UNCONDITIONAL UNIQUENESS OF HIGHER ORDER NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We show the existence of weak solutions in the extended sense of the Cauchy problem for the cubic fourth order nonlinear Schrödinger equation with the initial data $u_0 \in X$, where $X \in \{M_{2,q}^s(\mathbb{R}), H^\sigma(\mathbb{T}), H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})\}$ and $q \in [1, 2]$, $s \geqslant 0$, or $\sigma \geqslant 0$, or $s_2 \geqslant s_1 \geqslant 0$. Moreover, if $M_{2,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, or if $\sigma \geqslant \frac{1}{6}$, or if $s_1 \geqslant \frac{1}{6}$ and $s_2 > \frac{1}{2}$ we show that the Cauchy problem is unconditionally wellposed in X. Similar results hold true for all higher order nonlinear Schrödinger equations and mixed order NLS due to a factorization property of the corresponding phase factors. For the proof we employ the normal form reduction via the differentiation by parts technique and build upon our previous work.

Keywords: normal form method; modulation space; unconditional uniqueness; higher order nonlinear Schrödinger

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1. Introduction and main results

We consider the Cauchy problem associated to the cubic fourth order nonlinear Schrödinger equation, also known as the cubic biharmonic NLS, given by

$$(1) \qquad \begin{cases} \mathrm{i}\partial_t u - \partial_x^4 u \pm |u|^2 u = 0, \\ u(0, x) = u_0(x) \end{cases}$$

with the initial data $u_0 \in X$ for $X \in \{M_{2,q}^s(\mathbb{R}), H^s(\mathbb{T}), H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})\}$. The biharmonic NLS provides an important model case for nonlinear partial differential equations of super-quadratic order. The study of the biharmonic NLS goes back to [22] and [23], where the partial differential equation was introduced to take into

account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity (for applications of higher order NLS, such as the sixth and eighth order one, see [9], [21], [33] and [36]).

A large amount of work has been devoted to the Cauchy problem (1) with the initial data u_0 in the Sobolev spaces $H^s(\mathbb{R})$ or $H^s(\mathbb{T})$. In both settings solutions to the problem enjoy mass and energy conservation. For a solution u to the continuous setting this reads

$$\|u(t,\cdot)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \tag{2}$$

$$E(u(t,\cdot)) := \frac{1}{2}\int_{\mathbb{R}} |\Delta u|^2 \, \mathrm{d}x \mp \frac{1}{4}\int_{\mathbb{R}} |u|^4 \, \mathrm{d}x = E(u_0) \tag{3}$$

and it is known that in the mass subcritical cases (with the nonlinearity $|u|^{\alpha-1}u$, $\alpha \in [1, 1+8/d)$) the Cauchy problem (1) is globally wellposed in $L^2(\mathbb{R}^d)$ via Strichartz type estimates (similar results hold in $H^2(\mathbb{R}^d)$ for the energy subcritical cases), see [14] as well as [2], [4], [31], [32] and the references therein. For the results outside of $L^2(\mathbb{R}^d)$ (see [34]) and for the results, where the dispersive estimates were used in the modulation space setting, see [16].

In the periodic setting it is known that the Cauchy problem (1) is globally well-posed in $H^s(\mathbb{T})$ for $s > -\frac{1}{3}$ (see [27] and [28]), where the proof is done via the short-time Fourier restriction norm method. For more results we refer the interested reader to [8], [12], [29] and the references therein.

From [3] and [26] it is known that the (semi-)group $S(t) = \mathrm{e}^{\mathrm{i}t\Delta^2}$, $t \in \mathbb{R}$, defined as a Fourier multiplier operator with symbol

$$\mathcal{F}(S(t))(\xi) := \mathrm{e}^{\mathrm{i}t\xi^4} \tag{4}$$

is not bounded on $M^s_{p,q}(\mathbb{R})$ (for the definition of these modulation spaces see Section 2) unless $p = 2$ in which case it is an isometry. If in addition we assume that either $q = 1$ and $s \geqslant 0$, or $q > 1$ and $s > 1/q'$, then the modulation space is a Banach algebra. Hence, for the initial data $u_0 \in M^s_{2,q}(\mathbb{R})$ an easy Banach contraction argument implies that the Cauchy problem (1) is locally wellposed with the solution u being the fixed point of the operator

$$T(u) := S(t)u_0 \pm \mathrm{i} \int_0^t S(t-\tau)|u|^2 u \, \mathrm{d}\tau \tag{5}$$

in the space $M(R,T) := \{u \in C([0,T], M^s_{2,q}(\mathbb{R})) \colon \|u\| \leqslant R \approx 2\|u_0\|_{M^s_{2,q}}\}$ for $T > 0$ sufficiently small. We should also mention that in [7] it was shown that $S(t)$ is bounded from $M^s_{p',q}(\mathbb{R})$ into $M^s_{p,q}(\mathbb{R})$ for $p \geqslant 2$ and as a result small data global existence was obtained still in the case that the modulation spaces are Banach algebras.

One of the goals of this paper is to consider similar questions in the case, where the modulation space $M_{2,q}^s(\mathbb{R})$ does not belong to the previously mentioned Banach algebra family, i.e., in the case, where $q > 1$ and $s \in [0, 1/q']$. This will be achieved with the use of the differentiation by parts technique which was inspired by the periodic case in [17] and was used in [5], [6] and [30] to study similar questions for the cubic NLS in one dimension

$$
(6) \qquad \begin{cases} i\partial_t u - \partial_x^2 u \pm |u|^2 u = 0, & (t,x) \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}
$$

Here let us remark that the (semi-)group of (6), namely the Schrödinger operator $e^{it\Delta}$, is bounded on all modulation spaces $M_{p,q}^s(\mathbb{R})$, $p, q \in [1, \infty]$, $s \in \mathbb{R}$, and not only in the special case $p = 2$, see again [3], [26].

In the periodic setting, as in [1] or [17], the differentiation by parts technique transforms the PDE into a countable system of ODEs for the Fourier coefficients of the solution. In the approach described in [5], [6] and [30] the authors replaced the Fourier coefficients of periodic functions by the isometric decomposition operators $\Box_k$, in order to have a similar localization in the Fourier space. Using these "box" operators for localization yields a unified approach to the periodic and the continuous settings. According to this, a proof using normal form reduction via differentiation by parts for the initial data $u_0 \in H^s(\mathbb{T})$ can be transformed to a proof for the initial data $u_0 \in M_{2,q}^s(\mathbb{R})$, $q \in [1, 2]$. This is even possible for initial data in the "tooth problem" space $H^s(\mathbb{R}) + H^s(\mathbb{T})$ (we refer to [5] for the modifications to be made and also for the explanation why we call this a "*tooth problem*"). The second goal of this paper is to emphasize these relations. Taking this into account we shall only present the proofs of main Theorems 4 and 6 in the case of $u_0 \in M_{2,q}^s(\mathbb{R})$. A more detailed explanation is given in Remark 7.

As it was done in [30], in order to give a meaning to solutions of the biharmonic NLS in $C([0,T], M_{2,q}^s(\mathbb{R}))$ and to the nonlinearity $\mathcal{N}(u) := u\bar{u}u$ we need the following definitions which first appeared in [10], [11], where power series solutions to the cubic NLS were studied (see also [15] for similar considerations for the KdV).

Definition 1. A *sequence of Fourier cutoff operators* is a sequence of Fourier multiplier operators $\{T_N\}_{N \in \mathbb{N}}$ on $\mathcal{S}'(\mathbb{R})$ with multipliers $m_N \colon \mathbb{R} \to \mathbb{C}$ such that

▷ m_N has compact support on $\mathbb{R}$ for every $N \in \mathbb{N}$,

▷ m_N is uniformly bounded, i.e., $\sup\limits_{x,N} |m_N(x)| < \infty$,

▷ $\lim\limits_{N \to \infty} m_N(x) = 1$ for any $x \in \mathbb{R}$.

Definition 2. Let $u \in C([0,T], M_{2,q}^s(\mathbb{R}))$. We say that $\mathcal{N}(u)$ *exists and is equal to a distribution* $w \in \mathcal{S}'((0,T) \times \mathbb{R})$ if for every sequence $\{T_N\}_{N \in \mathbb{N}}$ of Fourier cutoff operators we have

$$\lim_{N \to \infty} \mathcal{N}(T_N u) = w \tag{7}$$

in the sense of distributions on $(0,T) \times \mathbb{R}$.

Definition 3. We say that $u \in C([0,T], M_{2,q}^s(\mathbb{R}))$ is a *weak solution in the extended sense of NLS* (1) if it holds

▷ $u(0,x) = u_0(x)$,
▷ the nonlinearity $\mathcal{N}(u)$ exists in the sense of Definition 2,
▷ u satisfies (1) in the sense of distributions on $(0,T) \times \mathbb{R}$, where the nonlinearity $\mathcal{N}(u) = u|u|^2$ is interpreted as above.

Our main result which guarantees the existence of weak solutions in the extended sense is the following theorem.

Theorem 4. *Let* $1 \leqslant q \leqslant 2$ *and* $s \geqslant 0$. *For* $u_0 \in M_{2,q}^s(\mathbb{R})$ *there exists a weak solution in the extended sense* $u \in C([0,T]; M_{2,q}^s(\mathbb{R}))$ *of NLS* (1) *with the initial condition* u_0, *where the time* T *of existence depends only on* $\|u_0\|_{M_{2,q}^s}$. *Moreover, the solution map is locally Lipschitz continuous.*

Remark 5. The restriction on the range of q appears when estimating the resonant operator R_2^t. See Lemma 15 in Section 3.

The next theorem is about the unconditional wellposedness of NLS (1) in modulation spaces, that is, uniqueness of solutions in $C([0,T], M_{2,q}^s(\mathbb{R}))$ without intersecting with any auxiliary function space (see [24], where this notion first appeared).

Theorem 6. *For* $u_0 \in M_{2,q}^s(\mathbb{R})$ *with either* $s \geqslant 0$ *and* $1 \leqslant q \leqslant \frac{3}{2}$ *or* $\frac{3}{2} < q \leqslant 2$ *and* $s > \frac{2}{3} - 1/q$, *the solution* u *with the initial condition* u_0 *constructed in Theorem 4 is unique in* $C([0,T], M_{2,q}^s(\mathbb{R}))$.

Remark 7. Having stated Theorems 4 and 6 for $u_0 \in M_{2,q}^s(\mathbb{R})$ let us explain what happens in the cases of $H^s(\mathbb{T})$, $s \geqslant 0$, and $H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$, $s_2 \geqslant s_1 \geqslant 0$. In the latter case we consider functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as periodic functions on $\mathbb{R}$.

In the periodic setting, Definition 1 remains the same, whereas in Definition 2 the limit is taken in the sense of distributions on $(0,T) \times \mathbb{T}$. Then the proof follows the calculations presented in the next sections, where instead of considering the quantities $\square_n u$ we have the Fourier coefficients of the periodic function u, i.e.,

$$u_n(t) := \int_0^1 e^{-2\pi i n x} u(t,x) \, dx, \quad n \in \mathbb{Z}. \tag{8}$$

In [28] the authors study (1) in the periodic setting but they are interested in the energy and therefore, they use differentiation by parts for quadrilinear forms whereas we aim for the existence of solutions and thus, we have to study trilinear operators. The corresponding tree structures and the estimates for the multilinear expressions in [28] are different from the ones in the present paper. Existence of local solutions is proved in [28] by the Fourier restriction norm method and even includes negative Sobolev spaces. The fact that the periodic cubic fourth order NLS is unconditionally wellposed in $H^s(\mathbb{T})$ for $s \geqslant \frac{1}{6}$ was already observed in [28] without including the proof.

In the tooth problem space setting, i.e., that of the initial data $u_0 = v_0 + w_0 \in H^{s_1}(\mathbb{R}) + H^{s_2}(\mathbb{T})$, having the result for the cubic biharmonic periodic NLS with the initial data $w_0 \in H^{s_2}(\mathbb{T})$ at hand (from the previous paragraph), we consider the cubic modified biharmonic NLS given by

(9)
$$\begin{cases} i\partial_t v - \partial_x^4 v \pm G(w, v) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ v(0, x) = v_0(x) \in H^{s_1}(\mathbb{R}), & x \in \mathbb{R}, \end{cases}$$

where $G(w, v)$ is the nonlinearity

(10)　$G(w, v) = |w + v|^2(w + v) - |w|^2 w = |v|^2 v + v^2 \overline{w} + w^2 \overline{v} + 2w|v|^2 + 2v|w|^2.$

Then Definition 1 remains the same and Definitions 2 and 3 are the same as Definitions 4 and 5 given in [5]. The proof of the existence of a weak solution in the extended sense v of (9) can then be done by modifying the calculations presented in the next sections and combining them with what has been done in [5].

