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LOCALIZATION AND COLOCALIZATION IN TILTING TORSION THEORY FOR COALGEBRAS

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Abstract. Tilting theory plays an important role in the representation theory of coalgebras. This paper seeks how to apply the theory of localization and colocalization to tilting torsion theory in the category of comodules. In order to better understand the process, we give the (co)localization for morphisms, (pre)covers and special precovers. For that reason, we investigate the (co)localization in tilting torsion theory for coalgebras.

Keywords: (pre)cover; tilting comodule; (co)localization; torsion theory

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1. Introduction

As is well-known, (co)localization is an important tool in the representation theory of algebras. From different perspectives, many scholars researched the localization, the most famous of which is the localization in rings as a systematic method of adding multiplicative inverses to a ring. Gabriel in [7] abstractly described the localization in abelian and Grothendieck categories. Since the category $\mathcal{M}^C$ of right C-comodules over a coalgebra C is a locally finite Grothendieck category, it is natural to consider how to apply the localization to the category of comodules. Here, by a coalgebra we mean a K-coalgebra, where K is a field.

Following the localization for rings, Năstăsescu and Torrecillas in [19] developed a theory of localization for coalgebras. More precisely, if C is a coalgebra over a field K and $\mathcal{T}$ is a dense subcategory or a Serre class of the category $\mathcal{M}^C$ of right C-comodules, Năstăsescu and Torrecillas considered the quotient category $\mathcal{M}^C/\mathcal{T}$ and the canonical functor $T\colon \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$. More importantly, Năstăsescu and

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Torrecillas considered the colocalizing subcategory $\mathcal{T}$, i.e., the functor T has a left adjoint H, instead of considering the localizing subcategory $\mathcal{T}$, i.e., $\mathcal{T}$ is closed under arbitrary direct sums, or equivalently, T has a right adjoint S. Later, Navarro in [20] developed the ideas of Gabriel in the category of comodules by replacing the quotient category with a comodule category, which makes it easier to understand the localization for modules over an arbitrary algebra. The theory of localization for coalgebras has been developed by some scholars, see [8], [10], [11], [20], [21], [22], with the development of the representation theory of coalgebras, see [2], [5], [9], [12]–[18], [23]–[34], [39].

Tilting theory which Simson in [25] hoped to develop in the categories of comodules is a critical part of the representation theory of coalgebras. It is natural to consider the following question.

Question. How to apply the theory of localization and colocalization to tilting comodules and torsion pairs in the categories of comodules?

We present our main results as follows.

Theorem 1.1. *Let C be a K-coalgebra, $C^* = \mathrm{Hom}_K(C, K)$ be its K-dual K-algebra with the multiplication given by the convolution product, and $e \in C^*$ be an idempotent defining a perfect localization. Assume that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is a tilting eCe-comodule if and only if $S(M)$ is a tilting C-comodule.*

Theorem 1.2. *Assume that C is a basic coalgebra and an idempotent $e \in C^*$ defines a perfect localization. If M is a tilting eCe-comodule, then the following holds for the C-comodule $S(M)$:*

(a) $\mathcal{F}_C(S(M)) = \mathrm{Cogen}(S(M))$;

(b) $(\mathcal{T}_C(S(M)), \mathcal{F}_C(S(M)))$ *is a torsion pair in* $\mathcal{M}^C$.

This paper is organized as follows. Section 2 gives a brief overview of localization and colocalization in the categories of comodules. Section 3 analyses the localization and colocalization in morphisms. Section 4 presents the localization and colocalization in precovers, covers and special precovers. Section 5, Section 6 and Section 7 investigate the questions of the localization and colocalization in tilting comodules, comodule classes $\mathrm{Cogen}_n M$, $\mathrm{Cogen}_\infty M$ and torsion pairs.

2. Preliminaries

Throughout, let (C, Δ, ε) be a coalgebra over a field K, where Δ_C (denoted by Δ) is its comultiplication and ε is its counit. For any coalgebra C over a field K, we denote by $\mathcal{M}^C$ the categories of right C-comodules and by $C^* = \operatorname{Hom}_K(C, K)$ the K-dual algebra with respect to the convolution product, see [4]. The counit $\varepsilon\colon C \to K$ of C is the identity element of the algebra C^*.

Following [11], we call a full subcategory $\mathcal{T}$ of $\mathcal{M}^C$ dense if for every exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{M}^C$, the comodule M lies in $\mathcal{T}$ if and only if each of the comodules M' and M'' lies in $\mathcal{T}$. In other words, $\mathcal{T}$ is closed under extensions in $\mathcal{M}^C$. For any dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$, there exists an abelian category $\mathcal{M}^C/\mathcal{T}$ and an exact functor $T\colon \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$ such that $T(M) = 0$ for every $M \in \mathcal{T}$, satisfying the following universal property: for any exact functor $F\colon \mathcal{M}^C \to \mathcal{C}$ such that $F(M) = 0$ for each $M \in \mathcal{T}$, there exists a unique functor $\overline{F}\colon \mathcal{M}^C/\mathcal{T} \to \mathcal{C}$ verifying that $F = \overline{F}T$, where $\mathcal{C}$ is an arbitrary abelian category. The category $\mathcal{M}^C/\mathcal{T}$ is called the *quotient category* of $\mathcal{M}^C$ with respect to $\mathcal{T}$ and T is known as the *quotient functor*.

A dense subcategory $\mathcal{T}$ of $\mathcal{M}^C$ is said to be localizing if the quotient functor $T\colon \mathcal{M}^C \to \mathcal{M}^C/\mathcal{T}$ has a right adjoint functor $S\colon \mathcal{M}^C/\mathcal{T} \to \mathcal{M}^C$, called the *section functor*. If the section functor S is exact, a localizing subcategory $\mathcal{T}$ is called *perfect localizing*. The subcategory $\mathcal{T}$ is said to be colocalizing if T has a left adjoint functor $H\colon \mathcal{M}^C/\mathcal{T} \to \mathcal{M}^C$, called the *colocalizing functor*. The subcategory $\mathcal{T}$ is called a *perfect colocalizing subcategory* if the colocalizing functor H is exact.

In [3], [11], [38], localizing subcategories of the comodule category $\mathcal{M}^C$ are described by means of idempotents $e \in C^*$ of the K-dual K-algebra C^*. In addition, it is proved that the quotient category is the category of right comodules over the coalgebra eCe, where e is an idempotent associated to the localizing subcategory. The coalgebra structure of eCe is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \varepsilon_{eCe}(exe) = e(x) \quad \text{for any } x \in C,$$

where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ is the sigma notation of [35]. If M is a right C-comodule, then eM has a natural structure of the right eCe-comodule given by

$$\varrho(ex) = \sum_{(x)} ex_{(0)} \otimes ex_{(1)}e, \quad \text{where } \varrho_M(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}$$

for any $x \in M$ by the sigma notation of [35].

The following two lemmas (cf. [7], [11] and [19]) list the properties of the (co)localizing functor.

Lemma 2.1. *Let $\mathcal{T}$ be a dense subcategory of the category of right comodules $\mathcal{M}^C$ over a coalgebra C. Then the following statements hold.*

(1) *The quotient functor T is exact.*

(2) *If $\mathcal{T}$ is localizing, then the section functor S is left exact and the equivalence $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*

(3) *If $\mathcal{T}$ is colocalizing, then the colocalizing functor H is right exact and the equivalence $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ holds.*

Lemma 2.2. *Let C be a coalgebra and e be an idempotent in C^*. Then the following statements hold.*

(1) *The quotient functor $T\colon \mathcal{M}^C \to \mathcal{M}^{eCe}$ is naturally equivalent to the functor $e(-)$. The quotient functor T is also equivalent to the cotensor functor $- \,\square\, CeC$.*

(2) *The section functor $S\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the cotensor functor $- \,\square_{eCe}\, Ce$.*

(3) *If $\mathcal{T}$ is a colocalizing subcategory of $\mathcal{M}^C$, then the colocalizing functor H: $\mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the functor* $\mathrm{Cohom}_{eCe}(eC, -)$.

