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ROW HADAMARD MAJORIZATION ON $\mathbf{M}_{m,n}$

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Abstract. An $m \times n$ matrix R with nonnegative entries is called row stochastic if the sum of entries on every row of R is 1. Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$ real matrices. For $A, B \in \mathbf{M}_{m,n}$, we say that A is row Hadamard majorized by B (denoted by $A \prec_{RH} B$) if there exists an $m \times n$ row stochastic matrix R such that $A = R \circ B$, where $X \circ Y$ is the Hadamard product (entrywise product) of matrices $X, Y \in \mathbf{M}_{m,n}$. In this paper, we consider the concept of row Hadamard majorization as a relation on $\mathbf{M}_{m,n}$ and characterize the structure of all linear operators $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ preserving (or strongly preserving) row Hadamard majorization. Also, we find a theoretic graph connection with linear preservers (or strong linear preservers) of row Hadamard majorization, and we give some equivalent conditions for these linear operators on $\mathbf{M}_n$.

Keywords: linear preserver; row Hadamard majorization; row stochastic matrix

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1. Introduction

Let $\mathbf{M}_{m,n}$ be the set of all $m \times n$ real matrices. For $X, Y \in \mathbf{M}_{m,n}$ it is said that X is *matrix majorized* by Y (denoted by $X \prec Y$), if there exists a row stochastic matrix $R \in \mathbf{M}_n$ such that $X = RY$, see [2] and [3]. The linear preservers and strong linear preservers of matrix majorization have been characterized in [4] and [5]. The *Hadamard product* (Schur product) of two matrices $X = [x_{ij}], Y = [y_{ij}] \in \mathbf{M}_{m,n}$ is their entrywise product $X \circ Y = [x_{ij} y_{ij}]$. In this paper, following the form of [6], we replace the ordinary product by the Hadamard product on $\mathbf{M}_{m,n}$ and introduce a new kind of majorization that is called *row Hadamard majorization* or, in brief, R-Hadamard majorization.

Definition 1.1. Let $X, Y \in \mathbf{M}_{m,n}$. We say that X is R-*Hadamard majorized by* Y (denoted by $X \prec_{RH} Y$), if there exists a row stochastic matrix $R \in \mathbf{M}_{m,n}$ such that $X = R \circ Y$.

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For a linear operator $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{p,q}$, it is said that T *preserves* (or *strongly preserves*) R-*Hadamard majorization* if $T(X) \prec_{RH} T(Y)$ whenever $X \prec_{RH} Y$ (or $T(X) \prec_{RH} T(Y)$ if and only if $X \prec_{RH} Y$). Throughout the paper, we denote by $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ the standard basis of $\mathbf{M}_{m,n}$. We also denote by $\mathbf{J}$ the $m \times n$ matrix of all ones. In this paper, we find some interesting properties of linear operators preserving R-Hadamard majorization and a connection with graph theory. In particular, we completely determine the structure of all linear and strong linear preservers of R-Hadamard majorization on $\mathbf{M}_{m,n}$ as follows:

Theorem 1.1. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. Then*
(1) *If $n = 1$, T is a linear preserver of $\prec_{RH}$.*
(2) *If $n \geqslant 2$, T is a linear preserver of $\prec_{RH}$ if and only if there exists $A \in \mathbf{M}_{m,n}$ and permutation matrices $Q_1, \ldots, Q_m \in \mathbf{M}_n$ such that*

$$T(X) = \begin{pmatrix} X_{k_1}Q_1 \\ X_{k_2}Q_2 \\ \vdots \\ X_{k_m}Q_m \end{pmatrix} \circ A \quad \forall\, X \in \mathbf{M}_{m,n}, \tag{1.1}$$

where $X_{k_1}, \ldots, X_{k_m}$ are some rows of X (not necessarily distinct).

Theorem 1.2. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. Then:*
(1) *If $n = 1$, T is a strong linear preserver of $\prec_{RH}$ if and only if T is invertible.*
(2) *If $n \geqslant 2$, T is a strong linear preserver of $\prec_{RH}$ if and only if there exists $A \in \mathbf{M}_{m,n}$ with no zero entries and permutation matrices $P \in \mathbf{M}_m$ and $Q_1, \ldots, Q_m \in \mathbf{M}_n$ such that*

$$T(X) = P \begin{pmatrix} X_1Q_1 \\ X_2Q_2 \\ \vdots \\ X_mQ_m \end{pmatrix} \circ A \quad \forall\, X \in \mathbf{M}_{m,n}, \tag{1.2}$$

where $X_1, \ldots, X_m$ are the rows of X.

2. Linear preservers of R-Hadamard majorization

In this section, first we state some properties of R-Hadamard majorization and its linear preservers. Then we find all linear operators that preserve R-Hadamard majorization. The next remark gives some properties of R-Hadamard majorization on $\mathbf{M}_{m,n}$.