Remark 8. Notice that for $q = 2$ in Theorem 6 we obtain that the cubic fourth order NLS is unconditionally wellposed in $H^s(\mathbb{R})$ for $s \geqslant \frac{1}{6}$.

Remark 9. Theorems 4 and 6 (and Remarks 7, 8) remain true for the following mixed order nonlinear Schrödinger equations

(11)
$$\begin{cases} i\partial_t u - \sum_{j=1}^{M} (-1)^j \varepsilon_j \partial_x^{2j} u \pm |u|^2 u = 0, \\ u(0, x) = u_0(x), \end{cases}$$

where $M \in \mathbb{N}$, $\varepsilon_j \in \mathbb{R}_{\geqslant 0}$ for $j \in \{1, \ldots, M\}$ and $\sum_{j=1}^{M} \varepsilon_j > 0$. This is the case because the phase factors Φ_{2j}, $j \in \{1, \ldots, M\}$, (see (30) for their definition) enjoy a special factorization property (see Proposition 30). For a more detailed argument we refer to Section 6. The question whether such a factorization exists was asked in [18], Remark 1.5.

Remark 10. A recent preprint [25] deals with the unconditional uniqueness of other dispersive PDEs on the multidimensional torus with the use of differentiation by parts in a more abstract framework. It seems not to be clear how to adapt this to the situation of the present paper.

We should also mention [20], where the authors use a different approach to the unconditional uniqueness of the cubic NLS (6) which applies to various spatial domains. The main idea is to exploit the relation of solutions of the cubic NLS to solutions of the Gross-Pitaevskii hierarchy.

The paper is organized as follows: In Section 2 we present the preliminaries and Section 3 contains the first few steps of the iteration process together with the estimates for the first resonant and non-resonant operators that appear. Section 4 describes the tree notation and the induction step, and Section 5 finishes the argument of the proofs of Theorems 4 and 6. Finally, in Section 6 we deal with the higher order NLS (11).

The following notation is used throughout the paper: For a number $1 \leqslant p \leqslant \infty$ we write p' for its dual exponent, that is the number that satisfies $1/p + 1/p' = 1$. For two quantities A, B (they can be functions or numbers) whenever we write $A \lesssim B$ we mean that there is a universal constant $C > 0$ such that $A \leqslant CB$. For a set A we use $\#(A)$ and $|A|$ to denote its cardinality.

2. Preliminaries

Let us denote by $S(\mathbb{R})$ the Schwartz class and by $S'(\mathbb{R})$ the tempered distributions.

Definition 11. Let $Q_0 = [-\frac{1}{2}, \frac{1}{2})$ and its translations $Q_k = Q_0 + k$ for all $k \in \mathbb{Z}$. Consider a partition of unity $\{\sigma_k = \sigma_0(\cdot - k)\}_{k \in \mathbb{Z}} \subset C^\infty(\mathbb{R})$ satisfying

▷ $\exists c > 0 \colon \forall \eta \in Q_0 \colon |\sigma_0(\eta)| \geqslant c$,

▷ $\mathrm{supp}(\sigma_0) \subseteq \{\xi \in \mathbb{R} \colon |\xi| < 1\}$.

Note that this implies $1 = \sigma_0(0) = \sigma_k(k)$ for all $k \in \mathbb{Z}$. Given a partition of unity as above, we define the *isometric decomposition operators* (box operators) as

$$(12) \qquad \Box_k := \mathcal{F}^{(-1)} \sigma_k \mathcal{F} \quad \forall k \in \mathbb{Z}.$$

Then the norm of a tempered distribution $f \in S'(\mathbb{R})$ in the modulation space $M_{p,q}^s(\mathbb{R})$, $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$, is

$$(13) \qquad \|f\|_{M_{p,q}^s} := \|\{\langle k \rangle^s \| \Box_k f\|_{L^p(\mathbb{R})}\}_{k \in \mathbb{Z}}\|_{l^q(\mathbb{Z})},$$

where we denote the Japanese bracket by $\langle k \rangle := (1 + |k|^2)^{1/2}$ and

$$(14) \qquad M_{p,q}^s(\mathbb{R}) := \{f \in S'(\mathbb{R}) \colon \|f\|_{M_{p,q}^s} < \infty\}.$$

It can be proved that different choices of such sequences of functions $\{\sigma_k\}_{k \in \mathbb{Z}}$ lead to equivalent norms in $M_{p,q}^s(\mathbb{R})$. When $s = 0$ we denote the space $M_{p,q}^0(\mathbb{R})$ by $M_{p,q}(\mathbb{R})$. In the special case, where $p = q = 2$ we have $M_{2,2}^s(\mathbb{R}) = H^s(\mathbb{R})$ the usual Sobolev spaces

$$(15) \qquad H^s(\mathbb{R}) := \left\{ f \in S'(\mathbb{R}) \colon \|f\|_{H^s(\mathbb{R})} := \left(\int_{\mathbb{R}} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty \right\}.$$

In this paper we use that for $s > 1/q'$ and $1 \leqslant p, q \leqslant \infty$, the embedding

$$(16) \qquad M_{p,q}^s(\mathbb{R}) \hookrightarrow C_b(\mathbb{R}) = \{ f \colon \mathbb{R} \to \mathbb{C} \colon f \text{ continuous and bounded} \}$$

and for $(1 \leqslant p_1 \leqslant p_2 \leqslant \infty,\ 1 \leqslant q_1 \leqslant q_2 \leqslant \infty,\ s_1 \geqslant s_2)$ or $(1 \leqslant p_1 \leqslant p_2 \leqslant \infty,\ 1 \leqslant q_2 < q_1 \leqslant \infty,\ s_1 > s_2 + 1/q_2 - 1/q_1)$, the embedding

$$(17) \qquad\qquad M_{p_1,q_1}^{s_1}(\mathbb{R}) \hookrightarrow M_{p_2,q_2}^{s_2}(\mathbb{R})$$

are both continuous and can be found in [13], Propositions 6.8 and 6.5. Also, by [35] it is known that for any $1 < p \leqslant \infty$ we have the embedding $M_{p,1}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which together with the fact that $M_{2,2}(\mathbb{R}) = L^2(\mathbb{R})$ and interpolation imply that for any $p \in [2, \infty]$ we have the embedding $M_{p,p'}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$. Later in the proof of Theorem 6 we use this fact for $p = 3$, that is

$$(18) \qquad\qquad M_{3,3/2}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R}).$$

The following facts are useful in the calculations presented in the next sections. Firstly, notice that for $S(t) = e^{it\Delta^2}$ being the biharmonic Schrödinger (semi-)group we have the equality

$$(19) \qquad\qquad \|S(t)f\|_2 = \|f\|_2.$$

Secondly, we need the multiplier estimate which follows from Young's inequality and is known as Bernstein's inequality.

Lemma 12. *Let $1 \leqslant p \leqslant \infty$ and $\sigma \in C_c^\infty(\mathbb{R})$. Then the multiplier operator $T_\sigma \colon S(\mathbb{R}) \to S'(\mathbb{R})$ defined by*

$$(T_\sigma f) = \mathcal{F}^{-1}(\sigma \cdot \hat{f}) \quad \forall f \in S(\mathbb{R})$$

is bounded on $L^p(\mathbb{R})$ and

$$\|T_\sigma\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim \|\check{\sigma}\|_{L^1(\mathbb{R})}.$$

An immediate consequence is that for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ we have

$$(20) \qquad \| \Box_k f \|_{p_2} \lesssim \| \Box_k f \|_{p_1},$$

where the implicit constant is independent of k and the function f (for a proof see e.g. [30]).

Lastly, let us recall the following number theoretic fact (see [19], Theorem 315) which is going to be used in the proof of Theorem 4.

Proposition 13. *Given an integer m, let $d(m)$ denote the number of divisors of m. Then we have*

$$(21) \qquad d(m) \lesssim e^{c(\log m / \log \log m)} = o(m^{\varepsilon}) \quad \forall \varepsilon > 0.$$

3. Description of the iteration process

The proof follows the same steps as in [30] but the operators that originate from applying the differentiation by parts technique are different and have to be estimated differently in order to control them in the appropriate spaces. For this reason our proof is detailed only in those steps, where a different approach is needed.

In the space $M_{2,q}^s(\mathbb{R})$ there is a more convenient expression for its norm which is the one we are going to use in our calculations. Let us denote by $\widetilde{\Box}_k$ the frequency projection operator $\mathcal{F}^{(-1)} 1_{[k,k+1]} \mathcal{F}$, where $1_{[k,k+1]}$ is the characteristic function of the interval $[k, k+1]$, $k \in \mathbb{Z}$. It can be proved that

$$(22) \qquad \| f \|_{M_{2,q}^s} \approx \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{sq} \| \widetilde{\Box}_k f \|_2^q \right)^{1/q},$$

that is, the two norms are equivalent in $M_{2,q}^s(\mathbb{R})$. We are going to use expression (22) for the norm in $M_{2,q}^s(\mathbb{R})$ and for convenience we still write $\Box_n$ instead of $\widetilde{\Box}_n$ and σ_k instead of $1_{[k,k+1]}$.

From here on, we consider only the case $s = 0$ in Theorem 4 since for $s > 0$ similar considerations apply. See Remark 27 for a more detailed explanation.

The next notations are essential for the analysis that follows. For $n \in \mathbb{Z}$ let us put

$$(23) \qquad u_n(t,x) = \Box_n \, u(t,x),$$

$$(24) \qquad v(t,x) = e^{it\partial_x^4} u(t,x),$$

$$(25) \qquad v_n(t,x) = e^{it\partial_x^4} u_n(t,x) = \Box_n \, [(e^{it\partial_x^4} u(t,x)] = \Box_n \, v(t,x).$$

Also for $(\xi, \xi_1, \xi_2, \xi_3) \in \mathbb{R}^4$ we define the function

(26)
$$\Phi_4(\xi, \xi_1, \xi_2, \xi_3) = \xi^4 - \xi_1^4 + \xi_2^4 - \xi_3^4,$$

which is equal to (see [27], Lemma 3.1)

(27) $\Phi_4(\xi, \xi_1, \xi_2, \xi_3) = (\xi - \xi_1)(\xi - \xi_3)(\xi^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + 2(\xi_1 + \xi_3)^2)$

if $\xi = \xi_1 - \xi_2 + \xi_3$. Notice that if we let

(28)
$$\Phi_2(\xi, \xi_1, \xi_2, \xi_3) = \xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2$$

then under the assumption $\xi = \xi_1 - \xi_2 + \xi_3$, $\Phi_2 = 2(\xi - \xi_1)(\xi - \xi_3)$ and the relation

(29)
$$|\Phi_4| \sim \max\{|\xi|, |\xi_1|, |\xi_2|, |\xi_3|\}^2 |\xi - \xi_1| |\xi - \xi_3| \gtrsim |\Phi_2|^2$$

holds. More generally, for $k \in \mathbb{Z}_+$ we introduce the function

(30)
$$\Phi_{2k}(\xi, \xi_1, \xi_2, \xi_3) := \xi^{2k} - \xi_1^{2k} + \xi_2^{2k} - \xi_3^{2k}$$

which is studied in more detail in Section 6.

The main equation (1) implies that

(31)
$$\mathrm{i}\partial_t u_n - \partial_x^4 u_n \pm \Box_n\left(|u|^2 u\right) = 0$$

and by using the expansion $u = \sum_k \Box_k u$ it is immediate that

$$\Box_n\left(u\bar{u}u\right) = \Box_n \sum_{n_1, n_2, n_3} u_{n_1} \bar{u}_{n_2} u_{n_3} = \sum_{n_1 - n_2 + n_3 \approx n} \Box_n\left[u_{n_1} \bar{u}_{n_2} u_{n_3}\right],$$

where by $\approx n$ we mean $= n$ or $= n+1$ or $= n-1$. During the calculations we also write $\xi \approx n$, where ξ is going to be a continuous variable and n an integer. By that we mean that $\xi \in [n, n+1)$ or, more generally, that ξ is in a suitable interval around n.

