminus \{w\}) = f(V_1 \setminus \{w\}) \leqslant 0.$$

On the other hand, $\gamma_b(G) \geqslant 0$. Therefore, $\gamma_b(G) = 0$, i.e., $G$ is $d$-balanced. $\square$

For any two disjoint graphs $G$ and $H$, the *Cartesian product* of $G$ and $H$, denoted by $G \times H$, is the graph defined as follows:

▷ $V_{G\times H} = \{(u,v)\colon\ u \in V_G,\ v \in V_H\}$,

▷ $(u,v)$ and $(u',v')$ are adjacent if and only if $u = u'$ and $vv' \in E_H$ or $v = v'$ and $uu' \in E_G$.

Let $P_n \times P_2$ be the Cartesian product of $P_n$ and $P_2$ with the vertex set $\{u_1,\dots,u_n,v_1,\dots,v_n\}$ and edge set $\{u_iu_{i+1},\ v_iv_{i+1}\colon\ 1 \leqslant i \leqslant n-1\} \cup \{u_iv_i\colon 1 \leqslant i \leqslant n\}$. A graph $G$ is said to be a *generalized ladder graph* if $G$ can be obtained from $P_n \times P_2$ by deleting arbitrary edges in $\{u_iv_i\colon\ 2 \leqslant i \leqslant n-1\}$, see [2].

**Proposition 3.5.** *If $G$ is a generalized ladder graph with $2n$ vertices, then $G$ is $d$-balanced.*

P r o o f. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$. Obviously, $\gamma_b(G) = s - t$. If $f(u_i) + f(v_i) = 0$ for $1 \leqslant i \leqslant n$, then $f(V_G) = 0$ and therefore $\gamma_b(G) = 0$, as desired. So, in what follows, we may assume that there exists at least one $j$ $(1 \leqslant j \leqslant n)$ such that $f(u_j) + f(v_j) \neq 0$. Hence, $f \neq 0$. In order to show that $G$ is $d$-balanced, it suffices to show $s = t$. We begin with the following claim.

**Claim 3.6.** $|N_G(v) \cap A_{-1}| = 1$ *for each vertex* $v \in A_1$.

P r o o f of Claim 3.6. Let $v$ be an arbitrary vertex in $A_1$. Notice that $f(N_G[v])=0$ and $d_G(v) = 2$ or 3. If $d_G(v) = 2$, then $|N_G(v) \cap A_{-1}| = 1$ and $|N_G(v) \cap A_0| = 1$, as desired. If $d_G(v) = 3$, then either $|N_G(v) \cap A_{-1}| = 2$ and $|N_G(v) \cap A_1| = 1$, or $|N_G(v) \cap A_{-1}| = 1$ and $|N_G(v) \cap A_0| = 2$.

Suppose to the contrary that there exists a vertex of degree 3, say $u_l$ $(2 \leqslant l \leqslant n-1)$, in $A_1$ such that $|N_G(u_l) \cap A_{-1}| = 2$ and $|N_G(u_l) \cap A_1| = 1$. Up to isomorphism, we have $(f(u_{l-1}), f(u_{l+1}), f(v_l)) \in \{(-1,-1,1),(-1,1,-1)\}$. If $(f(u_{l-1}), f(u_{l+1}), f(v_l)) = (-1,-1,1)$, then $f(v_{l-1}) = f(v_{l+1}) = -1$. That is, $f(u_i) = f(v_i)$ for $i \in \{l-1, l, l+1\}$. By the symmetry of $u_i$ and $v_i$, it is routine to check that $f(u_i) = f(v_i)$ for $1 \leqslant i \leqslant n$. In particular, $f(u_1) = f(v_1)$ and $f(u_2) = f(v_2)$. Hence, $f(N_G[u_1]) = 2f(u_1) + f(u_2) = 0$. Notice that $f(v) \in \{-1,0,1\}$ for each vertex $v \in V_G$. Therefore, $f(u_1) = f(u_2) = 0$. According to the structure of the generalized ladder graph $G$, we have $f = 0$, a contradiction.