Remark 2.1. Let $A, B, C \in \mathbf{M}_{m,n}$. The following statements hold:

(i) $A \prec_{RH} A$ if and only if $A = A \circ R$ for some $(0,1)$-row stochastic matrix R.

(ii) For arbitrary permutation matrices $P \in \mathbf{M}_m$ and $Q \in \mathbf{M}_n$, $P(B \circ C)Q = (PBQ) \circ (PCQ)$ and hence a linear operator $X \mapsto T(X)$ preserves $\prec_{RH}$ if and only if the linear operator $X \mapsto PT(X)Q$ preserves $\prec_{RH}$.

(iii) If A has no zero entry, a linear operator $X \mapsto T(X)$ is a linear preserver of $\prec_{RH}$ if and only if the linear operator $X \mapsto T(X) \circ A$ is a linear preserver of $\prec_{RH}$.

Now we can prove the following theorem.

Theorem 2.1. *Let $n \geqslant 2$. If $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a linear preserver of $\prec_{RH}$, then the following conditions hold:*

(1) *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \leqslant r, p \leqslant m$ and $1 \leqslant s, q \leqslant n$ with $(r,s) \neq (p,q)$.*

(2) *For every $1 \leqslant p \leqslant m$ and $1 \leqslant q \leqslant n$ there exists a $(0,1)$-row stochastic matrix R such that $T(E_{pq}) = T(E_{pq}) \circ R$.*

(3) *For every $1 \leqslant p, r \leqslant m$ and $1 \leqslant q, s \leqslant n$ with $p \neq r$, $T(E_{pq})$ and $T(E_{rs})$ do not simultaneously have a nonzero entry in any row.*

P r o o f. (1) Assume if possible that $T(E_{pq}) \circ T(E_{rs}) \neq 0$ for some $(p,q) \neq (r,s)$. Then $[T(E_{pq})]_{ij} = \lambda \neq 0$ and $[T(E_{rs})]_{ij} = \mu \neq 0$ for some $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Let $Y = \lambda^{-1}E_{pq} - \mu^{-1}E_{rs}$ and $X = R \circ Y$, where R is a row stochastic matrix such that the (p,q)th and (r,s)th entries of R are $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Now $X \prec_{RH} Y$ but $T(X) \not\prec_{RH} T(Y)$, which is a contradiction.

(2) We have $E_{pq} \prec_{RH} E_{pq}$, so by the assumption and part (i) of Remark 2.1 the result is trivial.

(3) For arbitrary but fixed $1 \leqslant p, r \leqslant m$ and $1 \leqslant q, s \leqslant n$ with $p \neq r$, let $A = [a_{ij}] = T(E_{pq})$ and $B = [b_{ij}] = T(E_{rs})$. We show that A and B do not simultaneously have a nonzero entry in any row. If $A = 0$ or $B = 0$, there is nothing to prove. Let $A \neq 0$ and by part (ii) of Remark 2.1, without loss of generality assume that $a_{11} \neq 0$. We show that the first row of B is zero. By part (1), $b_{11} = 0$. Assume if possible that $b_{1j} \neq 0$ for some $2 \leqslant j \leqslant n$, then by part (1), $a_{1j} = 0$. Set $E = E_{pq} + E_{rs}$. Since $p \neq r$, there exists a $(0,1)$-row stochastic matrix R such that $E = R \circ E$ and hence $E \prec_{RH} E$. Now by the assumption we conclude that $T(E) = A + B \prec_{RH} T(E) = A + B$ and by part (i) of Remark 2.1, there exists a $(0,1)$-row stochastic matrix S such that $A + B = S \circ (A + B)$ which is impossible. Consequently, the first row of B is a zero row. □

In the following, $\mathbb{R}_n$ is the set of all $1 \times n$ real (row) vectors, and for a linear operator $L\colon \mathbb{R}_n \to \mathbb{R}_n$, $[L]$ is the matrix representation of L with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}_n$. The next lemma characterizes all linear operators on $\mathbb{R}_n$ which preserve $\prec_{RH}$. It is said that $A \in \mathbf{M}_m$ is *dominated by a permutation matrix* if there exists a permutation matrix $P \in \mathbf{M}_m$ such that $A = A \circ P$.