Next we do the change of variables $u_n(t, x) = \mathrm{e}^{-\mathrm{i}t\partial_x^4} v_n(t, x)$ and arrive at the expression

(32) $\partial_t v_n = \pm\mathrm{i} \sum_{n_1 - n_2 + n_3 \approx n} \Box_n\left(\mathrm{e}^{\mathrm{i}t\partial_x^4}[\mathrm{e}^{-\mathrm{i}t\partial_x^4} v_{n_1} \cdot \mathrm{e}^{\mathrm{i}t\partial_x^4} \bar{v}_{n_2} \cdot \mathrm{e}^{-\mathrm{i}t\partial_x^4} v_{n_3}]\right).$

The 1st generation operators are given by

(33) $Q_n^{1,t}(v_{n_1}, \bar{v}_{n_2}, v_{n_3})(x) = \Box_n\left(\mathrm{e}^{\mathrm{i}t\partial_x^4}[\mathrm{e}^{-\mathrm{i}t\partial_x^4} v_{n_1} \cdot \mathrm{e}^{\mathrm{i}t\partial_x^4} \bar{v}_{n_2} \cdot \mathrm{e}^{-\mathrm{i}t\partial_x^4} v_{n_3}]\right)$

or in other words

$$(34) \qquad \partial_t v_n = \pm \mathrm{i} \sum_{n_1 - n_2 + n_3 \approx n} Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}).$$

Below we describe the first few steps of the iteration procedure known as differentiation by parts technique. We define many operators, R_1^t, R_2^t, N_{11}^t, $\widetilde{Q}_n^{1,t}$, N_{21}^t, N_4^t, N_{31}^t, and we need to be able to control all of them in the appropriate norms. This is done in Lemmata 15, 16, 17, 19, 20 and 21.

To move forward we use the splitting

$$(35) \qquad \partial_t v_n = \pm \mathrm{i} \sum_{n_1 - n_2 + n_3 \approx n} Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}) = \sum_{\substack{n_1 \approx n \\ \text{or} \\ n_3 \approx n}} \dots + \sum_{n_1 \not\approx n \not\approx n_3} \dots$$

and we define the resonant operator part

$$(36) \qquad R_2^t(v)(n) - R_1^t(v)(n) = \left(\sum_{n_1 \approx n} Q_n^{1,t} + \sum_{n_3 \approx n} Q_n^{1,t} \right) - \sum_{\substack{n_1 \approx n \\ \text{and} \\ n_3 \approx n}} Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})$$

with R_2^t being equal to the sum of the first two summands and R_1^t being equal to the last summand, and the non-resonant operator part

$$(37) \qquad N_1^t(v)(n) = \sum_{n_1 \not\approx n \not\approx n_3} Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}).$$

This implies the following expression for our biharmonic NLS (we drop the factor $\pm \mathrm{i}$ in front of the sum since it plays no role in our analysis)

$$(38) \qquad \partial_t v_n = R_2^t(v)(n) - R_1^t(v)(n) + N_1^t(v)(n).$$

We have to split the non-resonant part N_1^t further as

$$(39) \quad N_1^t(v)(n) = N_{11}^t(v)(n) + N_{12}^t(v)(n), \quad \text{where } N_{11}^t(v)(n) = \sum_{A_N(n)} Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}),$$

$(40) \quad A_N(n)$
$$= \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi_4(n, n_1, n_2, n_3)| \leqslant N\}$$

and

$(41) \quad A_N(n)^c$
$$= \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi_4(n, n_1, n_2, n_3)| > N\}.$$

The number $N > 0$ is considered to be large and will be fixed at the end of the proof.

At the N_{12}^t part we have to split even further, keeping in mind that we are on $A_N(n)^c$. We perform all formal calculations assuming that v is a sufficiently smooth solution. Later, we justify these formal computations. From (33) we know that

$$\mathcal{F}(Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}))(\xi)$$
$$= \sigma_n(\xi) \int_{\mathbb{R}^2} e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3$$

and by the usual product rule for the derivative we can write the previous integral as the sum of the expressions

$$(42) \quad \partial_t \left(\sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3 \right)$$
$$- \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \partial_t(\widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3)) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3.$$

Hence, we have the splitting

$$(43) \qquad \mathcal{F}(Q_n^{1,t}) = \partial_t \mathcal{F}(\widetilde{Q}_n^{1,t}) - \mathcal{F}(T_n^{1,t})$$

or equivalently

$$(44) \qquad Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}) = \partial_t(\widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})) - T_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}),$$

which allows us to write

$$(45) \qquad N_{12}^t(v)(n) = \partial_t(N_{21}^t(v)(n)) + N_{22}^t(v)(n),$$

where

$$(46) \qquad N_{21}^t(v)(n) = \sum_{A_N(n)^c} \widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})$$

and

$$(47) \qquad N_{22}^t(v)(n) = \sum_{A_N(n)^c} T_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}).$$

From the definition of $\widetilde{Q}_n^{1,t}$ we have

$$\mathcal{F}(\widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3}))(\xi) = e^{it\xi^4} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\widehat{u}_{n_1}(\xi_1) \widehat{\overline{u}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{u}_{n_3}(\xi_3)}{\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3$$

and we define the operator $R_n^{1,t}$ by

$$(48) \quad \mathcal{F}(R_n^{1,t}(u_{n_1}, \overline{u}_{n_2}, u_{n_3}))(\xi) = \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{\widehat{u}_{n_1}(\xi_1) \widehat{\overline{u}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{u}_{n_3}(\xi_3)}{\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3$$

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or, in other words,

$$(49) \quad R_n^{1,t}(w_{n_1}, \overline{w}_{n_2}, w_{n_3})(x)$$
$$= \int_{\mathbb{R}^3} e^{ix\xi} \sigma_n(\xi) \frac{\widehat{w}_{n_1}(\xi_1)\widehat{\overline{w}}_{n_2}(\xi - \xi_1 - \xi_3)\widehat{w}_{n_3}(\xi_3)}{\Phi_4(\xi, \xi_1, \xi - \xi_1 - \xi_3, \xi_3)} \, d\xi_1 \, d\xi_3 \, d\xi.$$

Writing out the Fourier transforms of the functions inside the integral it is immediate that

$$(50) \quad R_n^{1,t}(w_{n_1}, \overline{w}_{n_2}, w_{n_3})(x)$$
$$= \int_{\mathbb{R}^3} K_n^{(1)}(x, x_1, y, x_3) w_{n_1}(x) \overline{w}_{n_2}(y) w_{n_3}(x_3) \, dx_1 \, dy \, dx_3,$$

where

$$K_n^{(1)}(x, x_1, y, x_3) = \int_{\mathbb{R}^3} e^{i\xi_1(x-x_1)+i\eta(x-y)+i\xi_3(x-x_3)} \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{\Phi_4(\xi_1 + \eta + \xi_3, \xi_1, \eta, \xi_3)} \, d\xi_1 \, d\eta \, d\xi_3$$
$$= \mathcal{F}^{-1} \varrho_n^{(1)}(x - x_1, x - y, x - x_3)$$

and

$$\varrho_n^{(1)}(\xi_1, \eta, \xi_3) = \frac{\sigma_n(\xi_1 + \eta + \xi_3)}{\Phi_4(\xi_1 + \eta + \xi_3, \xi_1, \eta, \xi_3)}.$$

For the remaining part N_{22}^t we have to make use of equality (38) depending on whether the derivative falls on $\widehat{v}_{n_1}$, on $\widehat{\overline{v}}_{n_2}$ or on $\widehat{v}_{n_3}$. The expression we obtain is given by

$$N_{22}^t(v)(n) =$$
$$-2i \sum_{A_N(n)^c} [\widetilde{Q}_n^{1,t}(R_2^t(v)(n_1) - R_1^t(v)(n_1), \overline{v}_{n_2}, v_{n_3}) + \widetilde{Q}_n^{1,t}(N_1^t(v)(n_1), \overline{v}_{n_2}, v_{n_3})]$$
$$-i \sum_{A_N(n)^c} [\widetilde{Q}_n^{1,t}(v_{n_1}, R_2^t(\overline{v})(n_2) - R_1^t(\overline{v})(n_2), v_{n_3}) + \widetilde{Q}_n^{1,t}(v_{n_1}, N_1^t(\overline{v})(n_2), v_{n_3})]$$

(the number 2 appears in front of the first sum because the expression is symmetric with respect to v_{n_1} and v_{n_3}). Therefore, we can write N_{22}^t as a sum

$$(51) \quad N_{22}^t(v)(n) = N_4^t(v)(n) + N_3^t(v)(n),$$

where $N_4^t(v)(n)$ is the sum including the resonant parts $R_2^t - R_1^t$.

In order to continue, the non-resonant part N_3^t needs to be decomposed even further. It consists of 3 sums depending on, where the operator N_1^t acts. One of them is (similar considerations apply for the remaining sums, too)

$$(52) \quad \sum_{A_N(n)^c} \widetilde{Q}_n^{1,t}(N_1^t(v)(n_1), \overline{v}_{n_2}, v_{n_3}),$$

where

$$N_1^t(v)(n_1) = \sum_{m_1 \not\approx n_1 \not\approx m_3} Q_{n_1}^{1,t}(v_{m_1}, \overline{v}_{m_2}, v_{m_3})$$

and $n_1 \approx m_1 - m_2 + m_3$. Here we have to consider new restrictions on the frequencies $(m_1, m_2, m_3, n_2, n_3)$, where the "new" triple of frequencies m_1, m_2, m_3 appears as a "child" of the frequency n_1. Putting by $\varphi_1 = \Phi_4(n, n_1, n_2, n_3)$ and $\varphi_2 = \Phi_4(n_1, m_1, m_2, m_3)$, we define the set

(53) $$C_1 = \{|\varphi_1 + \varphi_2| \leqslant 5^3 |\varphi_1|^{1-1/100}\}$$

and split the sum in (52) as

(54) $$\sum_{A_N(n)^c} \sum_{C_1} \ldots + \sum_{A_N(n)^c} \sum_{C_1^c} \ldots = N_{31}^t(v)(n) + N_{32}^t(v)(n).$$

For the N_{32}^t part we have to apply differentiation by parts again which creates the 2nd generation operators. Our first 2nd generation operator $Q_n^{2,t}$ consists of three sums

$$q_{1,n}^{2,t} = \sum_{A_N(n)^c} \sum_{C_1^c} \widetilde{Q}_n^{1,t}(N_1^t(v)(n_1), \overline{v}_{n_2}, v_{n_3}),$$

$$q_{2,n}^{2,t} = \sum_{A_N(n)^c} \sum_{C_1^c} \widetilde{Q}_n^{1,t}(v_{n_1}, \overline{N_1^t(v)}(n_2), v_{n_3}),$$

$$q_{3,n}^{2,t} = \sum_{A_N(n)^c} \sum_{C_1^c} \widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, N_1^t(v)(n_3)).$$

Let us have a look at the first sum $q_{1,n}^{2,t}$ (we treat the other two in a similar manner). Its Fourier transform is equal to

$$\sum_{A_N(n)^c} \sum_{C_1^c} \sigma_n(\xi) \int_{\mathbb{R}^2} \frac{e^{it\Phi_4(\xi, \xi_1, \xi-\xi_1-\xi_3, \xi_3)}}{\Phi_4(\xi, \xi_1, \xi - \xi_1 - \xi_3, \xi_3)}$$
$$\times \mathcal{F}(N_1^t(v)(n_1))(\xi_1)\widehat{\overline{v}}_{n_2}(\xi - \xi_1 - \xi_3)\widehat{v}_{n_3}(\xi_3) \, d\xi_1 \, d\xi_3,$$

where

$$\mathcal{F}(N_1^t(v)(n_1))(\xi_1)$$

equals

$$\sum_{\substack{n_1 \approx m_1 - m_2 + m_3 \\ m_1 \not\approx n_1 \not\approx m_3}} \sigma_{n_1}(\xi_1) \int_{\mathbb{R}^2} e^{it\Phi_4(\xi_1, \xi_1', \xi_1 - \xi_1' - \xi_3', \xi_3')}\widehat{v}_{m_1}(\xi_1')\widehat{\overline{v}}_{m_2}(\xi_1 - \xi_1' - \xi_3')\widehat{v}_{m_3}(\xi_3') \, d\xi_1' \, d\xi_3'.$$