For the convenience of understanding, we present the following diagram

$$\begin{tikzcd}
\mathcal{M}^C \arrow[r, "T=e(-)=-\square CeC"] & \mathcal{M}^{eCe} \arrow[l, bend right=40, "S=-\square eCeCe"'] \arrow[l, bend left=40, "H=\mathrm{Cohom}_{eCe}(eC,-)"]
\end{tikzcd}$$

Throughout we denote by $\mathcal{T}_e$ the localizing subcategory associated to e for an idempotent $e \in C^*$ for any coalgebra C over a field K. For the convenience of writing, we use $T\cdot$ instead of $T(\cdot)$. We assume, unless otherwise stated, that all comodules are right comodules in this paper.

3. (Co)localization in morphisms

In this section, we apply the (co)localization technique to morphisms.

Lemma 3.1 ([6])**.** *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. We have the following:*

(1) *if M is a zero C-comodule, then eM is a zero eCe-comodule;*

(2) *if $\varrho\colon A \to B$ is a C-comodule monomorphism, then the induced map ϱ': $eA \to eB$ is an eCe-comodule monomorphism;*

(3) *if $\tau\colon B \to A$ is a C-comodule epimorphism, then the induced map $\tau'\colon eB \to eA$ is an eCe-comodule epimorphism.*

Corollary 3.2. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If $\varrho\colon A \to B$ is a C-comodule isomorphism, then the induced map $\varrho'\colon eA \to eB$ is an eCe-comodule isomorphism.*

Let (M, ϱ) be a right C-comodule. Following [20], there exists a unique minimal subcoalgebra $\mathrm{cf}(M)$ of C such that $\varrho(M) \subseteq M \otimes \mathrm{cf}(M)$, i.e., M is a right $\mathrm{cf}(M)$-comodule. This coalgebra $\mathrm{cf}(M)$ is called the *coefficient space* of M.

Definition 3.3 ([35]). Let C be a K-coalgebra, A and B subcoalgebras of C. The *wedge product* of subspaces A and B of C is $A \wedge B = \triangle^{-1}(A \otimes C + C \otimes B)$.

Definition 3.4. Let C be a K-coalgebra. A subcoalgebra A of C is called *coidempotent* if $A \wedge A = A$.

Lemma 3.5. *Let C be a K-coalgebra and $e \in C^*$ be idempotent. A left C-comodule M is zero if and only if the eCe-comodule $eM = 0$ for any idempotent $e \in C^*$.*

P r o o f. Necessity is clear. Sufficiency: Assume that the C-comodule $M \neq 0$. By [20], there is a bijective correspondence between localizing subcategories $\mathcal{T}$ of $\mathcal{M}^C$ and coidempotent subcoalgebras A of C, i.e., any localizing subcategories are related to the coalgebra $\mathcal{T}_C = \sum\limits_{M \in \mathcal{T}} \mathrm{cf}(M)$ and any coidempotent subcoalgebra A of C is related to the closed subcategory $\mathcal{T}_A$ which consists of objects $\{M \in \mathcal{M}^C \mid \mathrm{cf}(M) \subseteq A\}$. By the assumption, $M \neq 0$ and there exists $0 \neq m \in M$ such that

$$m = \sum_i m_{0i}\varepsilon(m_{1i}), \quad \text{where } \varrho(m) = \sum_i m_{0i} \otimes m_{1i},$$

where $m_{0i} \in M$, $m_{1i} \in C$, m_{0i} is a linearly independent basis (it always exists because of the tensor properties) and $\{m_{1i}\} \neq 0$ for $i = 1, \ldots, n$. For convenience, we write $m_{11} = x_1, m_{12} = x_2, \ldots$. From the previous description, we know $0 \neq m_{1i} \in \mathrm{cf}(M) \subseteq A \subseteq C$. Since every injective right C-comodule E is of the form $E = Ce$ for some idempotent $e \in C^*$, we get the form $C = E_1 \oplus E_2 \oplus \ldots$. For the convenience of writing, we denote by $C = C\varepsilon = C(e \oplus e') = C(e) \oplus C(e') = E_1 \oplus E_1'$, where e, e' are the idempotents of C^*, ε is a counit of C and $e \oplus e' = \varepsilon$. Note that the counit ε is an identity in C^*. In this case, $E_1 = Ce$ and $E_2 = Ce'$. Let $e_1(x_i) = \varepsilon(x_i)$ and $e_1(y) = 0$ for $y \in E_1'$. It is easy to check that e_1 is an idempotent in C^*. Without loss of generality, we put $e = e_1$. As a consequence,

$$e \rightharpoonup m = e(m) = \sum_i m_{0i} \cdot e(m_{1i}) = \sum_i m_{0i} \cdot \varepsilon(m_{1i}) = m \neq 0.$$

Consequently, $eM \neq 0$ and we get a contradiction. $\square$

Lemma 3.6. *Suppose that C is a K-coalgebra and an idempotent $e \in C^*$ defines a localization.*

(1) *If M is a zero eCe-comodule, then $S(M)$ is a zero C-comodule.*

(2) *If $\varrho\colon A \to B$ is an eCe-comodule monomorphism, then the induced map $\varrho'\colon S(A) \to S(B)$ is a C-comodule monomorphism.*

(3) *If $\tau\colon A \to B$ is an eCe-comodule epimorphism, then the induced map $\tau'\colon S(A) \to S(B)$ is a C-comodule epimorphism.*

(4) *If $\tau\colon A \to B$ is an eCe-comodule isomorphism, then the induced map $\tau'\colon S(A) \to S(B)$ is a C-comodule isomorphism.*

Proof. (1) It is clear.

(2) We have an exact sequence $0 \to A \to B$ in $\mathcal{M}^{eCe}$ because $\varrho\colon A \to B$ is an eCe-comodule monomorphism. Since S is a left exact functor, we get the exact sequence $0 \to S(A) \xrightarrow{\varrho'} S(B)$ in $\mathcal{M}^C$. Hence, ϱ' is a C-comodule monomorphism.

(3) If $\tau\colon A \to B$ is an eCe-comodule epimorphism, then we have $D = \operatorname{Coker}\tau = B/\operatorname{Im}\tau = 0$, i.e., $B = \operatorname{Im}\tau$. By (1), we know $S(D) = 0$. Thus, $S(B/\operatorname{Im}\tau) = S(B)/S(\operatorname{Im}\tau) = S(B)/\operatorname{Im}S(\tau) = 0$, i.e., $S(B) = \operatorname{Im}S(\tau)$. As a consequence, $\tau' = S(\tau)\colon S(A) \to S(B)$ is a C-comodule epimorphism.

(4) The statement follows from (2) and (3) immediately. □

Lemma 3.7. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. Then the following statements hold:*

(1) *$\varrho\colon A \to B$ is a C-comodule monomorphism if and only if the induced map $\varrho' = e(\varrho)\colon e(A) \to e(B)$ is an eCe-comodule monomorphism for each idempotent $e \in C^*$;*

(2) *$\varrho\colon A \to B$ is a C-comodule epimorphism if and only if the induced map $\varrho' = e(\varrho)\colon e(A) \to e(B)$ is an eCe-comodule epimorphism for each idempotent $e \in C^*$;*

(3) *$\varrho\colon A \to B$ is a C-comodule isomorphism if and only if the induced map $\varrho' = e(\varrho)\colon e(A) \to e(B)$ is an eCe-comodule isomorphism for each idempotent $e \in C^*$.*

Proof. (1) The necessity follows from Lemma 3.1 (2). Sufficiency: Since $\varrho' = e(\varrho)$ is an eCe-comodule monomorphism for each idempotent $e \in C^*$, then $\operatorname{Ker}\varrho' = e(\operatorname{Ker}\varrho) = 0$. By Lemma 3.5, we know $\operatorname{Ker}\varrho = 0$. Consequently, ϱ is a C-comodule monomorphism.

(2) The necessity is obtained from Lemma 3.1 (3). Sufficiency: Since $\varrho' = e(\varrho)$ is an eCe-comodule epimorphism for each idempotent $e \in C^*$, then $\operatorname{Coker}\varrho' = e(\operatorname{Coker}\varrho) = 0$. It follows from Lemma 3.5 that $\operatorname{Coker}\varrho = 0$. Consequently, ϱ is a C-comodule epimorphism.