Lemma 2.1. *Let $L\colon \mathbb{R}_n \to \mathbb{R}_n$ be a linear operator. Then L preserves $\prec_{RH}$ if and only if $[L]$ is dominated by a permutation matrix. In other words, L preserves $\prec_{RH}$ if and only if there exist an $n \times n$ permutation matrix P and $a \in \mathbb{R}_n$ such that $Lx = (xP) \circ a$ for all $x \in \mathbb{R}_n$.*

P r o o f. Let $[L] = A = [a_{ij}]$. Then $L(x) = xA$ for all $x \in \mathbb{R}_n$. First assume that A is dominated by a permutation matrix P. If $x \prec_{RH} y$ for some $x, y \in \mathbb{R}_n$, there exists a real $1 \times n$ row stochastic matrix $R = [r_1 \dots r_n]$ such that $x = R \circ y$. Let σ be the permutation corresponding to P. Then we have $yA = [y_{\sigma(1)} a_{\sigma(1)1} \dots y_{\sigma(n)} a_{\sigma(n)n}]$, and hence

$$\begin{aligned} L(x) = L(R \circ y) &= [r_{\sigma(1)} a_{\sigma(1)1} y_{\sigma(1)} \dots r_{\sigma(n)} a_{\sigma(n)n} y_{\sigma(1)}] \\ &= [r_{\sigma(1)} \dots r_{\sigma(n)}] \circ [a_{\sigma(1)1} y_{\sigma(1)} \dots a_{\sigma(n)n} y_{\sigma(n)}] \\ &= [r_{\sigma(1)} \dots r_{\sigma(n)}] \circ L(y). \end{aligned}$$

Since σ is a permutation, $[r_{\sigma(1)} \dots r_{\sigma(n)}]$ is a real $1 \times n$ row stochastic matrix. Therefore, $L(x) \prec_{RH} L(y)$. Conversely, assume that L preserves $\prec_{RH}$. By part (1) of Theorem 2.1, we have $L(e_q) \circ L(e_s) = 0$ for all $q \neq s$ $(1 \leqslant q, s \leqslant n)$. Thus, the rows of A have mutually disjoint supports. Since $e_i \prec_{RH} e_i$ and L preserves $\prec_{RH}$, we have $L(e_i) \prec_{RH} L(e_i)$. Then by part (i) of Remark 2.1, $L(e_i)$ has at most one nonzero entry. Therefore, A is dominated by a permutation matrix. $\square$

The notation $[X_1/\dots/X_m]$ is used for the matrix $X \in \mathbf{M}_{m,n}$ whose rows are $X_1, \dots, X_m \in \mathbb{R}_n$. It is well known that every linear operator T on $\mathbf{M}_{m,n}$ has the following form:

$$T(X) = T[X_1/\dots/X_m] = \left[\sum_{j=1}^{m} T_{1j}(X_j)/\dots/\sum_{j=1}^{m} T_{mj}(X_j)\right], \tag{2.1}$$

where $T_{ij} = \alpha^i T \alpha_j$ and $\alpha^i\colon \mathbf{M}_{m,n} \to \mathbb{R}_n$, $\alpha_j\colon \mathbb{R}_n \to \mathbf{M}_{m,n}$ are defined by

$$\alpha^i(X) = e_i X, \quad \alpha_j(x) = e_j^t x$$

for each $i, j = 1, \dots, m$, $X \in \mathbf{M}_{m,n}$ and $x \in \mathbb{R}_n$.

Now we are ready to prove Theorem 1.1.

P r o o f of Theorem 1.1. (1) For $X, Y \in \mathbf{M}_{m,1}$, $X \prec_{RH} Y$ is equivalent to $X = Y$ and hence every linear operator $T\colon \mathbf{M}_{m,1} \to \mathbf{M}_{m,1}$ preserves $\prec_{RH}$.

(2) If T is of the form (1.1), it is easy to show that T preserves $\prec_{RH}$. Conversely, assume that $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ is a linear preserver of $\prec_{RH}$. By the above, T has the form (2.1). We show that for each i ($1 \leqslant i \leqslant m$), at most one element of T_{ij} ($1 \leqslant j \leqslant m$) is nonzero. Assume if possible that T_{ir} and T_{is} are nonzero for some $1 \leqslant i, r, s \leqslant m$ and $r \neq s$. By Lemma 2.1, there exist nonzero vectors $a, b \in \mathbb{R}_n$ and $n \times n$ permutation matrices P_1, P_2 such that $T_{ir}(x) = (xP_1) \circ a$ and $T_{is}(x) = (xP_2) \circ b$, where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. Since a and b are nonzero, there exist two integer numbers k and l ($1 \leqslant k, l \leqslant n$) such that $a_k \neq 0$ and $b_l \neq 0$. Consider the following two cases:

Case 1: Let $k \neq l$. Put $X = e_r^t e_k P_1^t + e_s^t e_l P_2^t$ and hence $X \prec_{RH} X$. Since the ith row of $T(X)$ has two nonzero components, $T(X) \not\prec_{RH} T(X)$, which is a contradiction.

Case 2: Let $k = l$. Put $X = e_r^t e_k P_1^t - a_k b_k^{-1} e_s^t e_k P_2^t$ and $Y = e_r^t e_k P_1^t$. Thus, $Y \prec_{RH} X$. Since the ith row of $T(Y)$ is nonzero and the ith row of $T(X)$ is zero, $T(Y) \not\prec_{RH} T(X)$, which is a contradiction.