If $(f(u_{l-1}), f(u_{l+1}), f(v_l)) = (-1,1,-1)$, then $f(v_{l-1}) = -f(v_{l+1}) = 1$ and $u_{l+1}v_{l+1} \in E_G$. Notice that $f(N_G[v]) = 0$ for each vertex $v \in V_G$. Then,

$$\begin{cases} 0 = f(N_G[u_{l-1}]) = f(u_{l-1}) + f(u_l) + f(u_{l-2}) + g(u_{l-1}v_{l-1})f(v_{l-1}), \\ 0 = f(N_G[v_{l-1}]) = f(v_{l-1}) + f(v_l) + f(v_{l-2}) + g(u_{l-1}v_{l-1})f(u_{l-1}), \end{cases}$$

where $g(u_{l-1}v_{l-1}) = 1$ if $u_{l-1}v_{l-1} \in E_G$ and 0 otherwise. According to the above two equations, we get $f(u_{l-2}) + f(v_{l-2}) = 0$. Similarly, we obtain that $f(u_i) + f(v_i) = 0$ for each $i = 1, 2, \ldots, n$, a contradiction. Hence, $|N_G(v) \cap A_{-1}| = 1$ for each vertex $v \in A_1$. This completes the proof of Claim 3.6. □

By a similar reasoning like in the proof of Claim 3.6, we derive that $|N_G(v) \cap A_1| = 1$ for each vertex $v \in A_{-1}$. Thus, $s = t$ and so $\gamma_b(G) = 0$, i.e., $G$ is $d$-balanced. □

**Proposition 3.7.** *The join graphs $P_m \vee P_n$, $P_m \vee C_n$ and $C_m \vee C_n$ are $d$-balanced, respectively.*

Proof. If $\min\{m, n\} \leqslant 3$, then $\Delta(G) = m + n - 1$ for $G \in \{P_m \vee P_n, P_m \vee C_n, C_m \vee C_n\}$. By Proposition 3.2, our desired results hold. So, in what follows, we assume that $\min\{m, n\} \geqslant 4$. Let $f$ be the maximum BDF of $G$, i.e., $f(V_G) = \gamma_b(G)$.

We consider firstly that $G \cong P_m \vee P_n$. Assume that $V_{P_m} = \{v_1, v_2, \ldots, v_m\}$, $V_{P_n} = \{u_1, u_2, \ldots, u_n\}$, $f(V_{P_m}) = k$ and $f(V_{P_n}) = l$. Obviously, $\gamma_b(G) = k + l$. Notice that $f(N_G[v]) = 0$ for each vertex $v \in V_G$. Hence, $f(N_G[v] \cap V_{P_m}) = -l$ for each vertex $v \in V_{P_m}$ and $f(N_G[v] \cap V_{P_n}) = -k$ for each vertex $v \in V_{P_n}$. That is,

$$(A(P_m) + I_m)\mathbf{x}_1 = -l\mathbf{1}_m \quad \text{and} \quad (A(P_n) + I_n)\mathbf{x}_2 = -k\mathbf{1}_n, \tag{3.2}$$

where $\mathbf{x}_1 = (f(v_1), f(v_2), \ldots, f(v_m))^{\mathrm{T}}$, $\mathbf{x}_2 = (f(u_1), f(u_2), \ldots, f(u_n))^{\mathrm{T}}$; $I_m$ and $\mathbf{1}_m$ denote the identity matrix and the all-ones column vector of order $m$, respectively. By solving (3.2), we have:

(a) $f(v_{3p-q}) = k_q$ for $1 \leqslant p \leqslant \lceil \frac{m}{3} \rceil$ and $q \in \{0, 1, 2\}$ $(1 \leqslant 3p - q \leqslant m)$, where $k_1 + k_2 = -l$ and $k_0 = 0$,

(b) $f(u_{3p-q}) = l_q$ for $1 \leqslant p \leqslant \lceil \frac{n}{3} \rceil$ and $q \in \{0, 1, 2\}$ $(1 \leqslant 3p - q \leqslant n)$, where $l_1 + l_2 = -k$ and $l_0 = 0$,

(c) $f(v_{m-2}) = 0$ and $f(u_{n-2}) = 0$.

It is routine to check that $f(V_{P_m}) = -l\lceil \frac{m}{3} \rceil = k$ and $f(V_{P_n}) = -k\lceil \frac{n}{3} \rceil = l$. Therefore, $l\lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil = l$. Note that $\min\{m, n\} \geqslant 4$. Hence, $l = 0$ and $k = 0$. Thus, $\gamma_b(G) = 0$. That is, $P_m \vee P_n$ is $d$-balanced.