Putting everything together and applying differentiation by parts we can write the integrals inside the sums as

$$\partial_t \left(\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{\mathrm{e}^{-it(\varphi_1+\varphi_2)}}{\varphi_1(\varphi_1+\varphi_2)} \right.$$
$$\left. \times \widehat{v}_{m_1}(\xi_1') \widehat{\overline{v}}_{m_2}(\xi_1 - \xi_1' - \xi_3') \widehat{v}_{m_3}(\xi_3') \widehat{\overline{v}}_{n_2}(\xi - \xi_1 - \xi_3) \widehat{v}_{n_3}(\xi_3) \, \mathrm{d}\xi_1' \, \mathrm{d}\xi_3' \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3 \right)$$

minus

$$\sigma_n(\xi) \int_{\mathbb{R}^4} \sigma_{n_1}(\xi_1) \frac{\mathrm{e}^{-it(\varphi_1+\varphi_2)}}{\varphi_1(\varphi_1+\varphi_2)}$$
$$\times \partial_t(\widehat{v}_{m_1}(\xi_1') \widehat{\overline{v}}_{m_2}(\xi_1 - \xi_1' - \xi_3') \widehat{v}_{m_3}(\xi_3') \widehat{\overline{v}}_{n_2}(\xi - \xi_1 - \xi_3) \widehat{v}_{n_3}(\xi_3)) \, \mathrm{d}\xi_1' \, \mathrm{d}\xi_3' \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3,$$

where $\varphi_1 = \Phi_4(\xi, \xi_1, \xi - \xi_1 - \xi_3, \xi_3)$ and $\varphi_2 = \Phi_4(\xi_1, \xi_1', \xi_1 - \xi_1' - \xi_3', \xi_3')$. Equivalently,

$$(55) \qquad \mathcal{F}(q_{1,n}^{2,t}) = \partial_t(\widetilde{q}_{1,n}^{2,t}) - \mathcal{F}(\tau_{1,n}^{2,t}).$$

Thus, doing the same at the remaining two sums of $Q_n^{2,t}$, namely $q_{2,n}^{2,t}$, $q_{3,n}^{2,t}$, we obtain the splitting

$$(56) \qquad \mathcal{F}(Q_n^{2,t}) = \partial_t \mathcal{F}(\widetilde{Q}_n^{2,t}) - \mathcal{F}(T_n^{2,t}).$$

These new operators $\widetilde{q}_{i,n}^{2,t}$, $i = 1, 2, 3$, act on the following "types" of sequences

$$\widetilde{q}_{1,n}^{2,t}(v_{m_1}, \overline{v}_{m_2}, v_{m_3}, \overline{v}_{n_2}, v_{n_3})$$

with $m_1 - m_2 + m_3 \approx n_1$ and $n_1 - n_2 + n_3 \approx n$,

$$\widetilde{q}_{2,n}^{2,t}(v_{n_1}, \overline{v}_{m_1}, v_{m_2}, \overline{v}_{m_3}, v_{n_3})$$

with $m_1 - m_2 + m_3 \approx n_2$ and $n_1 - n_2 + n_3 \approx n$, and

$$\widetilde{q}_{3,n}^{2,t}(v_{n_1} \overline{v}_{n_2}, v_{m_1}, \overline{v}_{m_2}, v_{m_3})$$

with $m_1 - m_2 + m_3 \approx n_3$ and $n_1 - n_2 + n_3 \approx n$.

At this point let us stop the procedure and present how all these operators can be estimated.

Remark 14. In the following part of the paper a series of lemmata is presented. Unless stated otherwise we always assume that $1 \leqslant q \leqslant 2$.

Lemma 15. *For $j = 1, 2$,*

$$\|R_j^t(v)\|_{l^q L^2} \lesssim \|v\|_{M_{2,q}}^3$$

and

$$\|R_j^t(v) - R_j^t(w)\|_{l^q L^2} \lesssim (\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2)\|v - w\|_{M_{2,q}}.$$

P r o o f. It is the same as the one given in [30], Lemma 10. At exactly this point the requirement $1 \leqslant q \leqslant 2$ is essential. □

Lemma 16.

$$\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{1/2q'+}\|v\|_{M_{2,q}}^3$$

and

$$\|N_{11}^t(v) - N_{11}^t(w)\|_{l^q L^2} \lesssim N^{1/2q'+}(\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2)\|v - w\|_{M_{2,q}}.$$

P r o o f. The proof is similar to [30], Lemma 11 but with a small twist.

Obviously,

$$\|N_{11}^t(v)\|_{L^2} \leqslant \sum_{A_N(n)} \|Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})\|_{L^2},$$

which from (19), Lemma 12 and Hölder's inequality is estimated from above by

$$\sum_{A_N(n)} \|u_{n_1}\overline{u}_{n_2}u_{n_3}\|_{L^2} \leqslant \sum_{A_N(n)} \|u_{n_1}\|_{L^6}\|u_{n_2}\|_{L^6}\|u_{n_3}\|_{L^6}.$$

Here we make use of (20) and Hölder's inequality in the discrete variable to obtain the upper bound

$$\sum_{A_N(n)} \|u_{n_1}\|_{L^2}\|u_{n_2}\|_{L^2}\|u_{n_3}\|_{L^2} \leqslant \left(\sum_{A_N(n)} 1^{q'}\right)^{1/q'} \left(\sum_{A_N(n)} \|u_{n_1}\|_{L^2}^q\|u_{n_2}\|_{L^2}^q\|u_{n_3}\|_{L^2}^q\right)^{1/q}$$

$$= [\#(A_N(n))]^{1/q'} \left(\sum_{A_N(n)} \|u_{n_1}\|_{L^2}^q\|u_{n_2}\|_{L^2}^q\|u_{n_3}\|_{L^2}^q\right)^{1/q}.$$

Observe that from (29) we have the inclusion

$$A_N(n) \subset \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 - n_2 + n_3 \approx n, n_1 \not\approx n \not\approx n_3, |\Phi_2(n, n_1, n_2, n_3)| \leqslant N^{1/2}\}$$

and from the proof of [30], Lemma 11 we have that the cardinality of this last set is $o(N^{(1/2)+})$. Thus, we have

$$\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{1/2q'+}\left(\sum_{n \in \mathbb{Z}} \sum_{A_N(n)} \|u_{n_1}\|_{L^2}^q\|u_{n_2}\|_{L^2}^q\|u_{n_3}\|_{L^2}^q\right)^{1/q}$$

and this final summation is estimated by applying Young's inequality in $l^1(\mathbb{Z})$ providing us with the bound ($\|u\|_{M_{2,q}} = \|v\|_{M_{2,q}}$) $\|N_{11}^t(v)\|_{l^q L^2} \lesssim N^{1/2q'+}\|v\|_{M_{2,q}}^3$. □

Lemma 17.

$$(57) \qquad \|\widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})\|_2 \lesssim \frac{\|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3)|}.$$

P r o o f. It is the same as the one given in [30], Lemma 12. It is simply a duality argument that uses the localization of Fourier transforms of the functions v_{n_1}, v_{n_2} and v_{n_3}. The denominator turns out to be the absolute value of

$$\Phi_4(n_1 - n_2 + n_3, n_1, -n_2, n_3) = \Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3).$$

$\square$

Remark 18. Notice that Lemma 17 (this observation applies to Lemma 23, too) is true for any triple of functions $f, g, h \in M_{2,q}(\mathbb{R})$ and the only important property is that they are nicely localised on the Fourier side since we consider their box operators $\square_{n_1} f$, $\square_{n_2} g$ and $\square_{n_3} h$. Also, the same proof implies that the operator $Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})$ satisfies the estimate

$$(58) \qquad \|Q_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})\|_2 \lesssim \|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2.$$

These observations play an important role later on.

Lemma 19.
$$\|N_{21}^t(v)\|_{l^q L^2} \lesssim N^{1/q'-1} \|v\|_{M_{2,q}}^3$$

and
$$\|N_{21}^t(v) - N_{21}^t(w)\|_{l^q L^2} \lesssim N^{1/q'-1}(\|v\|_{M_{2,q}}^2 + \|w\|_{M_{2,q}}^2)\|v - w\|_{M_{2,q}}.$$

P r o o f. From Lemma 17 we have

$$\|N_{21}^t(v)\|_2 \leqslant \sum_{A_N(n)^c} \|\widetilde{Q}_n^{1,t}(v_{n_1}, \overline{v}_{n_2}, v_{n_3})\|_2 \lesssim \sum_{A_N(n)^c} \frac{\|v_{n_1}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3)|}$$

and by Hölder's inequality the upper bound

$$(59) \qquad \left(\sum_{A_N(n)^c} \frac{1}{|\Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3)|^{q'}} \right)^{1/q'} \left(\sum_{A_N(n)^c} \|v_{n_1}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{1/q}$$

$$(60) \qquad \sim \left(\sum_{A_N(n)^c} \frac{1}{(|n - n_1||n - n_3|)^{q'} n_{\max}^{2q'}} \right)^{1/q'} \left(\sum_{A_N(n)^c} \|v_{n_1}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{1/q},$$

where (29) was used and $n_{\max} := \max\{|n|, |n_1|, |n_2|, |n_3|\}$. The first sum of (60) behaves like $N^{1/q'-1}$ and to the remaining part we apply Young's inequality. $\square$

Lemma 20.

$$\|N_4^t(v)\|_{l^q L^2} \lesssim N^{1/q'-1}\|v\|_{M_{2,q}}^5$$

and

$$\|N_4^t(v) - N_4^t(w)\|_{l^q L^2} \lesssim N^{1/q'-1}(\|v\|_{M_{2,q}}^4 + \|w\|_{M_{2,q}}^4)\|v - w\|_{M_{2,q}}.$$

P r o o f. We repeat the proof of Lemma 19 and apply Lemma 15 to the part $R_2^t(v)(n_1) - R_1^t(v)(n_1)$. □

Lemma 21.

$$\|N_{31}^t(v)\|_{l^q L^2} \lesssim \|v\|_{M_{2,q}}^5$$

and

$$\|N_{31}^t(v) - N_{31}^t(w)\|_{l^q L^2} \lesssim (\|v\|_{M_{2,q}}^4 + \|w\|_{M_{2,q}}^4)\|v - w\|_{M_{2,q}}.$$

P r o o f. Using of Lemma 17, Remark 18 and Hölder's inequality we have

$$(61) \quad \|N_{31}^t(v)\|_2 \leqslant \sum_{A_N(n)^c} \sum_{C_1} \|\widetilde{Q}_n^{1,t}(Q_{n_1}^{1,t}(v_{m_1}, \overline{v}_{m_2}, v_{m_3}), \overline{v}_{n_2}, v_{n_3})\|_2$$

$$\lesssim \sum_{A_N(n)^c} \sum_{C_1} \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3)|}$$

$$\leqslant \left(\sum_{A_N(n)^c} \sum_{C_1} \frac{1}{|\varphi_1|^{q'}} \right)^{1/q'} \left(\sum_{A_N(n)^c} \sum_{C_1} \|v_{m_1}\|_2^q \|v_{m_2}\|_2^q \|v_{m_3}\|_2^q \|v_{n_2}\|_2^q \|v_{n_3}\|_2^q \right)^{1/q}.$$

At $q' \geqslant 2$, the first sum of (61) is controlled by the series

$$(62) \quad \left(\sum_{A_N(n)^c} \sum_{C_1} \frac{1}{|\varphi_1|^2} \right)^{1/2}.$$

Observe that by the definition of the set C_1 in (53) we have that

$$|\varphi_2| := |\Phi_4(n_1, m_1, m_2, m_3)| \sim |\varphi_1|.$$

Since $|\mu_j| \lesssim (n_{\max}^{(j)})^2$ for $j = 1, 2$, where

$$n_{\max}^{(1)} = \max\{|n|, |n_1|, |n_2|, |n_3|\}, \quad n_{\max}^{(2)} = \max\{|n_1|, |m_1|, |m_2|, |m_3|\},$$

by setting $\mu_1 = \Phi_2(n, n_1, n_2, n_3)$, $\mu_2 = \Phi_2(n_1, m_1, m_2, m_3)$ we may estimate (62) further by the expression

$$\left(\sum_{A_N(n)^c} \sum_{C_1} \frac{1}{|\mu_1 \mu_2|(n_{\max}^{(1)} n_{\max}^{(2)})^2} \right)^{1/2} \lesssim \left(\sum_{A_N(n)^c} \sum_{C_1} \frac{1}{|\mu_1 \mu_2|^{1+}} \right)^{1/2} \lesssim 1.$$

Hence, Young's inequality applied to the second sum of (61) finishes the proof. □

The following lemma should be compared to Lemma 17.