Therefore, for every i ($1 \leqslant i \leqslant m$) there exists k_i ($1 \leqslant k_i \leqslant m$) such that $T_{ij} = 0$ for all j ($1 \leqslant j \leqslant m$) with $j \neq k_i$. Then there exist vectors $a_1, \dots, a_m \in \mathbb{R}_n$ and $n \times n$ permutation matrices $Q_1, \dots, Q_m$ such that for all i ($1 \leqslant i \leqslant m$)

$$T_{ik_i}(x) = (xQ_i) \circ a_i \quad \forall\, x \in \mathbb{R}_n.$$

Now, let $A = [a_1/\dots/a_m]$. Therefore,

$$T(X) = \begin{pmatrix} X_{k_1}Q_1 \\ X_{k_2}Q_2 \\ \vdots \\ X_{k_m}Q_m \end{pmatrix} \circ A \quad \forall\, X \in \mathbf{M}_{m,n},$$

and the proof is completed. □

For a subset Ω of $\mathbf{M}_{m,n}$, the set of extreme points of Ω is denoted by $\operatorname{ext}(\Omega)$. In the following, $\mathbf{R}_{m,n}$ is the set of all $m \times n$ row stochastic matrices.

Proposition 2.1. *The set of all $m \times n$ row stochastic matrices is a convex set whose extreme points are $m \times n$, $(0,1)$-row stochastic matrices, i.e.*

$$\operatorname{ext}(\mathbf{R}_{m,n}) = \{A \in \mathbf{R}_{m,n}\colon\ A \textit{ is a } (0,1)\textit{-row stochastic matrix}\}.$$

P r o o f. It is easy to see that every $m \times n$, $(0,1)$-row stochastic matrix is an extreme point of $\mathbf{R}_{m,n}$. Now we show that if $R \in \mathbf{R}_{m,n}$ is not a $(0,1)$-row stochastic matrix, then R is not an extreme point of $\mathbf{R}_{m,n}$. Without loss of generality we may assume that the first row of R has k nonzero components with $k \geqslant 2$. Let

$$R = \begin{pmatrix} r_{11} \dots r_{1n} \\ A \end{pmatrix},$$

and let $r_{1j_1}, \dots, r_{1j_k}$ be the nonzero components of the first row of R. Put

$$R_{j_1} = E_{j_1} + \begin{pmatrix} 0 \\ A \end{pmatrix}, \dots, R_{j_k} = E_{j_k} + \begin{pmatrix} 0 \\ A \end{pmatrix}.$$

Then $R_{j_1}, \dots, R_{j_k} \in \mathbf{R}_{m,n}$, and we have $R = r_{j_1} R_{j_1} + \dots + r_{j_k} R_{j_k}$. Since $k \geqslant 2$, R is not an extreme point of $\mathbf{R}_{m,n}$ and the proof is complete. □

In the following lemma, we mention some useful results.

Lemma 2.2. *Let $n \geqslant 2$ and let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. Assume that $T(\mathbf{J})$ is a $(0,1)$-matrix and $T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \leqslant r, p \leqslant m$ and $1 \leqslant s, q \leqslant n$ with $(r,s) \neq (p,q)$. Then the following statements hold:*

(i) *If R is a $(0,1)$-matrix, then $T(R)$ is a $(0,1)$-matrix.*
(ii) *If $Z \circ T(\mathbf{J}) = 0$ and R is a $(0,1)$-matrix, then $Z \circ T(R) = 0$.*
(iii) *$T(X \circ Y) = T(X) \circ T(Y)$ for all $X, Y \in \mathbf{M}_{m,n}$.*

P r o o f. (i) It is enough to show that $T(E_{pq})$ is a $(0,1)$-matrix. Since T is a linear operator on $\mathbf{M}_{m,n}$, $T(\mathbf{J}) = \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} T(E_{ij})$. For each $(p,q) \in \mathbb{N}_m \times \mathbb{N}_n$, $T(\mathbf{J}) \circ T(E_{pq}) = T(E_{pq}) \circ T(E_{pq})$. Therefore, $T(E_{pq})$ is a $(0,1)$-matrix.

(ii) Since $T(E_{pq})$ is a $(0,1)$-matrix, we have $T(E_{pq}) \circ T(E_{pq}) = T(E_{pq})$. And if $Z \circ T(\mathbf{J}) = 0$, then $Z \circ T(E_{pq}) = Z \circ (T(\mathbf{J}) \circ T(E_{pq})) = (Z \circ (T(\mathbf{J})) \circ T(E_{pq}) = 0$.