(3) It follows from (1) and (2) immediately. □

Lemma 3.8. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a localization. Then the following assertions hold:*

(1) *M is a zero eCe-comodule if and only if $S(M)$ is a zero C-comodule;*

(2) *$\varrho\colon A \to B$ is an eCe-comodule monomorphism if and only if the induced map $\varrho' = S(\varrho)\colon S(A) \to S(B)$ is a C-comodule monomorphism;*

(3) *$\varrho\colon A \to B$ is an eCe-comodule epimorphism if and only if the induced map $\varrho' = S(\varrho)\colon S(A) \to S(B)$ is a C-comodule epimorphism;*

(4) *$\varrho\colon A \to B$ is an eCe-comodule isomorphism if and only if the induced map $\varrho' = S(\varrho)\colon S(A) \to S(B)$ is a C-comodule isomorphism.*

Proof. (1) The necessity follows from Lemma 3.6 (1). Sufficiency: Suppose that the eCe-comodule $M \neq 0$. Since the C-comodule $S(M) = 0$, we get $TS(M) = 0$ by Lemma 3.1 (1). From Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it follows that the eCe-comodule $M = 0$ and we get a contradiction.

(2) The necessity is obtained from Lemma 3.6 (2). Sufficiency: If $\varrho' = S(\varrho)$ is a C-comodule monomorphism, then $T(S(\varrho))$ is an eCe-comodule monomorphism by Lemma 3.1 (2). From Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that ϱ is an eCe-comodule monomorphism.

(3) The necessity follows from Lemma 3.6 (3). Sufficiency: Assume that $\varrho' = S(\varrho)$ is a C-comodule epimorphism. It follows from Lemma 3.1 (3) that $T(S(\varrho))$ is an eCe-comodule epimorphism. By Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we know that ϱ is an eCe-comodule epimorphism.

(4) It is obvious. □

Lemma 3.9. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a colocalization. Then the following assertions hold:*

(1) *if M is a zero eCe-comodule, then $H(M)$ is a zero C-comodule;*

(2) *if $\varrho\colon A \to B$ is an eCe-comodule monomorphism, then the induced map $\varrho'\colon H(A) \to H(B)$ is a C-comodule monomorphism;*

(3) *if $\tau\colon A \to B$ is an eCe-comodule epimorphism, then the induced map $\tau'\colon H(A) \to H(B)$ is a C-comodule epimorphism;*

(4) *if $\tau\colon A \to B$ is an eCe-comodule isomorphism, then the induced map $\tau'\colon H(A) \to H(B)$ is a C-comodule isomorphism.*

Proof. (1) It is clear.

(2) Since $\varrho\colon A \to B$ is an eCe-comodule monomorphism, we get $D = \operatorname{Ker}\varrho \cong A/\operatorname{Im}\varrho = 0$, i.e., $A = \operatorname{Im}\varrho$. It follows from (1) that $H(D) = 0$. Furthermore, $H(A/\operatorname{Im}\varrho) = H(A)/\operatorname{Im} H(\varrho)$. Consequently, $H(A) = \operatorname{Im} H(\varrho)$ and $\varrho' = H(\varrho)\colon H(A) \to H(B)$ is a C-comodule monomorphism.

(3) If $\tau\colon A \to B$ is an eCe-comodule epimorphism, then we have an exact sequence $A \to B \to 0$ in $\mathcal{M}^{eCe}$. Since H is a right exact functor, we obtain the exact sequence $H(A) \xrightarrow{\varrho'} H(B) \to 0$ in $\mathcal{M}^C$. Hence, ϱ' is a C-comodule epimorphism.

(4) The statement follows from (2) and (3) immediately. $\square$

Lemma 3.10. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a colocalization. Then the following assertions hold:*

(1) *M is a zero eCe-comodule if and only if $H(M)$ is a zero C-comodule;*

(2) *$\varrho\colon A \to B$ is an eCe-comodule monomorphism if and only if the induced map $\varrho' = H(\varrho)\colon H(A) \to H(B)$ is a C-comodule monomorphism;*

(3) *$\varrho\colon A \to B$ is an eCe-comodule epimorphism if and only if the induced map $\varrho' = H(\varrho)\colon H(A) \to H(B)$ is a C-comodule epimorphism;*

(4) *$\varrho\colon A \to B$ is an eCe-comodule isomorphism if and only if the induced map $\varrho' = H(\varrho)\colon H(A) \to H(B)$ is a C-comodule isomorphism.*

P r o o f. (1) The necessity follows from Lemma 3.9 (1). Sufficiency: Suppose that the eCe-comodule $M \neq 0$. Since the C-comodule $H(M) = 0$, the eCe-comodule $TH(M) = 0$ by Lemma 3.1 (1). From Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, it follows that the eCe-comodule $M = 0$ and we get a contradiction.

(2) The necessity is obtained from Lemma 3.9 (2). Sufficiency: If $\varrho' = H(\varrho)$ is a C-comodule monomorphism, then $T(H(\varrho))$ is an eCe-comodule monomorphism by Lemma 3.1 (2). By Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we have that $T(H(\varrho)) \cong \varrho$ is an eCe-comodule monomorphism.

(3) The necessity is obtained from Lemma 3.9 (3). Sufficiency: If $\varrho' = H(\varrho)$ is a C-comodule epimorphism, then $T(H(\varrho))$ is an eCe-comodule epimorphism by Lemma 3.1 (3). From Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that $T(H(\varrho)) \cong \varrho$ is an eCe-comodule epimorphism.

(4) It is obvious. $\square$

4. (Co)localization in (pre)covers

In this section, we introduce the concepts of (pre)covers and special precovers for comodules. In addition, we investigate their (co)localization.

Definition 4.1. Let $\mathcal{F}$ be a class of comodules in $\mathcal{M}^C$ and $M \in \mathcal{M}^C$, then $\varphi \in \mathrm{Hom}_C(X, M)$ with $X \in \mathcal{F}$ is an $\mathcal{F}$-precover of M if it satisfies that $\mathrm{Hom}_C(F, \varphi)\colon \mathrm{Hom}_C(F, X) \to \mathrm{Hom}_C(F, M)$ is surjective for each $F \in \mathcal{F}$.

Definition 4.2. Let $\varphi \in \mathrm{Hom}_C(X, M)$ be an $\mathcal{F}$-precover of M.

(1) φ is said to be an *$\mathcal{F}$-cover* of M, if $\varphi g = \varphi$ and $g \in \mathrm{End}(X)$ imply that g is an automorphism of X;
(2) φ is called *special* if $\varphi \in \mathrm{Hom}_C(X, M)$ is surjective and $\mathrm{Ker}\,\varphi \in \mathcal{F}^{\perp}$.

We call $\mathcal{F} \subseteq \mathcal{M}^C$ a *precover (cover) class* if each comodule has an $\mathcal{F}$-precover ($\mathcal{F}$-cover).

Proposition 4.3. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$. If a right C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is an $\mathcal{F}$-cover of M, then the right eCe-comodule homomorphism $e\varphi \in \mathrm{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-cover of eM.*

Proof. Since the right C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is an $\mathcal{F}$-cover of M, we get that $\varphi^* = \mathrm{Hom}_C(F, \varphi)\colon \mathrm{Hom}_C(F, X) \to \mathrm{Hom}_C(F, M)$ is surjective for each $F \in \mathcal{F}$, that is, $\mathrm{Hom}_C(F, X) \xrightarrow{\varphi^*} \mathrm{Hom}_C(F, M) \to 0$ in $\mathcal{M}^C$ is exact. Moreover, $\varphi g = \varphi$ and $g \in \mathrm{End}_C(X)$ imply that g is an automorphism of X. By Lemma 3.1 (3), the eCe-comodule homomorphism $e\varphi^*$ is surjective, i.e., $\mathrm{Hom}_{eCe}(eF, eX) \xrightarrow{e\varphi^*} \mathrm{Hom}_{eCe}(eF, eM) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $eF \in e\mathcal{F}$. Since the C-comodule endomorphism g is an automorphism of X, it follows that for any eCe-comodule endomorphism $eg\colon eX \to eX$, eg is an automorphism of eX. As a consequence, we obtain that the eCe-comodule homomorphism $e\varphi$ is an $e\mathcal{F}$-cover of eM. □

Remark 4.4. Assume that C is a K-coalgebra and an idempotent $e \in C^*$. If a right C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is an $\mathcal{F}$-precover of M,
(1) then the right eCe-comodule homomorphism $e\varphi \in \mathrm{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-precover of eM;
(2) in addition, if $\varphi \in \mathrm{Hom}_C(X, M)$ is a special $\mathcal{F}$-precover of M, then the right eCe-comodule homomorphism $e\varphi$ is a special $e\mathcal{F}$-precover.