Lemma 22.

$$(63) \qquad \|\tilde{q}_{1,n}^{2,t}(v_{m_1}, \overline{v}_{m_2}, v_{m_3}, \overline{v}_{n_2}, v_{n_3})\|_2 \lesssim \frac{\|v_{m_1}\|_2 \|v_{m_2}\|_2 \|v_{m_3}\|_2 \|v_{n_2}\|_2 \|v_{n_3}\|_2}{|\varphi_1||\varphi_1 + \varphi_2|},$$

where $\varphi_1 = \Phi_4(n_1 - n_2, n_3, n_1, n_2, n_3)$ and $\varphi_2 = \Phi_4(m_1 - m_2 + m_3, m_1, m_2, m_3)$.

P r o o f. Similar to that of Lemma 17 and [30], Lemma 17. $\qquad \square$

Having described the first steps of the iteration process it is time to introduce the correct vocabulary in order to be able to express much more complicated operators. This is done in the next section with the use of a suitable tree notation (see [5] for a more sophisticated version).

4. Tree notation and induction step

The trees used here are similar to the ones described in [17] and are exactly the same as the ones used in [30], the only difference being the phase factors μ_j, described in [30], Equation (60) which we replace here by quantities of the form $\Phi_4(n_1 - n_2 + n_3, n_1, n_2, n_3)$. Since this is the heart of the argument and since Lemmata 24 and 26 have different proofs than the corresponding ones from [30], Lemmata 22 and 23, we describe the whole procedure again.

A tree T is a finite, partially ordered set with the following properties:

▷ For any $a_1, a_2, a_3, a_4 \in T$, if $a_4 \leqslant a_2 \leqslant a_1$ and $a_4 \leqslant a_3 \leqslant a_1$ then $a_2 \leqslant a_3$ or $a_3 \leqslant a_2$.

▷ There exists a maximum element $r \in T$, that is $a \leqslant r$ for all $a \in T$, which is called the *root*.

We call the elements of T the nodes of the tree and in this context we say that $b \in T$ is a child of $a \in T$ (or, equivalently, that a is the parent of b) if $b \leqslant a$, $b \neq a$ and for all $c \in T$ such that $b \leqslant c \leqslant a$ we have either $b = c$ or $c = a$.

A node $a \in T$ is called *terminal* if it has no children. A nonterminal node $a \in T$ is a node with exactly 3 children a_1, the left child, a_2, the middle child, and a_3, the right child. We define the sets

$$(64) \qquad\qquad T^0 = \{\text{all nonterminal nodes}\},$$

and

$$(65) \qquad\qquad T^\infty = \{\text{all terminal nodes}\}.$$

Obviously, $T = T^0 \cup T^\infty$, $T^0 \cap T^\infty = \emptyset$ and if $|T^0| = j \in \mathbb{Z}_+$ we have $|T| = 3j + 1$ and $|T^\infty| = 2j + 1$. We denote the collection of trees with j parental nodes by

(66) $$T(j) = \{T \text{ is a tree with } |T| = 3j + 1\}.$$

Next, we say that a sequence of trees $\{T_j\}_{j=1}^J$ is a chronicle of J generations if:

▷ $T_j \in T(j)$ for all $j = 1, 2, \ldots, J$.
▷ For all $j = 1, 2, \ldots, J-1$, the tree T_{j+1} is obtained by changing one of the terminal nodes of T_j into a nonterminal node with exactly 3 children.

Let us also denote by $\mathcal{I}(J)$ the collection of trees of the Jth generation. It is easily checked by an induction argument that

(67) $$|\mathcal{I}(J)| = 1 \cdot 3 \cdot 5 \ldots (2J - 1) =: (2J - 1)!!.$$

Given a chronicle $\{T_j\}_{j=1}^J$ of J generations we refer to T_J as an ordered tree of the Jth generation. We should keep in mind that the notion of ordered trees comes with associated chronicles. It includes not only the shape of the tree but also how it "grew".

Given an ordered tree T we define an index function $n \colon T \to \mathbb{Z}$ such that

▷ $n_a \approx n_{a_1} - n_{a_2} + n_{a_3}$ for all $a \in T^0$, where a_1, a_2, a_3 are the children of a,
▷ $n_a \not\approx n_{a_1}$ and $n_a \not\approx n_{a_3}$ for all $a \in T^0$,
▷ $|\varphi_1| := |\Phi_4(n_{r_1} - n_{r_2} + n_{r_3}, n_{r_1}, n_{r_2}, n_{r_3})| > N$, where r is the root of T,

and we denote the collection of all such index functions by $\mathcal{R}(T)$.

Given an ordered tree T with the chronicle $\{T_j\}_{j=1}^J$ and associated index functions $n \in \mathcal{R}(T)$, we need to keep track of the generations of frequencies. Fix an $n \in \mathcal{R}(T)$ and consider the very first tree T_1. Its nodes are the root r and its children r_1, r_2, r_3. We define the first generation of frequencies by

$$(n^{(1)}, n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) := (n_r, n_{r_1}, n_{r_2}, n_{r_3}).$$

From the definition of the index function we have

$$n^{(1)} \approx n_1^{(1)} - n_2^{(1)} + n_3^{(1)}, \quad n_1^{(1)} \not\approx n^{(1)} \not\approx n_3^{(1)}.$$

The ordered tree T_2 of the second generation is obtained from T_1 by changing one of its terminal nodes $a = r_k \in T_1^\infty$ for some $k \in \{1, 2, 3\}$ into a nonterminal node. Then, the second generation of frequencies is defined by

$$(n^{(2)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3}).$$

Thus, we have $n^{(2)} = n_k^{(1)}$ for some $k \in \{1, 2, 3\}$ and from the definition of the index function we have

$$n^{(2)} \approx n_1^{(2)} - n_2^{(2)} + n_3^{(2)}, \quad n_1^{(2)} \not\approx n^{(2)} \not\approx n_3^{(2)}.$$

After $j-1$ steps, the ordered tree T_j of the jth generation is obtained from T_{j-1} by changing one of its terminal nodes $a \in T_{j-1}^\infty$ into a nonterminal node. Then, the jth generation frequencies are defined as

$$(n^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)}) := (n_a, n_{a_1}, n_{a_2}, n_{a_3})$$

and we have $n^{(j)} = n_k^{(m)} (= n_a)$ for some $m \in \{1, 2, \ldots, j-1\}$ and $k \in \{1, 2, 3\}$, since this corresponds to the frequency of some terminal node in T_{j-1}. In addition, from the definition of the index function we have

$$n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, \quad n_1^{(j)} \napprox n^{(j)} \napprox n_3^{(j)}.$$

Finally, we use φ_j to denote the corresponding phase factor introduced at the jth generation. That is,

$$(68) \qquad \varphi_j = \Phi_4(n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)})$$

and we also introduce the quantities

$$(69) \qquad \widetilde{\varphi}_J = \sum_{j=1}^{J} \varphi_j, \quad \widehat{\varphi}_J = \prod_{j=1}^{J} \widetilde{\varphi}_j.$$

Notice that for $\mu_j = \Phi_2(n_1^{(j)} - n_2^{(j)} + n_3^{(j)}, n_1^{(j)}, n_2^{(j)}, n_3^{(j)})$ we have the relation

$$(70) \qquad |\varphi_j| \sim (n_{\max}^{(j)})^2 |\mu_j| \gtrsim |\mu_j|^2,$$

where we put $n_{\max}^{(j)} := \max\{|n^{(j)}|, |n_1^{(j)}|, |n_2^{(j)}|, |n_3^{(j)}|\}$.

We should keep in mind that everytime we apply differentiation by parts and split into resonant and non-resonant parts, we need to control the new frequencies that arise from this procedure. For this reason we introduce the sets

$$(71) \quad C_J := \{|\widetilde{\varphi}_{J+1}| \leqslant (2J+3)^3 |\widetilde{\varphi}_J|^{1-1/100}\} \cup \{|\widetilde{\varphi}_{J+1}| \leqslant (2J+3)^3 |\varphi_1|^{1-1/100}\}.$$

Let us denote by T_α all the nodes of the ordered tree T that are descendants of the node $\alpha \in T^0$, i.e., $T_\alpha = \{\beta \in T : \beta \leqslant \alpha, \beta \neq \alpha\}$.

We also need to define the principal and final "signs" of a node $a \in T$ which are functions from the tree T into the set $\{\pm 1\}$:

$$(72) \qquad \mathrm{psgn}(a) = \begin{cases} +1, & a \text{ is not the middle child of his parent,} \\ +1, & a = r, \text{ the root node,} \\ -1, & a \text{ is the middle child of his parent,} \end{cases}$$

$$
(73) \qquad \mathrm{fsgn}(a) = \begin{cases} +1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an even number of middle} \\ & \text{predecessors,} \\ -1, & \mathrm{psgn}(a) = +1 \text{ and } a \text{ has an odd number of middle} \\ & \text{predecessors,} \\ -1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an even number of middle} \\ & \text{predecessors,} \\ +1, & \mathrm{psgn}(a) = -1 \text{ and } a \text{ has an odd number of middle} \\ & \text{predecessors,} \end{cases}
$$

where the root node $r \in T$ is not considered a middle parent.

In the general Jth step we have to deal with $|\mathcal{I}(J)|$ operators of the $\widetilde{q}_{T^0,\mathbf{n}}^{J,t}$ "type" each one corresponding to one of the ordered trees of the Jth generation, $T \in T(J)$, where $\mathbf{n}$ is an arbitrary fixed index function on T. We have the subindices T^0 and $\mathbf{n}$ because each one of these operators has its Fourier transform supported on the cubes with centers being the frequencies assigned to the nodes that belong to T^0.