(iii) Since $T(E_{ij}) \circ T(E_{ij}) = T(E_{ij})$, we have

$$T(X \circ Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} y_{ij} T(E_{ij}) = \sum_{i=1}^{m} x_{ij} T(E_{ij}) \circ \sum_{j=1}^{n} y_{ij} T(E_{ij}) = T(X) \circ T(Y).$$

□

Proposition 2.2. *Let $n \geqslant 2$ and let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. Then T preserves $\prec_{RH}$ if and only if T satisfies the following conditions:*

(1) *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \leqslant p, r \leqslant m$ and $1 \leqslant q, s \leqslant n$ with $(r,s) \neq (p,q)$.*
(2) *For every $(0,1)$-matrix $R \in \mathbf{R}_{m,n}$ there exists a $(0,1)$-matrix $Z \in \mathbf{M}_{m,n}$ such that $Z \circ T(\mathbf{J}) = 0$ and $T(R) + Z$ has exactly one nonzero entry in each row.*

P r o o f. First assume that T is a linear preserver of $\prec_{RH}$. Now part (1) of Theorem 2.1 implies (1) and part (2) of Theorem 1.1 implies (2). Conversely, assume that T satisfies the conditions (1) and (2). By part (iii) of Remark 2.1, without loss of generality we can assume that $T(\mathbf{J})$ is a $(0,1)$-matrix. Let $X, Y \in \mathbf{M}_{m,n}$ and $X \prec_{RH} Y$. Then there exists a row stochastic matrix $R \in \mathbf{M}_{m,n}$ such that $X = R \circ Y$, and hence by part (iii) of Lemma 2.2, $T(X) = T(R) \circ T(Y)$. Now by Proposition 2.1, $R = \sum\limits_{i=1}^{k} \lambda_i R_i$ for some $(0,1)$-row stochastic matrices $R_1, \ldots, R_k \in \mathbf{M}_{m,n}$ and some positive numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum\limits_{i=1}^{k} \lambda_i = 1$. By the use of (2), for each $1 \leqslant i \leqslant k$, we can find matrices $Z_i \in \mathbf{M}_{m,n}$ such that $Z_i \circ T(\mathbf{J}) = 0$ and $T(R_i) + Z_i$ is a matrix with exactly one nonzero entry in each row. By part (i) of Lemma 2.2, $Z_i \circ T(R_i) = 0$ and so $T(R_i) + Z_i$ is a $(0,1)$-matrix. Thus,

$$R' = \sum_{i=1}^{k} \lambda_i (T(R_i) + Z_i)$$

is a row stochastic matrix. Now we have

$$T(X) = T(R) \circ T(Y) = T\left(\sum_{i=1}^{k} \lambda_i R_i\right) \circ T(Y) = \left(\sum_{i=1}^{k} \lambda_i (T(R_i) + Z_i)\right) \circ T(Y) = R' \circ T(Y).$$

Therefore, T preserves $\prec_{RH}$. □

In the rest of this section, the graph characterization of linear preservers of R-Hadamard majorization is investigated. A directed graph (for short, a digraph) $G = (V, \mathcal{E})$ consists of a finite set V of elements called vertices and a set $\mathcal{E}$ of ordered pairs of vertices called (directed) edges. The order of the digraph G is the number $|V|$ (cardinal number of V) of its vertices. If $\alpha = (x, y)$ is an edge, then x is the initial vertex of α and y is the terminal vertex, and we say that α is an edge from x to y. In case $x = y$, α is a loop with initial and terminal vertices both equal to x. In a digraph G, a vertex has two degrees. The outdegree $d^+(v)$ of a vertex v is the number of edges of which v is an initial vertex and the indegree $d^-(v)$ of v is the number of edges of which v is a terminal vertex. A loop at a vertex contributes 1 to both its indegree and its outdegree. Two graphs $G_1 = (V, \mathcal{E}_1)$ and $G_2 = (V, \mathcal{E}_2)$ are edge-disjoint if $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$, see for more details [1]. In the following, $\mathbf{G}_n$ is the set of all digraphs of order n and $\mathbf{G}_n^1$ is the set of all digraphs of order n, where every vertex of these graphs has outdegree equal 1.

Let $A = [a_{ij}] \in \mathbf{M}_n$. Associate with A a digraph $\mathcal{D}(A) = (V, \mathcal{E})$, where $V = \{1, \ldots, n\}$ and $\mathcal{E} = \{(i, j) \colon a_{ij} \neq 0\}$. Then we have the map $\mathcal{D} \colon \mathbf{M}_n \to \mathbf{G}_n$ defined by $A \mapsto \mathcal{D}(A)$. Also, let $G = (V, \mathcal{E}) \in \mathbf{G}_n$. The adjacency matrix of G is $\mathcal{A}(G) = (a_{ij}) \in \mathbf{M}_n$, where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. So, we have the map