Proof. (1) It follows from Proposition 4.3.

(2) Since the C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is a special $\mathcal{F}$-precover of M, we get that $\varphi \in \mathrm{Hom}_C(X, M)$ is surjective and $\mathrm{Ker}\,\varphi \in \mathcal{F}^{\perp}$. Let $N = \mathrm{Ker}\,\varphi$, then there is an exact sequence $0 \to N \to X \xrightarrow{\varphi} M \to 0$ in $\mathcal{M}^C$. Since T is an exact functor, we get the short exact sequence $0 \to eN \to eX \xrightarrow{e\varphi} eM \to 0$ in $\mathcal{M}^{eCe}$. Since $\mathrm{Ker}\,\varphi \in \mathcal{F}^{\perp}$, we obtain $\mathrm{Ext}_C(F, N) = 0$ for any $F \in \mathcal{F}$. It follows from Lemma 3.1 (1) that $e\mathrm{Ext}_C(F, N) = 0$, i.e., $\mathrm{Ext}_{eCe}(eF, eN) = 0$. Hence, $eN = \mathrm{Ker}(e\varphi) \in (e\mathcal{F})^{\perp}$. This shows that the eCe-comodule homomorphism $e\varphi$ is a special $e\mathcal{F}$-precover. □

Corollary 4.5. *Let C be a coalgebra. If $\mathcal{F} \subseteq \mathcal{M}^C$ is a (pre)cover class, then $e\mathcal{F} \subseteq \mathcal{M}^{eCe}$ is also a (pre)cover class.*

Proof. It follows from Proposition 4.3 and Remark 4.4. □

Proposition 4.6. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a localization. If a right eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M, then the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-cover of $S(M)$.*

Proof. Since the right eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M, we get that $\varphi^* = \mathrm{Hom}_{eCe}(F, \varphi)\colon \mathrm{Hom}_{eCe}(F, X) \to \mathrm{Hom}_{eCe}(F, M)$ is surjective for each $F \in \mathcal{F}$, that is, $\mathrm{Hom}_{eCe}(F, X) \xrightarrow{\varphi^*} \mathrm{Hom}_{eCe}(F, M) \to 0$ in $\mathcal{M}^{eCe}$ is exact. In addition, $\varphi g = \varphi$ and $g \in \mathrm{End}_{eCe}(X)$ imply that the eCe-comodule endomorphism g is an automorphism of X. It follows from Lemma 3.6 (3) that the right C-comodule homomorphism $S(\varphi^*)$ is surjective, i.e., $\mathrm{Hom}_C(S(F), S(X)) \xrightarrow{S(\varphi^*)} \mathrm{Hom}_C(S(F), S(M)) \to 0$ in $\mathcal{M}^C$ is exact, where $S(F) \in S(\mathcal{F})$. Since the eCe-comodule endomorphism $g \in \mathrm{Hom}_{eCe}(X, X)$ is an automorphism of X, the C-comodule endomorphism $S(g) = \mathrm{Hom}_C(S(X), S(X))$ is also an automorphism of $S(X)$. Consequently, the C-comodule homomorphism $S(\varphi)$ is an $S(\mathcal{F})$-cover of $S(M)$. □

Remark 4.7. Let C be a coalgebra and $e \in C^*$ be an idempotent which defines a perfect localization. If a right eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-precover of M,

(1) then the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-precover of $S(M)$;

(2) in addition, if $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M, then the right C-comodule homomorphism $S(\varphi)$ is a special $S(\mathcal{F})$-precover.

Proof. (1) It follows from Proposition 4.6.

(2) Since the eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M, we get that $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is surjective and $\mathrm{Ker}\,\varphi \in \mathcal{F}^{\perp}$. It follows from Lemma 3.6 (3) that the C-comodule homomorphism

$$S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$$

is surjective. Let $N = \mathrm{Ker}\,\varphi$, then there is an exact sequence $0 \to N \to X \xrightarrow{\varphi} M \to 0$ in $\mathcal{M}^{eCe}$. Since $\mathrm{Ker}\,\varphi \in \mathcal{F}^{\perp}$, it follows that $\mathrm{Ext}_{eCe}(F, N) = 0$ for any $F \in \mathcal{F}$. Since S is an exact functor, we obtain the short exact sequence $0 \to S(N) \to S(X) \xrightarrow{S(\varphi)} S(M) \to 0$ in $\mathcal{M}^C$. Since S is fully faithful, we obtain the following commutative diagram (we denote $S(\cdot)$ by $S\cdot$ for convenience)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{eCe}(F,N) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(F,X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(F,M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_C(SF,SN) \arrow[r] & \mathrm{Hom}_C(SF,SX) \arrow[r] & \mathrm{Hom}_C(SF,SM) \arrow[r] & \mathrm{Ext}_C(SF,SN)
\end{tikzcd}$$

where $S(F) \in S(\mathcal{F})$. Hence, $\operatorname{Ext}_C(S(F), S(N)) = 0$ and $S(N) \in S(\mathcal{F})^{\perp}$. As a consequence, $S(N) = \operatorname{Ker}(S(\varphi)) \in (S(\mathcal{F}))^{\perp}$. This shows that the C-comodule homomorphism $S(\varphi)$ is a special $S(\mathcal{F})$-precover. $\square$

Proposition 4.8. *Let C be a coalgebra and $e \in C^*$ be an idempotent defining a colocalization. If a right eCe-comodule homomorphism $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M, then the right C-comodule homomorphism*

$$H(\varphi) \in \operatorname{Hom}_C(H(X), H(M))$$

is an $H(\mathcal{F})$-cover of $H(M)$.

P r o o f. Since the right eCe-comodule homomorphism $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M, we get that $\varphi^* = \operatorname{Hom}_{eCe}(F, \varphi)\colon \operatorname{Hom}_{eCe}(F, X) \to \operatorname{Hom}_{eCe}(F, M)$ is surjective for each $F \in \mathcal{F}$, that is, $\operatorname{Hom}_{eCe}(F, X) \xrightarrow{\varphi^*} \operatorname{Hom}_{eCe}(F, M) \to 0$ in $\mathcal{M}^{eCe}$ is exact. Moreover, $\varphi g = \varphi$ and $g \in \operatorname{End}_{eCe}(X)$ imply that the eCe-comodule endomorphism g is an automorphism of X. It follows from Lemma 3.9 (3) that the C-comodule homomorphism $H(\varphi^*)$ is surjective, where

$$H(\varphi^*)\colon \operatorname{Hom}_C(H(F), H(X)) \to \operatorname{Hom}_C(H(F), H(M)).$$

In addition, $H(F) \in H(\mathcal{F})$ by the fact that $X \in \mathcal{F}$. Since the eCe-comodule endomorphism $g \in \operatorname{Hom}_{eCe}(X, X)$ is an automorphism of X, the C-comodule endomorphism $H(g) = \operatorname{Hom}_C(H(X), H(X))$ is also an automorphism of $H(X)$. In addition, $H(X) \in H(\mathcal{F})$ by the fact that $X \in \mathcal{F}$. As a consequence, the C-comodule homomorphism $H(\varphi)$ is an $H(\mathcal{F})$-cover of $H(M)$. $\square$

Remark 4.9. Let C be a coalgebra and $e \in C^*$ be an idempotent which defines a perfect colocalization. If a right eCe-comodule homomorphism $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-precover of M,

(1) then the right C-comodule homomorphism $H(\varphi) \in \operatorname{Hom}_C(H(X), H(M))$ is an $H(\mathcal{F})$-precover of $H(M)$;

(2) in addition, if $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M, then the right C-comodule homomorphism $H(\varphi)$ is a special $H(\mathcal{F})$-precover.