The operators $\widetilde{q}_{T^0,\mathbf{n}}^{J,t}$ are defined through their Fourier transforms as

$$
(74) \qquad \mathcal{F}(\widetilde{q}_{T^0,\mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty}))(\xi) = \mathrm{e}^{-\mathrm{i}t\xi^4} \mathcal{F}(R_{T^0,\mathbf{n}}^{J,t}(\{\mathrm{e}^{-\mathrm{i}t\partial_x^4} w_{n_\beta}\}_{\beta \in T^\infty}))(\xi),
$$

where the operator $R_{T^0,\mathbf{n}}^{J,t}$ acts on the functions $\{w_{n_\beta}\}_{\beta \in T^\infty}$ as

$$
(75) \quad R_{T^0,\mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty})(x) = \int_{\mathbb{R}^{2J+1}} K_{T^0}^{(J)}(x, \{x_\beta\}_{\beta \in T^\infty}) \Big[\bigotimes_{\beta \in T^\infty} w_{n_\beta}(x_\beta) \Big] \prod_{\beta \in T^\infty} \mathrm{d}x_\beta
$$

and the kernel $K_{T^0,\mathbf{n}}^{(J)}$ is defined as

$$
(76) \qquad K_{T^0,\mathbf{n}}^{(J)}(x, \{x_\beta\}_{\beta \in T^\infty}) = \mathcal{F}^{-1}(\varrho_{T^0,\mathbf{n}}^{(J)})(\{x - x_\beta\}_{\beta \in T^\infty}).
$$

The formula for the function $\varrho_{T^0,\mathbf{n}}^{(J)}$ with $(|T^\infty| = 2J+1)$-variables ξ_β, $\beta \in T^\infty$, is

$$
(77) \qquad \varrho_{T^0,\mathbf{n}}^{(J)}(\{\xi_\beta\}_{\beta \in T^\infty}) = \Big[\prod_{\alpha \in T^0} \sigma_{n_\alpha} \Big(\sum_{\beta \in T^\infty \cap T_\alpha} \mathrm{fsgn}(\beta)\xi_\beta \Big) \Big] \frac{1}{\widehat{\varphi}_T},
$$

where we put

$$
(78) \qquad \widehat{\varphi}_T = \prod_{\alpha \in T^0} \widetilde{\varphi}_\alpha, \quad \widetilde{\varphi}_\alpha = \sum_{\beta \in T^0 \setminus T_\alpha} \varphi_\beta,
$$

and for $\beta \in T^0$ we have

$$
(79) \qquad \varphi_\beta = \Phi_4(\xi_{\beta_1} - \xi_{\beta_2} + \xi_{\beta_3}, \xi_{\beta_1}, \xi_{\beta_2}, \xi_{\beta_3}),
$$

where we impose the relation $\xi_\alpha = \xi_{\alpha_1} - \xi_{\alpha_2} + \xi_{\alpha_3}$ for every $\alpha \in T^0$ that appears in the calculations until we reach the terminal nodes of T^∞. This is because in the definition

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of the function $\varrho_{T^0}^{J,t}$ we need the variables "ξ" to be assigned only at the terminal nodes of the tree T. We use the notation φ_β in similarity to φ_j of equation (68) because this is the "continuous" version of the discrete quantity. In addition, the variables ξ_{α_1}, ξ_{α_2}, ξ_{α_3} that appear in expression (77) are such that $\xi_{\alpha_1} \approx n_{\alpha_1}$, $\xi_{\alpha_2} \approx n_{\alpha_2}$, $\xi_{\alpha_3} \approx n_{\alpha_3}$ since the functions σ_{n_α} are supported in such a way. Therefore, $|\widehat{\varphi}_T| \sim |\widehat{\varphi}_J|$.

For the induction step of our iteration process we need the following lemma which should be compared to Lemmata 17 and 22.

Lemma 23.

$$\tag{80} \|\widetilde{q}_{T^0,\mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty})\|_2 \lesssim \left(\prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2 \right) \frac{1}{|\widehat{\varphi}_T|}$$

for every tree $T \in T(J)$ and index function $\mathbf{n} \in \mathcal{R}(T)$.

P r o o f. We use (74), (75) and duality: For $g \in L^2(\mathbb{R})$ we have

$$|\langle R_{T^0,\mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty}), g\rangle| = \left| \int_\mathbb{R} \mathcal{F}(R_{T^0,\mathbf{n}}^{J,t}(\{w_{n_\beta}\}_{\beta \in T^\infty}))(\xi)\mathcal{F}(g)(\xi)\, \mathrm{d}\xi \right|$$

$$= \left| \int_{\mathbb{R}^{2J+1}} \varrho_{T,\mathbf{n}}^{(J)}\left(\xi_{\beta_1}, \dots, \xi_{\beta_{2J}}, \xi - \sum_{k=1}^{2J} \xi_{\beta_k} \right) \right.$$

$$\left. \times \prod_{k=1}^{2J} \widehat{w}_{n_{\beta_k}}(\xi_{\beta_k})\widehat{w}_{n_{\beta_{2J+1}}}\left(\xi - \sum_{k=1}^{2J} \xi_{\beta_k} \right)\widehat{g}(\xi) \prod_{k=1}^{2J} \mathrm{d}\xi_{\beta_k}\, \mathrm{d}\xi \right|.$$

The last expression equals

$$\left| \int_{\mathbb{R}^{2J+1}} \varrho_{T,\mathbf{n}}^{(J)}(\xi_{\beta_1}, \dots, \xi_{\beta_{2J}}, \eta) \prod_{k=1}^{2J} \widehat{w}_{n_{\beta_k}}(\xi_{\beta_k})\widehat{w}_{n_{\beta_{2J+1}}}(\eta)\widehat{g}\left(\eta + \sum_{k=1}^{2J} \xi_{\beta_k} \right) \prod_{k=1}^{2J} \mathrm{d}\xi_{\beta_k}\, \mathrm{d}\eta \right|$$

$$= \left| \int_{I_{n_{\beta_1}}} \cdots \int_{I_{n_{\beta_{2J+1}}}} \varrho_{T,\mathbf{n}}^{(J)}(\xi_{\beta_1}, \dots, \xi_{\beta_{2J}}, \eta) \right.$$

$$\left. \times \prod_{k=1}^{2J} \widehat{w}_{n_{\beta_k}}(\xi_{\beta_k})\widehat{w}_{n_{\beta_{2J+1}}}(\eta)\widehat{g}\left(\eta + \sum_{k=1}^{2J} \xi_{\beta_k} \right) \prod_{k=1}^{2J} \mathrm{d}\xi_{\beta_k}\, \mathrm{d}\eta \right|,$$

where the intervals $I_{n_{\beta_1}}, \dots, I_{n_{\beta_{2J+1}}}$ contain the compact supports of the functions $\widehat{w}_{n_{\beta_1}}, \dots, \widehat{w}_{n_{\beta_{2J+1}}}$. By Hölder's inequality we obtain the upper bound

$$\|\varrho_{T,\mathbf{n}}^{(J)}\|_\infty \prod_{k=1}^{2J+1} \|w_{n_{\beta_k}}\|_2 \left(\int_{I_{n_{\beta_1}}} \cdots \int_{I_{n_{\beta_{2J+1}}}} \left| \widehat{g}(\eta + \sum_{k=1}^{2J} \xi_{\beta_k}) \right|^2 \prod_{k=1}^{2J} \mathrm{d}\xi_{\beta_k}\, \mathrm{d}\eta \right)^{1/2}$$

$$= \|\varrho_{T,\mathbf{n}}^{(J)}\|_\infty \prod_{k=1}^{2J+1} \|w_{n_{\beta_k}}\|_2 \|g\|_2 \prod_{k=1}^{2J} |I_{n_{\beta_k}}|^{1/2},$$

which finishes the proof. $\square$

Given an index function $\mathbf{n}$, $2J+1$ functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ and $\alpha \in T^\infty$, we define the action of the operator N_1^t (see (37)) on the set $\{v_{n_\beta}\}_{\beta \in T^\infty}$ to be the same set as before but with the difference that we substituted for the function v_{n_α} the new function $N_1^t(v)(n_\alpha)$. We denote this new set of functions by $N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})$. Similarly, the action of the operator $R_2^t - R_1^t$ (see (36)) on the set of functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ is denoted by $(R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta \in T^\infty})$.

The operator of the Jth step, $J \geqslant 2$, that we want to estimate, is given by the formula

$$(81) \qquad N_2^{(J)}(v)(n) := \sum_{T \in T(J-1)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \widetilde{q}_{T^0}^{J-1,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})).$$

In the following will you keep in mind that from the splitting procedure we are on the sets $A_N(n)^c, C_1^c, \ldots, C_{J-1}^c$ and since $|\varphi_1| > N$, we trivially have for all $j \in \{2, \ldots, J\}$

$$(82) \qquad |\widetilde{\varphi}_j| \gg (2j+3)^3 \max\{|\widetilde{\varphi}_{j-1}|^{1-1/100}, |\varphi_1|^{1-1/100}\} > (2j+3)^3 N^{1-1/100}.$$

Applying differentiation by parts on the Fourier side we obtain the expression

$$(83) \qquad N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + N_r^{(J+1)}(v)(n) + N^{(J+1)}(v)(n),$$

where

$$(84) \quad N_0^{(J+1)}(v)(n) := \sum_{T \in T(J)} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \widetilde{q}_{T^0,\mathbf{n}}^{J,t}(\{v_{n_\beta}\}_{\beta \in T^\infty}),$$

$$(85) \quad N_r^{(J+1)}(v)(n) := \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \widetilde{q}_{T^0,\mathbf{n}}^{J,t}((R_2^{t,\alpha} - R_1^{t,\alpha})(\{v_{n_\beta}\}_{\beta \in T^\infty})),$$

and

$$(86) \quad N^{(J+1)}(v)(n) := \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \widetilde{q}_{T^0,\mathbf{n}}^{J,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta \in T^\infty})).$$

We also split the operator $N^{(J+1)}$ as the sum

$$(87) \qquad\qquad\qquad N^{(J+1)} = N_1^{(J+1)} + N_2^{(J+1)},$$

where $N_1^{(J+1)}$ is the restriction of $N^{(J+1)}$ onto C_J and $N_2^{(J+1)}$ onto C_J^c.

First we estimate the operators $N_0^{(J+1)}$ and $N_r^{(J+1)}$ by the following lemma.

Lemma 24.
$$\|N_0^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-(J/2)(1-1/100)}\|v\|_{M_{2,q}}^{2J+1}$$

and

$$\|N_0^{(J+1)}(v) - N_0^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-(J/2)(1-1/100)}(\|v\|_{M_{2,q}}^{2J} + \|w\|_{M_{2,q}}^{2J})\|v-w\|_{M_{2,q}}.$$

Further,

$$\|N_r^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-(J/2)(1-1/100)}\|v\|_{M_{2,q}}^{2J+3}$$

and

$$\|N_r^{(J+1)}(v) - N_r^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-(J/2)(1-1/100)}(\|v\|_{M_{2,q}}^{2J+2} + \|w\|_{M_{2,q}}^{2J+2})\|v-w\|_{M_{2,q}}.$$

P r o o f. From (78) we have that $|\varphi_j| \lesssim \max\{|\widetilde{\varphi}_{j-1}|, |\widetilde{\varphi}_j|\}$ which together with (82) implies

$$(88) \qquad (2j)^3 N^{1-1/100}|\varphi_j| \ll |\widetilde{\varphi}_{j-1}||\widetilde{\varphi}_j| \quad \forall\, j \in \{2, \ldots, J\}.$$

This together with the use of (82) again shows that

$$(89) \qquad \prod_{j=1}^J [(2j+3)^3 N^{1-1/100}|\varphi_j|] \ll |\varphi_1||\widetilde{\varphi}_J| \prod_{j=2}^J [(2j)^3 N^{1-1/100}|\varphi_j|] \ll \prod_{j=1}^J |\widetilde{\varphi}_j|^2.$$

Recalling that

$$(90) \qquad \frac{1}{|\varphi_j|} \sim \frac{1}{|\mu_j|(n_{\max}^{(j)})^2|} \lesssim \frac{1}{|\mu_j|^{1+}},$$

we obtain

$$(91) \qquad \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \prod_{j=1}^J \frac{1}{|\widetilde{\varphi}_j|^2} \lesssim \frac{N^{-J(1-1/100)}}{\prod_{j=1}^J (2j+3)^3} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \prod_{j=1}^J \frac{1}{|\varphi_j|}$$

$$\lesssim \frac{N^{-J(1-1/100)}}{\prod_{j=1}^J (2j+3)^3} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T)\\ \mathbf{n}_r=n}} \prod_{j=1}^J \frac{1}{|\mu_j|^{1+}}$$

where the last expression is bounded from above by

$$(92) \qquad \frac{C^J N^{-J(1-1/100)}}{\prod_{j=1}^J (2j+3)^3}$$

for some constant $C > 0$.