$\mathcal{A}\colon \mathbf{G}_n \to \mathbf{M}_n$ defined by $G \mapsto \mathcal{A}(G)$. For each linear operator $T\colon \mathbf{M}_n \to \mathbf{M}_n$, we associate the map $\varphi_T\colon \mathbf{G}_n \to \mathbf{G}_n$ defined by $\varphi_T = \mathcal{D} \circ T \circ \mathcal{A}$, i.e. the below diagram commutes:

$$\begin{array}{ccc} \mathbf{M}_n & \xrightarrow{T} & \mathbf{M}_n \\ {\scriptstyle\mathcal{A}}\big\uparrow & & \big\downarrow{\scriptstyle\mathcal{D}} \\ M' & \xrightarrow{\varphi_T} & \mathbf{G}_n. \end{array} \tag{2.2}$$

In the following theorem, we give a graph theoretic connection to the linear preservers of R-Hadamard majorization on $\mathbf{M}_n$. For every $1 \leqslant i, j \leqslant n$, let $G_{i,j} = \mathcal{D}(E_{ij})$.

Theorem 2.2. *Let T be a linear operator on $\mathbf{M}_n$. Then T preserves $\prec_{RH}$ if and only if φ_T preserves edge-disjoint graphs and for all $G \in \mathbf{G}_n^1$ there exists $H \in \mathbf{G}_n$ such that H and $\mathcal{D}(T(\mathbf{J}))$ are edge-disjoint and $\varphi_T(G) \cup H \in \mathbf{G}_n^1$.*

Proof. Assume that T preserves $\prec_{RH}$. Let $G_1, G_2 \in \mathbf{G}_n$ be two edge-disjoint graphs. Then $G_1 = \bigcup\limits_{(i,j)\in\alpha} G_{i,j}$ and $G_2 = \bigcup\limits_{(i,j)\in\beta} G_{i,j}$ for some $\alpha, \beta \subseteq \mathbb{N}_n \times \mathbb{N}_n$ such that $\alpha \cap \beta = \emptyset$. Therefore, $\mathcal{A}(G_1) = \sum\limits_{(i,j)\in\alpha} E_{ij}$ and $\mathcal{A}(G_2) = \sum\limits_{(i,j)\in\beta} E_{ij}$. These imply that $\varphi_T(G_1) = \bigcup\limits_{(i,j)\in\alpha} \mathcal{D}(T(E_{ij}))$ and $\varphi_T(G_2) = \bigcup\limits_{(i,j)\in\beta} \mathcal{D}(T(E_{ij}))$ and by the use of part (1) of Proposition 2.2, $\mathcal{D}(T(E_{rs}))$ and $\mathcal{D}(T(E_{pq}))$ are edge-disjoint graphs for every $(r,s) \in \alpha$ and $(p,q) \in \beta$. Thus, $\varphi_T(G_1)$ and $\varphi_T(G_2)$ are edge-disjoint graphs, and hence φ_T preserves edge-disjoint graphs. Now, let $G \in \mathbf{G}_n^1$ and $R = \mathcal{A}(G)$. Then $R \in \mathbf{M}_n$ is a $(0,1)$-row stochastic matrix. By part (2) of Proposition 2.2, there exists a $(0,1)$-matrix $Z \in \mathbf{M}_n$ such that $Z + T(R)$ is a matrix which in each row has exactly one nonzero entry and $Z \circ T(\mathbf{J}) = 0$. Put $H = \mathcal{D}(Z)$ and the proof is complete.

Conversely, let $(p,q) \neq (r,s)$. Then $G_{p,q}$ and $G_{r,s}$ are edge-disjoint graphs. Since φ_T preserves edge-disjoint graphs, $\varphi_T(G_{p,q})$ and $\varphi_T(G_{r,s})$ are edge-disjoint graphs, which implies that $T(E_{pq}) \circ T(E_{rs}) = 0$. Now, let $R \in \mathbf{M}_n$ be a $(0,1)$-row stochastic matrix. Then $\mathcal{D}(R) \in \mathbf{G}_n^1$, and by the assumption there exists $H \in \mathbf{G}_n$ such that H and $\mathcal{D}(T(\mathbf{J}))$ are edge-disjoint graphs and $\varphi_T(\mathcal{D}(R)) \cup H \in \mathbf{G}_n^1$. Let $Z = \mathcal{A}(H)$. It is easy to check that $Z + T(R)$ has exactly one nonzero entry in each row and $Z \circ T(\mathbf{J}) = 0$. Therefore, by Proposition 2.2, T preserves $\prec_{RH}$. $\square$

Example 2.1. Let $T\colon \mathbf{M}_2 \to \mathbf{M}_2$ be linear operator defined by:

$$T\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{22} & 0 \\ x_{12} & x_{11} \end{pmatrix}.$$