P r o o f. (1) It follows from Proposition 4.8.

(2) Since the eCe-comodule homomorphism $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M, we get that $\varphi \in \operatorname{Hom}_{eCe}(X, M)$ is surjective and $\operatorname{Ker}\varphi \in \mathcal{F}^{\perp}$. By Lemma 3.9 (3), we know that the C-comodule homomorphism

$$H(\varphi) \in \operatorname{Hom}_C(H(X), H(M))$$

is surjective. Let $N = \operatorname{Ker}\varphi$, then there is an exact sequence $0 \to N \to X \xrightarrow{\varphi} M \to 0$ in $\mathcal{M}^{eCe}$. It follows that $\operatorname{Ext}_{eCe}(F, N) = 0$ for any $F \in \mathcal{F}$, because $\operatorname{Ker}\varphi \in \mathcal{F}^{\perp}$. Since H is an exact functor, there is the following exact sequence $0 \to H(N) \to H(X) \xrightarrow{H(\varphi)} H(M) \to 0$ in $\mathcal{M}^C$. Since H is fully faithful, we obtain the following commutative diagram (we denote $H(\cdot)$ by $H\cdot$ for convenience)

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{eCe}(F,N) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_{eCe}(F,X) \arrow[r] \arrow[d, "\cong"] & \operatorname{Hom}_{eCe}(F,M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_C(HF,HN) \arrow[r] & \operatorname{Hom}_C(HF,HX) \arrow[r] & \operatorname{Hom}_C(HF,HM) \arrow[r] & \operatorname{Ext}_C(HF,HN),
\end{tikzcd}$$

where $HF \in H\mathcal{F}$. Therefore, $\operatorname{Ext}_C(H(F), H(N)) = 0$ and $H(N) \in H(\mathcal{F})^{\perp}$. As a consequence, $H(N) = \operatorname{Ker}(H(\varphi)) \in (H\mathcal{F})^{\perp}$. This shows that the C-comodule homomorphism $H(\varphi)$ is a special $H(\mathcal{F})$-precover. $\square$

Theorem 4.10. *Let C be a coalgebra and $e \in C^*$ be an idempotent.*

(1) *A C-comodule homomorphism $\varphi \in \operatorname{Hom}_C(X, M)$ is an $\mathcal{F}$-precover of M if and only if the right eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-precover of eM for each idempotent $e \in C^*$.*

(2) *A C-comodule homomorphism $\varphi \in \operatorname{Hom}_C(X, M)$ is an $\mathcal{F}$-cover of M if and only if the right eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-cover of eM for each idempotent $e \in C^*$.*

(3) *A C-comodule homomorphism $\varphi \in \operatorname{Hom}_C(X, M)$ is a special $\mathcal{F}$-precover of M if and only if the right eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is a special $e\mathcal{F}$-precover of eM for each idempotent $e \in C^*$.*

Proof. (1) The necessity is obtained from Remark 4.4 (1). Sufficiency: Let us suppose that the right eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-precover of eM, then

$$\operatorname{Hom}_{eCe}(eF, eX) \xrightarrow{e\varphi^*} \operatorname{Hom}_{eCe}(eF, eM) \to 0 \quad \text{in } \mathcal{M}^{eCe}$$

is exact, i.e., $\operatorname{Coker}(e\varphi^*) = e \operatorname{Coker}\varphi^* = 0$. Since $e \in C^*$ is an arbitrary idempotent, it follows from Lemma 3.5 that $\operatorname{Coker}\varphi^* = 0$, where the C-comodule homomorphism $\varphi^*\colon \operatorname{Hom}_C(F, X) \to \operatorname{Hom}_C(F, M)$ is given by $\varphi^*(f) = \varphi f$. Hence, $\operatorname{Hom}_C(F, X) \xrightarrow{\varphi^*} \operatorname{Hom}_C(F, M) \to 0$ in $\mathcal{M}^C$ is exact. As a consequence, the C-comodule homomorphism $\varphi \in \operatorname{Hom}_C(X, M)$ is an $\mathcal{F}$-precover of M.

(2) By Proposition 4.3, the necessity is obvious. Sufficiency: Let us suppose that the right eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-cover of eM. Firstly, the eCe-comodule homomorphism $e\varphi \in \operatorname{Hom}_{eCe}(eX, eM)$ is an $e\mathcal{F}$-precover

of eM. By (1), we know that the C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is an $\mathcal{F}$-precover of M. Furthermore, $(e\varphi)(eg) = (e\varphi)$ and $eg \in \mathrm{End}_{eCe}(eX, eX)$ imply that the eCe-comodule endomorphism eg is an automorphism of eX, i.e., $\mathrm{Ker}(eg) = e\,\mathrm{Ker}\, g = 0$ and $\mathrm{Coker}(eg) = e\,\mathrm{Coker}\, g = 0$. Since $e \in C^*$ is an arbitrary idempotent, it follows from Lemma 3.5 that $\mathrm{Ker}\, g = 0$ and $\mathrm{Coker}\, g = 0$. As a consequence, $\varphi g = \varphi$ and $g \in \mathrm{End}_C(X)$ imply that the C-comodule endomorphism g is an automorphism of X. Consequently, the C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is an $\mathcal{F}$-cover of M.

(3) The necessity follows from Remark 4.4 (2). Sufficiency: If the right eCe-comodule homomorphism $e\varphi \in \mathrm{Hom}_{eCe}(eX, eM)$ is a special $e\mathcal{F}$-precover of M, then $e\varphi$ is surjective and $\mathrm{Ker}(e\varphi) \in (e\mathcal{F})^{\perp}$, where $\varphi \in \mathrm{Hom}_C(X, M)$. Let $\mathrm{Ker}(e\varphi) = eN$, i.e., $0 \to eN \to eX \to eM \to 0$ in $\mathcal{M}^{eCe}$ is an exact sequence. Take $eF \in e\mathcal{F}$, then $\mathrm{Ext}^1_{eCe}(eF, eN) = e\mathrm{Ext}^1_C(F, N) = 0$. Since $e \in C^*$ is an arbitrary idempotent, it follows from Lemma 3.5 that $\mathrm{Ext}^1_C(F, N) = 0$, where $F \in \mathcal{F}$. Consequently, $N = \mathrm{Ker}\,\varphi \in (\mathcal{F})^{\perp}$, where $\varphi \in \mathrm{Hom}_C(X, M)$. As a consequence, the C-comodule homomorphism $\varphi \in \mathrm{Hom}_C(X, M)$ is a special $\mathcal{F}$-precover of M. □

Theorem 4.11. *Let C be a coalgebra and $e \in C^*$ be an idempotent defining a localization.*

(1) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-precover of M if and only if the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-precover of $S(M)$.*

(2) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M if and only if the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-cover of $S(M)$.*

(3) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M if and only if the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is a special $S(\mathcal{F})$-precover of $S(M)$.*

Proof. (1) The necessity follows from Remark 4.7 (1). Sufficiency: If the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-precover of $S(M)$, then the eCe-comodule homomorphism $T(S(\varphi))$ is a $TS(\mathcal{F})$-precover of $T(S(M))$ by Remark 4.4 (1). By Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we get that the eCe-comodule homomorphism $T(S(\varphi)) \cong \varphi$ is an $\mathcal{F}$-precover of M.

(2) The necessity is obtained from Proposition 4.6. Sufficiency: If the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is an $S(\mathcal{F})$-cover of $S(M)$, then the eCe-comodule homomorphism $T(S(\varphi))$ is a $TS(\mathcal{F})$-cover of $T(S(M))$ by Proposition 4.3. From Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that the eCe-comodule homomorphism $T(S(\varphi)) \cong \varphi$ is an $\mathcal{F}$-cover of M.