By Lemma 23 we obtain

$$(93) \qquad \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\widetilde{q}_{T^0,\mathbf{n}}^{J,t}(\{v_\beta\}_{\beta \in T^\infty})\|_2 \lesssim \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \left(\prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2 \right) \left(\prod_{k=1}^{J} \frac{1}{|\widetilde{\varphi}_k|} \right)$$

which by Hölder's inequality and (91), (92) is controlled by

$$(94) \qquad \left(\sum_{\substack{|\varphi_1| > N \\ |\widetilde{\varphi}_j| > (2j+1)^3 N^{1-1/100} \\ j=2,\ldots,J}} \prod_{j=1}^{J} \frac{1}{|\widetilde{\varphi}_j|^{q'}} \right)^{1/q'} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2^q \right)^{1/q}$$

$$\leqslant \left(\sum_{\substack{|\varphi_1| > N \\ |\widetilde{\varphi}_j| > (2j+1)^3 N^{1-1/100} \\ j=2,\ldots,J}} \prod_{j=1}^{J} \frac{1}{|\widetilde{\varphi}_j|^2} \right)^{1/2} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2^q \right)^{1/q}$$

$$\lesssim \frac{C^{J/2} N^{-J/2(1-1/100)}}{\prod_{j=1}^{J} (2j+3)^{3/2}} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty} \|v_{n_\beta}\|_2^q \right)^{1/q}.$$

Applying Young's inequality for the last summand implies the desired estimate. For the operator $N_r^{(J+1)}$ the proof is the same but in addition we use Lemma 15 for the operator $R_2^t - R_1^t$. $\qquad \square$

Remark 25. Note that there is an extra factor $\sim J$ when we estimate the differences $N_0^{(J+1)}(v) - N_0^{(J+1)}(w)$ since

$$|a^{2J+1} - b^{2J+1}| \lesssim \left(\sum_{j=1}^{2J+1} a^{2J+1-j} b^{j-1} \right) |a - b|$$

has $O(J)$ many terms. Also, we have $c_J = |\mathcal{I}(J)|$ many summands in the operator $N_0^{(J+1)}$ since there are c_J many trees of the Jth generation and c_J behaves like a double factorial in J, see (67). However, these extra terms do not cause any problem since the constant we obtain from (94) decays like a fractional power of a double factorial in J or, to be more precise, we have

$$(95) \qquad \frac{C^{J/2} \, c_J}{\prod_{j=2}^{J} (2j+3)^{3/2}} \sim \frac{1}{J^{J/2}}.$$

Here is the estimate of the operator $N_1^{(J+1)}$.

Lemma 26.

$$\|N_1^{(J+1)}(v)\|_{l^q L^2} \lesssim N^{-((J-1)/2)(1-1/100)}\|v\|_{M_{2,q}}^{2J+3}$$

and

$$\|N_1^{(J+1)}(v) - N_1^{(J+1)}(w)\|_{l^q L^2} \lesssim N^{-((J-1)/2)(1-1/100)}(\|v\|_{M_{2,q}}^{2J+2} + \|w\|_{M_{2,q}}^{2J+2})\|v-w\|_{M_{2,q}}.$$

P r o o f. Since we are on C_J the requirement

$$|\widetilde{\varphi}_{J+1}| = |\widetilde{\varphi}_J + \varphi_{J+1}| \lesssim (2J+3)^3 |\widetilde{\varphi}_J|^{1-1/100}$$

(similar considerations for the requirement $|\widetilde{\varphi}_{J+1}| \lesssim (2J+3)^3 |\varphi_1|^{1-1/100}$) implies that $|\varphi_{J+1}| \lesssim J^3 |\widetilde{\varphi}_J|$. With the use of (88) we obtain

$$(96) \qquad |\varphi_1||\varphi_{J+1}| \prod_{j=2}^{J}(2j+3)^3 N^{1-1/100}|\varphi_j| \lesssim J^3 \prod_{j=1}^{J}|\widetilde{\varphi}_j|^2.$$

Following the argument in (91) and (92) with the use of (90) we arrive at

$$(97) \qquad \sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \prod_{j=1}^{J}\frac{1}{|\widetilde{\varphi}_j|^2} \lesssim \frac{N^{-(J-1)(1-1/100)}}{\prod_{j=2}^{J-1}(2j+3)^3} \sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \prod_{j=1}^{J+1}\frac{1}{|\varphi_j|}$$

$$\lesssim \frac{C^{J+1}N^{-(J-1)(1-1/100)}}{\prod_{j=2}^{J-1}(2j+3)^3},$$

which finishes the proof since trivially

$$\sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}} \|\widetilde{q}_{T^0,\mathbf{n}}^{J,t}(N_1^{t,\alpha}(\{v_{n_\beta}\}_{\beta\in T^\infty}))\|_2$$

$$\lesssim \sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}}\left(\|v_{n_{\alpha_1}}\|_2\|v_{n_{\alpha_2}}\|_2\|v_{n_{\alpha_3}}\|_2 \prod_{\beta\in T^\infty\setminus\{\alpha\}}\|v_{n_\beta}\|_2\right)$$

$$\times\left(\prod_{k=1}^{J}\frac{1}{|\widetilde{\varphi}_k|}\right)\left(\sum_{\substack{|\varphi_1|>N \\ |\widetilde{\varphi}_j|>(2j+1)^3 N^{1-1/100} \\ j=2,\ldots,J}} \prod_{j=1}^{J}\frac{1}{|\widetilde{\varphi}_j|^{q'}}\right)^{1/q'}$$

$$\times\left(\sum_{\substack{\mathbf{n}\in\mathcal{R}(T) \\ \mathbf{n}_r=n}}\|v_{n_{\alpha_1}}\|_2^q\|v_{n_{\alpha_2}}\|_2^q\|v_{n_{\alpha_3}}\|_2^q \prod_{\beta\in T^\infty\setminus\{\alpha\}}\|v_{n_\beta}\|_2^q\right)^{1/q}.$$

The first summand is controlled by (97) and to the second summand we apply Young's inequality. $\qquad\square$

Remark 27. For $s > 0$ we observe that all the previous lemmata hold true if we replace the $l^q L^2$ norm by the $l_s^q L^2$ norm and the $M_{2,q}(\mathbb{R})$ norm by the $M_{2,q}^s(\mathbb{R})$ norm. To this end, notice that for large $n^{(j)}$ there exists at least one of $n_1^{(j)}$, $n_2^{(j)}$, $n_3^{(j)}$ such that $|n_k^{(j)}| \geqslant \frac{1}{3}|n^{(j)}|$, $k \in \{1, 2, 3\}$, since we have the relation $n^{(j)} \approx n_1^{(j)} - n_2^{(j)} + n_3^{(j)}$. Thus, in the estimates of the Jth generation, there exists at least one frequency $n_k^{(j)}$ for some $j \in \{1, \dots, J\}$ with the property

$$\langle n \rangle^s \leqslant 3^{js} \langle n_k^{(j)} \rangle^s \leqslant 3^{Js} \langle n_k^{(j)} \rangle^s.$$

This exponential growth does not affect our calculations due to the double factorial growth in the denominator of (95).

5. Existence and uniqueness

The calculations of this section are the same as the ones given in [30], Subsections 2.3 and 2.4. The only small difference is the following lemma which deals with the behaviour of the remainder operator N_2^J as $J \to \infty$.

Lemma 28. For $v \in M_{2,q}^s(\mathbb{R})$ with $s \geqslant 0$ and $q \in [1, 2]$, if $M_{2,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ then we have

$$\lim_{J \to \infty} \|N_2^{(J)}(v)\|_{l^\infty L^2} = 0.$$

P r o o f. By (83) we can write the remainder operator as the sum

$$(98) \quad N_2^{(J)}(v)(n) = \partial_t(N_0^{(J+1)}(v)(n)) + \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \widetilde{q}_{T^0,\mathbf{n}}^{J,t}(\partial_t^{(\alpha)}(\{v_{n_\beta}\}_{\beta \in T^\infty})),$$

where we define the action of $\partial_t^{(\alpha)}$ onto the set of functions $\{v_{n_\beta}\}_{\beta \in T^\infty}$ to be the same set of functions except for the α node, where we replace v_{n_α} by the function $\partial_t v_{n_\alpha}$.

We control the first summand $\partial_t(N_0^{(J+1)}(v)(n))$ by Lemma 24. For the last summand of the RHS of (98) we estimate its L^2 norm as

$$\sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \|\widetilde{q}_{T^0}^{J,t}(N_1^{t,\alpha}(\{w_{n_\beta}\}_{\beta \in T^\infty}))\|_2$$

$$\lesssim \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2 \frac{\|\partial_t v_{n_\alpha}\|_2}{\prod_{k=1}^J |\widetilde{\varphi}_k|},$$

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which by Hölder's inequality with exponents $1/q + 1/q' = 1$ and (95) implies the upper bound

$$\frac{1}{J^{J/2}} \sum_{T \in T(J)} \sum_{\alpha \in T^\infty} \left(\sum_{\substack{\mathbf{n} \in \mathcal{R}(T) \\ \mathbf{n}_r = n}} \prod_{\beta \in T^\infty \setminus \{\alpha\}} \|v_{n_\beta}\|_2^q \|\partial_t v_{n_\alpha}\|_2^q \right)^{1/q}.$$

Then to the sum inside the parentheses we apply Young's inequality in the discrete variable, where for the first $2J$ functions we take the l^1 norm and for the last, and the l^∞ norm ones we arrive at the estimate

$$\|v\|_{M_{2,q}}^{2J} \sup_{n \in \mathbb{Z}} \|\partial_t v_n\|_2 = \|v\|_{M_{2,q}}^{2J} \|\partial_t v_n\|_{l^\infty L^2}.$$

Since by (32) we have $\partial_t v_n = \mathrm{e}^{\mathrm{i}t\partial_x^4} \Box_n (|u|^2 u)$ it is straightforward to obtain

$$\|\partial_t v_n\|_{l^\infty L^2} \lesssim \|v\|_{M_{2,q}}^3.$$

Indeed, from (19) and since $\Box_n (|u|^2 u)$ is nicely localised it suffices to estimate

$$\| \Box_n (|u|^2 u)\|_2 \lesssim \| \Box_n (|u|^2 u)\|_1 \lesssim \||u|^2 u\|_1 = \|u\|_3^3 \lesssim \|u\|_{M_{2,q}^s}^3 = \|v\|_{M_{2,q}^s}^3,$$

where we used (20), Lemma 12 and the embedding $M_{2,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$. Therefore, putting everything together, we arrive at

$$\|N_2^{(J)}(v)\|_{l^\infty L^2} \lesssim \frac{1}{J^{J/2}} \|v\|_{M_{2,q}^s}^{2J+3},$$

which finishes the proof. $\qquad \square$

Observe that the assumption $M_{2,q}^s(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$ implies that if u is a solution of the biharmonic NLS (1) in the space $C([0,T], M_{2,q}^s(\mathbb{R}))$ then u and hence $v = \mathrm{e}^{\mathrm{i}t\partial_x^4}u$ are elements of $X_T \hookrightarrow C([0,T], L^3(\mathbb{R}))$. Thus, the nonlinearity of the biharmonic NLS (1) makes sense as an element of $C([0,T], L^1(\mathbb{R}))$ and by (32) we obtain that $\partial_t v_n \in C([0,T], L^1(\mathbb{R}))$. The next lemma justifies all the formal calculations that were performed in the previous sections (for a proof see e.g. [30], Lemma 27).

Lemma 29. *Let $f, \partial_t f \in C([0,T], L^1(\mathbb{R}^d))$ and define the distribution $\int_{\mathbb{R}^d} f(\cdot, x)\, \mathrm{d}x$ by*

$$\left\langle \int_{\mathbb{R}^d} f(\cdot, x)\, \mathrm{d}x, \varphi \right\rangle = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(t, x)\varphi(t)\, \mathrm{d}x\, \mathrm{d}t$$

for $\varphi \in C_c^\infty(\mathbb{R})$. Then, $\partial_t \int_{\mathbb{R}^d} f(\cdot, x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \partial_t f(\cdot, x)\, \mathrm{d}x$.

Here is an application of the lemma. Consider (42) for fixed n and ξ. We want to apply Lemma 29 to the function

$$f(t, \xi_1, \xi_3) = \sigma_n(\xi) \frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3),$$

where $\xi \approx n$, $\xi_1 \approx n_1$, $\xi_3 \approx n_3$, $\xi - \xi_1 - \xi_3 \approx -n_2$ and $(n, n_1, n_2, n_3) \in A_N(n)^c$ given by (41). Notice that $f, \partial_t f \in C([0,T], L^1(\mathbb{R}^2))$ since $v \in C([0,T], M^s_{2,q}(\mathbb{R}))$ and $\partial_t v_n \in C([0,T], L^1(\mathbb{R}))$ for all integers n. Thus,

$$\partial_t \left[\iint_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3) \, d\xi_1 \, d\xi_3 \right]$$

$$= \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3) \right] d\xi_1 \, d\xi_3$$

$$= \int_{\mathbb{R}^2} \sigma_n(\xi) \partial_t \left[\frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \right] \widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3) \, d\xi_1 \, d\xi_3$$

$$+ \int_{\mathbb{R}^2} \sigma_n(\xi) \frac{e^{it\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)}}{i\Phi_4(\xi,\xi_1,\xi-\xi_1-\xi_3,\xi_3)} \partial_t[\widehat{v}_{n_1}(\xi_1) \widehat{\overline{v}}_{n_2}(\xi-\xi_1-\xi_3) \widehat{v}_{n_3}(\xi_3)] \, d\xi_1 \, d\xi_3.$$

In the second equality we used the product rule which is applicable since $v \in C([0,T], L^3(\mathbb{R}))$ implies that $\partial_t v_n \in C([0,T], L^1(\mathbb{R}))$.