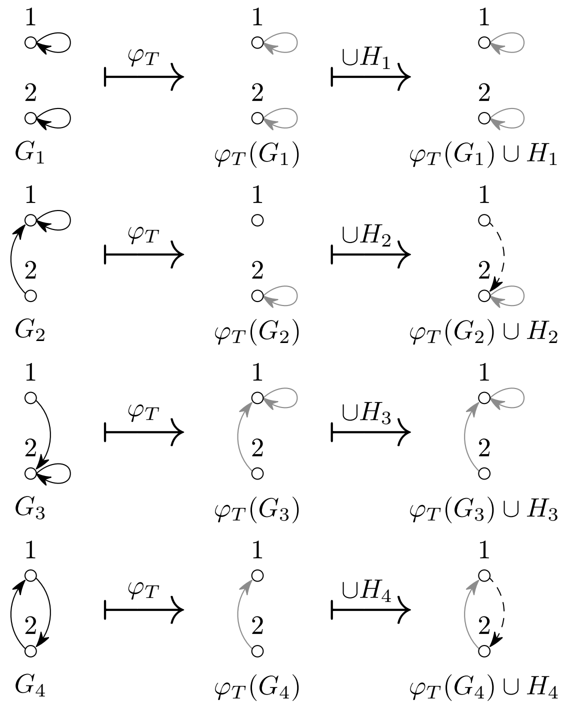

Figure 1.

Consider G_1, G_2, G_3 and G_4 as Figure 1. It is easy to see that for $1 \leqslant i \leqslant 4$, H_i and $\mathcal{D}(T(\mathbf{J}))$ are edge-disjoint graphs and $\varphi_T(G_i) \cup H_i \in \mathbf{G}_n^1$, where H_1, H_2, H_3 and H_4 are as Figure 2. Therefore, by Theorem 2.2, T is a linear preserver of $\prec_{RH}$.

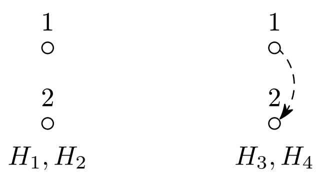

Figure 2.

In the next example, by using graphs, it is shown that the given linear operator T does not preserve R-Hadamard majorization.

Example 2.2. Define $T\colon\ \mathbf{M}_3 \to \mathbf{M}_3$ by

$$T(X) = \begin{pmatrix} x_{11}+x_{12} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}, \quad \forall\, X = [x_{ij}] \in \mathbf{M}_3. \tag{2.3}$$

Consider G_1 and G_2 as Figure 3. Then G_1 and G_2 are edge-disjoint graphs, but $\varphi_T(G_1)$ and $\varphi_T(G_2)$ are not edge-disjoint graphs. Therefore, by Theorem 2.2, T does not preserve $\prec_{RH}$ on $\mathbf{M}_3$.

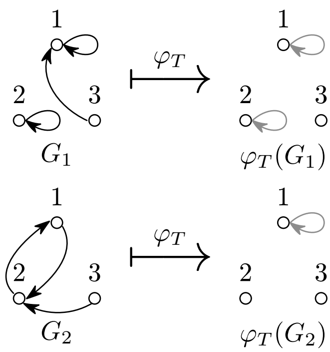

Figure 3.

3. Strong linear preservers of R-Hadamard majorization

In this section, we consider the linear operators that strongly preserve R-Hadamard majorization on $\mathbf{M}_{m,n}$. The following lemma can be obtained from the definition of R-Hadamard majorization.

Lemma 3.1. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. If T strongly preserves $\prec_{RH}$, then T is invertible.*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. (1) It is obtained by using part (1) of Theorem 1.1 and Lemma 3.1.

(2) First assume that T strongly preserves $\prec_{RH}$. By part (2) of Theorem 1.1, there are $A \in \mathbf{M}_{m,n}$ and permutation matrices $\widetilde{Q}_1, \ldots, \widetilde{Q}_1 \in \mathbf{M}_n$ such that

$$T(X) = \begin{pmatrix} X_{i_1}\widetilde{Q}_1 \\ X_{i_2}\widetilde{Q}_2 \\ \vdots \\ X_{i_m}\widetilde{Q}_m \end{pmatrix} \circ A \quad \forall\, X \in \mathbf{M}_{m,n},$$

where $X_{i_1}, \ldots, X_{i_m}$ are some rows of X. By Lemma 3.1, T is invertible and hence A

has no zero entry and $X_{i_1}, \ldots, X_{i_m}$ are distinct rows of X. Therefore,

$$T(X) = P \begin{pmatrix} X_1 Q_1 \\ X_2 Q_2 \\ \vdots \\ X_m Q_m \end{pmatrix} \circ A \quad \forall\, X \in \mathbf{M}_{m,n},$$

where P is an $m \times m$ permutation matrix so that $P(1, \ldots, m)^t = (i_1, \ldots, i_m)^t$ and $Q_{i_j} = \widetilde{Q}_j$ $(1 \leqslant j \leqslant m)$. Conversely, if T is of the form (1.2), we conclude that

$$T^{-1}(X) = P^{-1} \begin{pmatrix} X_1 Q_1^{-1} \\ X_2 Q_2^{-1} \\ \vdots \\ X_m Q_m^{-1} \end{pmatrix} \circ B \quad \forall\, X \in \mathbf{M}_{m,n},$$

where $B = [a_{ij}^{-1}] \in \mathbf{M}_{m,n}$. Now by Theorem 1.1, T and T^{-1} preserve $\prec_{RH}$. Therefore, T strongly preserves $\prec_{RH}$ and the proof is complete. □