(3) By Remark 4.7(2), the necessity is clear. Sufficiency: If the right C-comodule homomorphism $S(\varphi) \in \mathrm{Hom}_C(S(X), S(M))$ is a special $S(\mathcal{F})$-precover of $S(M)$, then the eCe-comodule homomorphism $T(S(\varphi))$ is a special $TS(\mathcal{F})$-precover of $T(S(M))$ by Remark 4.4 (2). From Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that the eCe-comodule homomorphism $T(S(\varphi)) \cong \varphi$ is a special $\mathcal{F}$-precover of M. □

Theorem 4.12. *Let C be a coalgebra and $e \in C^*$ be an idempotent which defines a colocalization.*

(1) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-precover of M if and only if the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is an $H(\mathcal{F})$-precover of $H(M)$.*

(2) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is an $\mathcal{F}$-cover of M if and only if the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is an $H(\mathcal{F})$-cover of $H(M)$.*

(3) *An eCe-comodule homomorphism $\varphi \in \mathrm{Hom}_{eCe}(X, M)$ is a special $\mathcal{F}$-precover of M if and only if the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is a special $H(\mathcal{F})$-precover of $H(M)$.*

P r o o f. (1) The necessity is obtained from Remark 4.9 (1). Sufficiency: If the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is an $H(\mathcal{F})$-precover of $H(M)$, then the eCe-comodule homomorphism $T(H(\varphi))$ is a $TH(\mathcal{F})$-precover of $T(H(M))$ by Remark 4.4 (1). By Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we get that the eCe-comodule homomorphism $T(H(\varphi)) \cong \varphi$ is an $\mathcal{F}$-precover of M.

(2) The necessity follows from Proposition 4.8. Sufficiency: If the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is an $H(\mathcal{F})$-cover of $H(M)$, then the eCe-comodule homomorphism $T(H(\varphi))$ is a $TH(\mathcal{F})$-cover of $T(H(M))$ by Proposition 4.3. By Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we obtain that the eCe-comodule homomorphism $T(H(\varphi)) \cong \varphi$ is an $\mathcal{F}$-cover of M.

(3) The necessity is obtained from Remark 4.9 (2). Sufficiency: If the right C-comodule homomorphism $H(\varphi) \in \mathrm{Hom}_C(H(X), H(M))$ is a special $H(\mathcal{F})$-precover of $H(M)$, then the eCe-comodule homomorphism $T(H(\varphi))$ is a special $TH(\mathcal{F})$-precover of $T(H(M))$ by Remark 4.4 (2). From Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that the eCe-comodule homomorphism $T(H(\varphi)) \cong \varphi$ is a special $\mathcal{F}$-precover of M. □

In this section, we recall the notion of tilting comodules and apply (co)localization technique to them.

Definition 5.1 ([37]). A right C-comodule M is called a *tilting comodule* if M satisfies the following three conditions:
(1) $\operatorname{inj.dim}(M) \leqslant 1$;
(2) $\operatorname{Ext}^1_C(M^X, M) = 0$ for any cardinal X;
(3) there exists an exact sequence $0 \to M_1 \to M_0 \to C \to 0$, where $M_i \in \operatorname{Prod} M$.

Proposition 5.2. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If M is a tilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then $T(M)$ is a tilting eCe-comodule.*

Proof. By [38], Theorem 1.13, we know that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the functor T is an equivalence.

(1) By the assumption, we have $\operatorname{inj.dim}(M) \leqslant 1$, that is, there is an exact sequence $0 \to M \to E_0 \to E_1 \to 0$ in $\mathcal{M}^C$, where E_0 and E_1 are injective C-comodules. Consequently, $0 \to T(M) \to T(E_0) \to T(E_1) \to 0$ is exact, where $T(E_0)$ and $T(E_1)$ are injective right eCe-comodules. Hence, we get $\operatorname{inj.dim}(T(M)) \leqslant 1$.

(2) Take an exact sequence $0 \to M \to E \to E/M \to 0$ in $\mathcal{M}^C$, where E is an injective C-comodule. By the assumption, we get $\operatorname{Ext}^1_C(M^X, M) = 0$. Since T is an equivalence, we obtain the commutative diagram, see Diagram 1 (we denote $T(\cdot)$ by $T\cdot$ for convenience) where $T(M^X) = (TM)^X$. As a consequence, we get $\operatorname{Ext}_{eCe}((TM)^X, TM) = 0$.

(3) It follows from our assumption that the exact sequence $0 \to M_1 \to M_0 \to C \to 0$ lies in $\mathcal{M}^C$, where $M_i \in \operatorname{Prod} M$. Since T is an equivalence, we obtain the short exact sequence $0 \to T(M_1) \to T(M_0) \to eCe \to 0$ in $\mathcal{M}^{eCe}$, where $T(M_i) \in \operatorname{Prod} T(M)$. $\square$

Proposition 5.3. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent defining a perfect localization. If M is a tilting eCe-comodule, then $S(M)$ is a tilting C-comodule.*

Proof. (1) By the assumption, we get $\operatorname{inj.dim}(M) \leqslant 1$, that is, there is an exact sequence $0 \to M \to E_0 \to E_1 \to 0$ in $\mathcal{M}^{eCe}$, where E_0 and E_1 are injective eCe-comodules. Since S is exact and preserves injective comodules, we have the short exact sequence $0 \to S(M) \to S(E_0) \to S(E_1) \to 0$ in $\mathcal{M}^C$, where $S(E_0)$ and $S(E_1)$ are injective right C-comodules. Therefore, we get $\operatorname{inj.dim}(S(M)) \leqslant 1$.

(2) Take an exact sequence $0 \to M \to E \to E/M \to 0$ in $\mathcal{M}^{eCe}$, where E is an injective eCe-comodule. It follows from the assumption that $\mathrm{Ext}^1_{eCe}(M^X, M) = 0$. Since S is fully faithful, we have the commutative diagram, see Diagram 2 (we denote $S(\cdot)$ by $S\cdot$ for convenience) where $S(M^X) = (SM)^X$ because S preserves products of comodules. As a consequence, $\mathrm{Ext}_C((SM)^X, SM) = 0$.

(3) It follows from our assumption that the exact sequence $0 \to M_1 \to M_0 \to eCe \to 0$ lies in $\mathcal{M}^{eCe}$, where $M_i \in \mathrm{Prod}\, M$. Since S is an exact functor and preserves products of comodules, we obtain the short exact sequence $0 \to S(M_1) \to S(M_0) \to C \to 0$ in $\mathcal{M}^C$, where $S(M_i) \in \mathrm{Prod}\, S(M)$. $\square$

Theorem 5.4. *Let C be a K-coalgebra, $C^* = \mathrm{Hom}_K(C, K)$ be its K-dual K-algebra with the multiplication given by the convolution product, and $e \in C^*$ be an idempotent defining a perfect localization. Assume that $X \cong Ce$ is a quasi-finite injective cogenerator, then M is a tilting eCe-comodule if and only if $S(M)$ is a tilting C-comodule.*

Proof. Necessity: It follows from Proposition 5.3. Sufficiency: It follows from [38], Theorem 1.13, that the functor T is an equivalence if and only if $X \cong Ce$ is a quasi-finite injective cogenerator.

(1) By the assumption, we obtain inj.dim$S(M) \leqslant 1$, that is, there is an exact sequence $0 \to S(M) \to S(E_0) \to S(E_1) \to 0$ in $\mathcal{M}^C$, where $S(E_0)$ and $S(E_1)$ are injective C-comodules. Since T is an equivalence, it follows that $0 \to TS(M) \to TS(E_0) \to TS(E_1) \to 0$ in $\mathcal{M}^{eCe}$ is exact, where $TS(E_0)$ and $TS(E_1)$ are injective right eCe-comodules. By Lemma 2.1 (2) $TS = 1_{\mathcal{M}/\mathcal{T}}$ and we obtain the short exact sequence $0 \to M \to E_0 \to E_1 \to 0$ in $\mathcal{M}^{eCe}$, where E_0 and E_1 are injective eCe-comodules. Consequently, we get inj.dim$(M) \leqslant 1$.