Note that $3f(V_{C_n}) = -nf(V_{P_m})$ if $G \cong P_m \vee C_n$, and $3f(V_{C_n}) = -nf(V_{C_m})$ if $G \cong C_m \vee C_n$. Then by a similar reasoning, we may show that $P_m \vee C_n$ and $C_m \vee C_n$ are also $d$-balanced, which claims are omitted here.

This completes the proof. □

**Remark 3.8.** Note that not all join graphs are $d$-balanced. For example, consider the two disjoint graphs $G$ and $H$ as shown in Figure 2.

Let $f$ be the given labelling in Figure 2. Clearly, $f(V_G) = 5$ and $f(V_H) = -3$. For every vertex $v \in V_G$, $f(N_G[v]) = 3$, and for every vertex $v \in V_H$, $f(N_H[v]) = -5$.

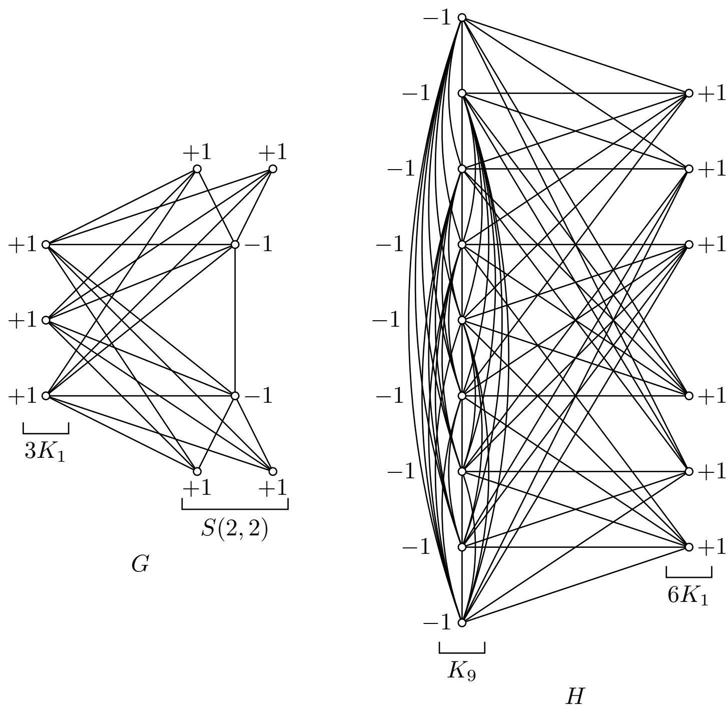


Figure 2. Graphs $G$ and $H$.

Hence, $f$ is the BDF of $G \vee H$. Therefore, $\gamma_b(G \vee H) \geqslant f(V_{G\vee H}) = f(V_G) + f(V_H) = 2 > 0$. That is, $G \vee H$ is not $d$-balanced.

## 4. Some open problems

In this paper, we proposed a novel invariant "balanced domination" of graphs. We determine some types of $d$-balanced graphs.

In fact, it is difficult to characterize all of the $d$-balanced graphs. In view of Corollary 2.5, the corona of any tree is $d$-balanced. Clearly, Theorem 2.6 implies that all trees with at most 5 vertices are $d$-balanced. However, Proposition 3.3 implies that not all trees are $d$-balanced. It is well known that a *caterpillar* is either a $K_2$ or a tree on at least 3 vertices such that deleting its leaves we obtain a path of order at least 1. It is natural to pose the following problem:

**Problem 4.1.** How to characterize all $d$-balanced caterpillar graphs? Furthermore, how to characterize all $d$-balanced trees?

In view of Proposition 3.5, we know that $P_m \times P_2$ is $d$-balanced. Another attractive question is posed as follows:

**Problem 4.2.** How to determine the exact value of $\gamma_b(P_m \times P_n)$ for $n > 2$? Is it true that $P_m \times P_n$ is $d$-balanced?

**Problem 4.3.** What about NP-completeness proof for determining whether a given graph has a balanced dominating set or not?

We will study the above problems in the near future.

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