The next proposition gives necessary and sufficient conditions for a linear operator T on $\mathbf{M}_{m,n}$ that strongly preserves R-Hadamard majorization.

Proposition 3.1. *Let $T\colon \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear operator. Then T strongly preserves $\prec_{RH}$ if and only if T is invertible and T satisfies the following conditions:*

(1) *$T(E_{rs}) \circ T(E_{pq}) = 0$ for every $1 \leqslant p, r \leqslant m$ and $1 \leqslant q, s \leqslant n$ with $(r, s) \neq (p, q)$.*

(2) *$T(R)$ has exactly one nonzero entry in each row for every $(0,1)$-row stochastic matrix $R \in \mathbf{M}_{m,n}$.*

Proof. Similar to the proof of Proposition 2.2, without loss of generality we can assume that $T(\mathbf{J})$ is a $(0,1)$-matrix. Assume that T strongly preserves $\prec_{RH}$. By Lemma 3.1, T is invertible and by part (1) of Proposition 2.2, (1) holds. Now by part (2) of Proposition 2.2 for every $(0,1)$-row stochastic matrix $R \in \mathbf{M}_n$ there exists a $(0,1)$-matrix $Y \in \mathbf{M}_n$ such that $Y \circ T(\mathbf{J}) = 0$ and $T(R) + Y$ has exactly one nonzero entry in each row. Since T is invertible, $T(\mathbf{J})$ has no zero entry. Hence $Y = 0$ and the conclusion is desired. Conversely, since T is invertible and satisfies (2), T^{-1} maps every $(0,1)$-row stochastic matrix to a $(0,1)$-row stochastic matrix and hence T^{-1} satisfies (2). For $1 \leqslant p, r \leqslant m$ and $1 \leqslant q, s \leqslant n$ with $(r, s) \neq (p, q)$, let $A = T^{-1}(E_{rs})$ and $B = T^{-1}(E_{pq})$. Thus, by using part (iii) of Lemma 2.2, $T(A \circ B) = T(A) \circ T(B) = E_{rs} \circ E_{pq} = 0$. This implies that $A \circ B = 0$ and hence T^{-1} satisfies (1). Therefore, by Theorem 2.2, T^{-1} preserves $\prec_{RH}$ and hence T strongly preserves $\prec_{RH}$. □

In the next theorem, we give a graph characterization for linear operators which are preservers of R-Hadamard majorization on $\mathbf{M}_n$.

Theorem 3.1. *Let T be a linear operator on $\mathbf{M}_n$. Then T strongly preserves $\prec_{RH}$ if and only if φ_T preserves edge-disjoint graphs and $\varphi_T(\mathbf{G}_n^1) \subseteq \mathbf{G}_n^1$.*

Proof. Let T strongly preserve $\prec_{RH}$. Then T preserves $\prec_{RH}$, and by Theorem 2.2, φ_T preserves edge-disjoint graphs. Assume that $G \in \mathbf{G}_n^1$ and $R = \mathcal{A}(G)$. Then $R \in \mathbf{M}_n$ is a $(0,1)$-row stochastic matrix and part (2) of Theorem 3.1 implies that $T(R)$ is a matrix with exactly one nonzero entry in each row. Therefore, $\mathcal{D}(T(R)) \in \mathbf{G}_n^1$ and hence $\varphi_T(\mathbf{G}_n^1) \subseteq \mathbf{G}_n^1$. Conversely, let φ_T preserve edge-disjoint graphs and $\varphi_T(\mathbf{G}_n^1) \subseteq \mathbf{G}_n^1$. By the proof of Theorem 2.2, $T(E_{rs})\circ T(E_{pq}) = 0$, where $(r,s) \neq (p,q)$. Assume that $R \in \mathbf{M}_n$ is a $(0,1)$-row stochastic matrix. So $\mathcal{D}(R) \in \mathbf{G}_n^1$, and by the assumption $\varphi_T(\mathcal{D}(R)) \in \mathbf{G}_n^1$. This implies that $\mathcal{D}(T(R)) \in \mathbf{G}_n^1$. Therefore, $T(R)$ has exactly one nonzero entry in each row and so by Theorem 3.1, T strongly preserves $\prec_{RH}$. □

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