(2) Take an exact sequence $0 \to S(M) \to S(E) \to S(E)/S(M) \to 0$ in $\mathcal{M}^C$, where $S(E)$ is an injective C-comodule. By the assumption, we have $\mathrm{Ext}^1_C((SM)^X, S(M)) = 0$. Since T is an equivalence, we get the commutative diagram, see Diagram 3 (we denote $S(\cdot)$ by $S\cdot$ for convenience) where $T((SM)^X) = (TSM)^X$. As a consequence, $\mathrm{Ext}_{eCe}(M^X, M) = 0$.

(3) It follows from our assumption that the exact sequence $0 \to S(M_1) \to S(M_0) \to C \to 0$ lies in $\mathcal{M}^C$, where $S(M_i) \in \mathrm{Prod}\, S(M)$. Since T is an equivalence, we obtain the short exact sequence $0 \to TS(M_1) \to TS(M_0) \to eCe \to 0$ in $\mathcal{M}^{eCe}$, where $TS(M_i) \in \mathrm{Prod}\, TS(M)$. From Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and it follows that there is the short exact sequence $0 \to M_1 \to M_0 \to eCe \to 0$ in $\mathcal{M}^{eCe}$, where $M_i \in \mathrm{Prod}\, M$. $\square$

We recall the concept of cotilting comodule introduced by Simson in [30].

Definition 5.5. M is called a *cotilting comodule* if M satisfies the following three conditions:

(a) M is quasi-finite;

(b) $\operatorname{inj.dim}(M) \leqslant 1$;

(c) $\operatorname{Ext}^1_C(M^X, M) = 0$ for any cardinal X;

(d) there exists an exact sequence $0 \to M_1 \to M_0 \to C \to 0$, where $M_i \in \operatorname{Prod} M$.

Remark 5.6. Definition 5.5 has one more condition as compared with Definition 5.1, i.e., M is quasi-finite. In this section, we adopt the Definition 5.1 of tilting comodule introduced by Wang in [37]. However, if we adopt Definition 5.5, then Propositions 5.2 and 5.3 and Theorem 5.4 also hold because S preserves quasi-finiteness and T preserves quasi-finiteness under the condition that T is an equivalence.

6. (Co)localization in $\operatorname{Cogen}_n M$

Given the classes $\operatorname{Cogen}_n M$ and $\operatorname{Cogen}_\infty M$ of comodules (cf. [15]), we research their (co)localizations, which contributes significantly to investigate the tilting theory for the categories of comodules.

Definition 6.1. For $M, U \in \mathcal{M}^C$, we denote by $\operatorname{Cogen}_n M$ the following class consisting of the C-comodules U

$$\operatorname{Cogen}_n M = \{U \in \mathcal{M}^C \colon \text{ there is an exact sequence } 0 \to U \to M^{X_1} \to M^{X_2} \\ \to \ldots \to M^{X_n}, \text{ where } X_i \text{ are cardinals for all } 1 \leqslant i \leqslant n\}.$$

In addition, we define $\operatorname{Cogen}_\infty M$ as

$$\operatorname{Cogen}_\infty M = \{U \in \mathcal{M}^C \colon \text{ there is an exact sequence } 0 \to U \to M^{X_1} \to M^{X_2} \\ \to \ldots \to M^{X_n} \to \ldots, \text{ where } X_i \text{ are cardinals for all } i \geqslant 1\}.$$

Lemma 6.2. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right C-comodule $U \in \operatorname{Cogen}_n M$, then the right eCe-comodule $T(U) \in \operatorname{Cogen}_n T(M)$.*

Proof. By [38], Theorem 1.13 we get that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the functor T is an equivalence.

Since the right C-comodule $U \in \operatorname{Cogen}_n M$, we have the exact sequence

$$0 \to U \to M^{X_1} \to M^{X_2} \to \ldots \to M^{X_n}$$

in $\mathcal{M}^C$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. Since T is an equivalence, there is the exact sequence

$$0 \to T(U) \to (T(M))^{X_1} \to (T(M))^{X_2} \to \ldots \to (T(M))^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. As a consequence, the right eCe-comodule $T(U) \in \operatorname{Cogen}_n T(M)$. $\square$

Lemma 6.3. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a localization. If a right eCe-comodule $U \in \operatorname{Cogen}_n M$, then the right C-comodule $S(U) \in \operatorname{Cogen}_n S(M)$.*

P r o o f. By the assumption, there is an exact sequence

$$0 \to U \to M^{X_1} \to M^{X_2} \to \ldots \to M^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. Since the section functor S is left exact and preserves products of comodules, there is the exact sequence

$$0 \to S(U) \to (S(M))^{X_1} \to (S(M))^{X_2} \to \ldots \to (S(M))^{X_n}$$

in $\mathcal{M}^C$, where X_i are cardinals for all $1 \leqslant i \leqslant n$, i.e., the C-comodule $S(U) \in \operatorname{Cogen}_n S(M)$. $\square$

Lemma 6.4. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right eCe-comodule $U \in \operatorname{Cogen}_n M$ if and only if the right C-comodule $S(U) \in \operatorname{Cogen}_n S(M)$.*

P r o o f. The necessity follows from Lemma 6.3. Sufficiency: By [38], Theorem 1.13 we get that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the quotient functor T is an equivalence.

Since the right C-comodule $S(U) \in \operatorname{Cogen}_n S(M)$, we have the exact sequence

$$0 \to S(U) \to (S(M))^{X_1} \to (S(M))^{X_2} \to \ldots \to (S(M))^{X_n}$$

in $\mathcal{M}^C$ because the quotient functor T is an equivalence, where X_i are cardinals for all $1 \leqslant i \leqslant n$. It follows that there is the exact sequence

$$0 \to TS(U) \to (TS(M))^{X_1} \to (TS(M))^{X_2} \to \ldots \to (TS(M))^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. By Lemma 2.1 (2) $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we get the exact sequence

$$0 \to U \to M^{X_1} \to M^{X_2} \to \ldots \to M^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. Consequently, the right eCe-comodule $U \in \operatorname{Cogen}_n M$. $\square$

Lemma 6.5. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a colocalization, then the colocalizing functor H preserves products of comodules.*

P r o o f. By Lemma 2.2, the colocalizing functor $H\colon \mathcal{M}^{eCe} \to \mathcal{M}^C$ is naturally equivalent to the functor $\mathrm{Cohom}_{eCe}(eC, -)$. As a consequence,

$$\begin{aligned}\mathrm{Cohom}_{eCe}\Big(eC, \prod_i M_i\Big) &= \mathrm{Cohom}_{eCe}\Big(eC, \prod_i M_i \,\square_{eC}\, eC\Big)\\ &= \prod_i M_i \,\square_{eC}\, \mathrm{Cohom}_{eCe}(eC, eC)\\ &= \prod_i (M_i \,\square_{eC}\, \mathrm{Cohom}_{eCe}(eC, eC))\\ &= \prod_i \mathrm{Cohom}_{eCe}(eC, M_i \,\square_{eC}\, eC)\\ &= \prod_i \mathrm{Cohom}_{eCe}(eC, M_i).\end{aligned}$$

It follows from [36], Proposition 1.14 that the second equality and the fourth equality hold because eC is quasi-finite. Since $\mathrm{Cohom}_{eCe}(eC, eC)$ is quasi-finite, it has a left adjoint and the third equality holds. $\square$

Lemma 6.6. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a perfect colocalization. If a right eCe-comodule $U \in \mathrm{Cogen}_n M$, then the right C-comodule $H(U) \in \mathrm{Cogen}_n H(M)$.*

P r o o f. By the assumption, there is an exact sequence

$$0 \to U \to M^{X_1} \to M^{X_2} \to \ldots \to M^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. Since the colocalizing functor H is exact and preserves products of comodules, there is the following exact sequence

$$0 \to H(U) \to (H(M))^{X_1} \to (H(M))^{X_2} \to \ldots \to (H(M))^{X_n}$$

in $\mathcal{M}^C$, where X_i are cardinals for all $1 \leqslant i \leqslant n$, i.e., the C-comodule $H(U) \in \mathrm{Cogen}_n H(M)$. $\square$

Lemma 6.7. *Suppose that C is a K-coalgebra and an idempotent $e \in C^*$ defines a perfect colocalization. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right eCe-comodule $U \in \mathrm{Cogen}_n M$ if and only if the right C-comodule $H(U) \in \mathrm{Cogen}_n H(M)$.*

P r o o f. The necessity follows from Lemma 6.6. Sufficiency: By [38], Theorem 1.13 we get that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the quotient functor T is an equivalence.