Finally it remains to justify the interchange of differentiation in time and summation in the discrete variable but this is done in exactly the same way as in [17], Lemma 5.1. Similar arguments justify the interchange in the Jth step of the infinite iteration procedure.

Having proved these lemmata we define the partial sum operators $\Gamma^{(J)}_{v_0}$ as

$$(99) \quad \Gamma^{(J)}_{v_0} v(t) = v_0 + \sum_{j=2}^{J} N_0^{(j)}(v)(n) - \sum_{j=2}^{J} N_0^{(j)}(v_0)(n)$$

$$+ \int_0^t R_1^\tau(v)(n) + R_2^\tau(v)(n) + \sum_{j=2}^{J} N_r^{(j)}(v)(n) + \sum_{j=1}^{J} N_1^{(j)}(v)(n) \, d\tau,$$

where we have $N_1^{(1)} := N_{11}^t$ from (39), $N_0^{(2)} := N_{21}^t$ from (45), $N_1^{(2)} := N_{31}^t$ from (54) and $N_r^{(2)} := N_4^t$ from (51), and $v_0 \in M_{2,q}(\mathbb{R})$ is a fixed function.

The argument from [30, Subsection 2.3] shows that for sufficiently large N and sufficiently small $T > 0$ these operators $\Gamma^{(J)}_{v_0}$ are well defined in $X_T := C([0,T], M_{2,q}(\mathbb{R}))$ for every $J \in \mathbb{N} \cup \{\infty\}$. We write Γ_{v_0} for $\Gamma^\infty_{v_0}$.

The differentiation by parts argument presented in the previous sections shows that if the function v is sufficiently smooth or if the modulation space $M^s_{2,q}(\mathbb{R})$

embeds in $L^3(\mathbb{R})$ then a solution v of the biharmonic NLS with the initial data v_0 is a fixed point of the operator Γ_{v_0}, i.e.,

$$(100) \qquad v(t) = v_0 + \mathrm{i} \int_0^t N_1^\tau(v) - R_1^\tau(v) + R_2^\tau(v) \, \mathrm{d}\tau$$

$$= v_0 + \sum_{j=2}^\infty N_0^{(j)}(v)(n) - \sum_{j=2}^\infty N_0^{(j)}(v_0)(n) + \int_0^t R_1^\tau(v)(n) + R_2^\tau(v)(n)$$

$$+ \sum_{j=2}^\infty N_r^{(j)}(v)(n) + \sum_{j=1}^\infty N_1^{(j)}(v)(n) \, \mathrm{d}\tau = \Gamma_{v_0} v.$$

The important property of the Γ_{v_0} operators is that if we are given two set of initial data $v_0^{(1)}$ and $v_0^{(2)}$ that are close in $M_{2,q}^s(\mathbb{R})$ then if $v^{(1)}$ is the solution to the biharmonic NLS with the initial data $v_0^{(1)}$ and $v^{(2)}$ is the solution with the initial data $v_0^{(2)}$ we have

$$(101) \qquad \|v^{(1)} - v^{(2)}\|_{X_T} = \|\Gamma_{v_0^{(1)}} v^{(1)} - \Gamma_{v_0^{(2)}} v^{(2)}\|_{X_T} \lesssim \|v_0^{(1)} - v_0^{(2)}\|_{M_{2,q}^s}.$$

As it was done in [6] and [30] for the cubic NLS (see also [5]), the proof of Theorem 4 consists of approximating the initial data $v_0 := u_0$ by smooth functions $\{v_0^{(m)}\}_{m \in \mathbb{N}}$ in $M_{2,q}^s(\mathbb{R})$, solving the biharmonic NLS for such $v_0^{(m)}$ in X_T, using the $\Gamma_{v_0^{(m)}}$ operators with a smooth solution $v^{(m)}$, showing that $v^{(m)}$ have a common time of existence for all $m \in \mathbb{N}$ and that the sequence $\{v^{(m)}\}_{m \in \mathbb{N}}$ is Cauchy in X_T. The limit v is the weak solution in the extended sense of the biharmonic NLS with the initial data v_0 that we were trying to find. The nonlinearity $\mathcal{N}(u)$ for $u = \mathrm{e}^{-\mathrm{i}t\partial_x^4} v$ is equal to $\lim_{n \to \infty} \mathcal{N}(u^{(m)})$ in the sense of distributions in $(0,T) \times \mathbb{R}$, where $u^{(m)} = \mathrm{e}^{-\mathrm{i}t\partial_x^4} v^{(m)}$.

The proof of Theorem 6 follows from (101) since if there are two solutions u_1 and u_2 with the same initial data u_0 we obtain

$$\|u_1 - u_2\|_{X_T} = \|\Gamma_{u_0} u_1 - \Gamma_{u_0} u_2\|_{X_T} \lesssim \|u_0 - u_0\|_{M_{2,q}^s} = 0.$$

6. The general higher order nonlinear Schrödinger equation

For $k \in \mathbb{Z}_+$, consider the following Cauchy problem of the higher order NLS

$$(102) \qquad \begin{cases} \mathrm{i}\partial_t u - (-1)^k \partial_x^{2k} u \pm |u|^2 u = 0, \\ u(0,x) = u_0(x). \end{cases}$$

It is straightforward to see that its phase factor is given by the function Φ_{2k} defined in (30) which enjoys the following factorization.

Proposition 30. *Under the assumption $\xi = \xi_1 - \xi_2 + \xi_3$ we have that*

$$(103) \qquad \Phi_{2k}(\xi, \xi_1, \xi_2, \xi_3) = (\xi_1 - \xi_2)(\xi_3 - \xi_2)P_k(\xi_1, \xi_2, \xi_3),$$

where $P_k(\xi_1, \xi_2, \xi_3) \in \mathbb{Z}[\xi_1, \xi_2, \xi_3]$ is a non-negative homogeneous polynomial of degree $2k - 2$ with only the trivial root. More precisely, we have the formula

$$(104) \qquad P_k(\xi_1, \xi_2, \xi_3) = \sum_{m=1}^{2k-2} \xi_3^m \sum_{q=0}^{2k-2-m} \xi_1^q \xi_2^{2k-2-m-q} + \sum_{q=0}^{2k-2} \xi_1^q \xi_2^{2k-2-q}$$
$$+ \sum_{m=1}^{2k-1} \xi_1^{2k-1-m} \sum_{p=0}^{m-1} \binom{m}{p} (\xi_1 - \xi_2)^{m-p-1} \xi_3^p.$$

As a consequence, if $\xi = \xi_1 - \xi_2 + \xi_3$ then

$$(105) \qquad |\Phi_{2k}| \sim \max\{|\xi|, |\xi_1|, |\xi_2|, |\xi_3|\}^{2k-2}|\xi - \xi_1||\xi - \xi_3| \gtrsim |\Phi_2|^k$$

for all $k \in \mathbb{N}$.

P r o o f. As $\xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 = 2(\xi_1 - \xi_2)(\xi_3 - \xi_2)$, we may write

$$(106) \qquad \frac{1}{2}P_k(\xi_1, \xi_2, \xi_3) = \frac{\xi^{2k} - \xi_1^{2k} + \xi_2^{2k} - \xi_3^{2k}}{\xi^2 - \xi_1^2 + \xi_2^2 - \xi_3^2}$$
$$= s^{2k-2} \frac{y^{2k} + (1-y)^{2k} - (x^{2k} + (1-x)^{2k})}{y^2 + (1-y)^2 - (x^2 + (1-x)^2)},$$

where we set $s = \xi + \xi_2 = \xi_1 + \xi_3$, $x = \xi_1/s$ and $y = \xi_2/s$. Without loss of generality, assume $s \neq 0$ else we see directly from the above expression that P_k can only be zero when $\xi_1 = \xi_2 = \xi_3 = 0$. Consider the function $f(t) = t^{2k} + (1-t)^{2k}$. Then $f(t)$ is symmetric around $t = \frac{1}{2}$ and a change of the coordinate $u = t - \frac{1}{2}$ shows that

$$(107) \qquad f(t(u)) = \left(u + \frac{1}{2}\right)^{2k} + \left(u - \frac{1}{2}\right)^{2k} = \sum_{l=0}^{2k} \binom{2k}{2l} u^{2(k-l)} 2^{-2l+1}.$$

This implies that f is convex and strictly decreasing (increasing) when $t < \frac{1}{2}$ ($t > \frac{1}{2}$, respectively). Hence, $f(x) = f(y)$ can only hold if $x = y$ or $x + y = 1$, which shows that $\xi_3 = \xi_2$ or $\xi_1 = \xi_2$, respectively. Using l'Hospital's rule, one easily sees that in either case $P_k \neq 0$ for nontrivial (ξ_1, ξ_2, ξ_3) whenever $k \geqslant 2$.

In order to prove (105), note that both Φ_{2k} and P_k are homogeneous polynomials and, therefore, it is enough to show the required estimate on the unit circle. Also note that, since there is at least one positive value (choose $(1, -1, 1)$ for example),

we have $P_k > 0$ on $\mathbb{R}^3 \setminus \{0\}$. Notice that $P_k \lesssim \max(|\xi_1|, |\xi_2|, |\xi_3|)^{2k-2}$ is true for any polynomial P_k of degree $2k-2$. The other direction in this estimate then holds because P_k attains the strictly positive minimum on the sphere. To conclude the proof, observe that $\Phi_2^{k-1} \lesssim \max(|\xi_1|, |\xi_2|, |\xi_3|)^{2k-2}$ since Φ_2^{k-1} is a polynomial of degree $2k-2$. $\square$

Remark 31. In the case where $k = 1$ the polynomial is $P_1(\xi_1, \xi_2, \xi_3) = 2$, whereas in the case of $k = 2$ the polynomial is

$$P_2(\xi_1, \xi_2, \xi_3) = \xi_1^2 + \xi_2^2 + \xi_3^2 + (\xi_1 - \xi_2 + \xi_3)^2 + 2(\xi_1 + \xi_3)^2,$$

which is proved in [27], Lemma 3.1. Using (104) for the case of $k = 3$ after some tedious but elementary calculations we obtain the following expression for P_3:

$$(108) \quad P_3(\xi_1, \xi_2, \xi_3) = \xi_2^4 + (\xi_1 - \xi_2 + \xi_3)^4$$
$$+ \frac{1}{2}(\xi_1^2 + \xi_3^2)[(\xi_1 - \xi_2 + \xi_3)^2 + \xi_1^2 + \xi_2^2 + \xi_3^2]$$
$$+ 2(\xi_1 + \xi_3)^2[(\xi_1 - \xi_2 + \xi_3)^2 + \xi_2^2] + 2(\xi_1 + \xi_3)(\xi_1^3 + \xi_3^3).$$

Notice that all the summands are positive since all of them are even powers except the very last summand which is also positive but for a different reason: $\xi_1 + \xi_3$ and $\xi_1^3 + \xi_3^3$ have the same sign. Thus, for the polynomial $P_3(\xi_1, \xi_2, \xi_3)$ the trivial root $(0, 0, 0)$ is the only real root.

Because of (105), it is evident that all the calculations presented in Sections 3, 4 and 5 go through for the general higher order NLS (102), that is Theorems 4 and 6 (and Remarks 7, 8) hold true. The same reasoning applies to the mixed order equation (11) described in Remark 9 since the phase factors sum up as

$$(109) \quad \sum_{j=1}^{M} \varepsilon_j \Phi_{2j}(\xi, \xi_1, \xi_2, \xi_3) = (\xi_1 - \xi_2)(\xi_1 - \xi_3)\left(\sum_{j=1}^{M} \varepsilon_j P_j(\xi_1, \xi_2, \xi_3)\right)$$

whenever $\xi = \xi_1 - \xi_2 + \xi_3$.

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