Since the right C-comodule $H(U) \in \mathrm{Cogen}_n H(M)$, we have the exact sequence

$$0 \to H(U) \to (H(M))^{X_1} \to (H(M))^{X_2} \to \ldots \to (H(M))^{X_n}$$

in $\mathcal{M}^C$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. It follows that there is the exact sequence

$$0 \to TH(U) \to (TH(M))^{X_1} \to (TH(M))^{X_2} \to \ldots \to (TH(M))^{X_n}$$

in $\mathcal{M}^{eCe}$ because the quotient functor T is an equivalence, where X_i are cardinals for all $1 \leqslant i \leqslant n$. By Lemma 2.1 (3) $TH \simeq 1_{\mathcal{M}^C/\mathcal{T}}$ and we get the exact sequence

$$0 \to U \to M^{X_1} \to M^{X_2} \to \ldots \to M^{X_n}$$

in $\mathcal{M}^{eCe}$, where X_i are cardinals for all $1 \leqslant i \leqslant n$. Consequently, the right eCe-comodule $U \in \mathrm{Cogen}_n M$. $\square$

Similarly, we have the following lemmas. We just state them and omit the proofs.

Lemma 6.8. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right C-comodule $U \in \mathrm{Cogen}_\infty M$, then the right eCe-comodule $T(U) \in \mathrm{Cogen}_\infty T(M)$.*

Lemma 6.9. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a localization. If a right eCe-comodule $U \in \mathrm{Cogen}_\infty M$, then the right C-comodule $S(U) \in \mathrm{Cogen}_\infty S(M)$.*

Lemma 6.10. *Let C be a K-coalgebra and $e \in C^*$ be an idempotent. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right eCe-comodule $U \in \mathrm{Cogen}_\infty M$ if and only if the right C-comodule $S(U) \in \mathrm{Cogen}_\infty S(M)$.*

Lemma 6.11. *Suppose that C is a K-coalgebra and an idempotent $e \in C^*$ defines a perfect colocalization. If a right eCe-comodule $U \in \mathrm{Cogen}_\infty M$, then the right C-comodule $H(U) \in \mathrm{Cogen}_\infty H(M)$.*

Lemma 6.12. *Assume that C is a K-coalgebra and an idempotent $e \in C^*$ defines a perfect colocalization. If $X \cong Ce$ is a quasi-finite injective cogenerator and a right eCe-comodule $U \in \mathrm{Cogen}_\infty M$ if and only if the right C-comodule $H(U) \in \mathrm{Cogen}_\infty H(M)$.*

7. (Co)localization in torsion theories

In this section, we apply the (co)localization technique to torsion pairs. We introduce the full subcategories of $\mathcal{M}^C$ as

$$\mathcal{T}_C(M) = \{X \in \mathcal{M}^C;\ \mathrm{Hom}_C(X, M) = 0\} \subseteq \mathcal{M}^C,$$
$$\mathcal{F}_C(M) = \{X \in \mathcal{M}^C;\ \mathrm{Ext}_C(X, M) = 0\} \subseteq \mathcal{M}^C.$$

We recall the notion of a torsion pair (i.e. torsion theory, cf. [1], Definition 1.1 in Section VI.1 of Chapter VI).

Definition 7.1. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{M}^C$ is called a *torsion pair* (or a *torsion theory*) if the following conditions hold:
(a) $\mathrm{Hom}_C(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$;
(b) $\mathrm{Hom}_C(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$;
(c) $\mathrm{Hom}_C(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

As described in previous Section 6, when n is equal to 1, $\mathrm{Cogen}_n M$ denotes the full subcategory of $\mathcal{M}^C$ consisting of all comodules U such that there is a monomorphism $U \to M^L$ for some index set L. In fact, in this case $\mathrm{Cogen}_n M$ is the class of comodules cogenerated by M and we also denote the class by $\mathrm{Cogen}\, M$, that is, $\mathrm{Cogen}_n M = \mathrm{Cogen}\, M$.

Lemma 7.2 ([30]). *Assume that C is a basic coalgebra. If M is a tilting C-comodule, then*
(a) *$\mathcal{F}_C(M) = \mathrm{Cogen}(M)$;*
(b) *$(\mathcal{T}_C(M), \mathcal{F}_C(M))$ is a torsion pair in $\mathcal{M}^C$.*

Lemma 7.3. *Assume that C is a basic coalgebra and $e \in C^*$ is an idempotent. If M is a tilting C-comodule and $X \cong Ce$ is a quasi-finite injective cogenerator, then*
(a) *$\mathcal{F}_{eCe}(T(M)) = \mathrm{Cogen}(T(M))$;*
(b) *$(\mathcal{T}_{eCe}(T(M)), \mathcal{F}_{eCe}(T(M)))$ is a torsion pair in $\mathcal{M}^{eCe}$.*

Proof. By [38], Theorem 1.13 we get that $X \cong Ce$ is a quasi-finite injective cogenerator if and only if the functor T is an equivalence.

(a) Let eCe-comodule $Z \in \mathrm{Cogen}(T(M))$, then there is a monomorphism $u\colon Z \to (T(M))^L$ for some L. There exists a C-comodule X such that $T(X) = Z$

and $u\colon T(X) \to (T(M))^L \cong T(M^L)$ because T is an equivalence. Since T is an equivalence, there is the commutative diagram

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "\cong"] & M^L \arrow[d, "\cong"] \\ T(X) \arrow[r, "u"] & T(M^L). \end{tikzcd}$$

Consequently, u is injective if and only if $v\colon X \to M^L$ is injective if and only if $\operatorname{Ext}^1_C(X, M) = 0$ if and only if $\operatorname{Ext}^1_{eCe}(T(X), T(M)) = 0$, i.e., $T(X) = Z \in \mathcal{F}_{eCe}(T(M))$.

(b) In order to prove $(\mathcal{T}_{eCe}(T(M)), \mathcal{F}_{eCe}(T(M)))$ is a torsion pair in $\mathcal{M}^{eCe}$, we need to show that the following three statements hold:

(b_1) $\operatorname{Hom}_C(Y, Z) = 0$ for all $Y \in \mathcal{T}_{eCe}(T(M))$ and $Z \in \mathcal{F}_{eCe}(T(M))$;

(b_2) $\operatorname{Hom}_C(Y, -)|_{\mathcal{F}} = 0$ implies $Y \in \mathcal{T}_{eCe}(T(M))$;

(b_3) $\operatorname{Hom}_C(-, Z)|_{\mathcal{T}} = 0$ implies $Z \in \mathcal{F}_{eCe}(T(M))$.

Since T is an equivalence, (b) is obvious. □

Theorem 7.4. *Assume that C is a basic coalgebra and an idempotent $e \in C^*$ defines a perfect localization. If M is a tilting eCe-comodule, then the following holds for the C-comodule $S(M)$:*

(a) $\mathcal{F}_C(S(M)) = \operatorname{Cogen}(S(M))$;

(b) $(\mathcal{T}_C(S(M)), \mathcal{F}_C(S(M)))$ *is a torsion pair in* $\mathcal{M}^C$.

Proof. It follows from Lemma 7.2 that the following conditions hold because M is a tilting eCe-comodule.

(1) $\mathcal{F}_{eCe}(M) = \operatorname{Cogen}(M)$.

(2) $(\mathcal{T}_{eCe}(M), \mathcal{F}_{eCe}(M))$ is a torsion pair in $\mathcal{M}^{eCe}$, that is,

(2′) $\operatorname{Hom}_{eCe}(Y, Z) = 0$ for all $Y \in \mathcal{T}_{eCe}(M)$ and $Z \in \mathcal{F}_{eCe}(M)$;

(2″) $\operatorname{Hom}_{eCe}(Y, -)|_{\mathcal{F}_{eCe}(M)} = 0$ implies $Y \in \mathcal{T}_{eCe}(M)$;

(2‴) $\operatorname{Hom}_{eCe}(-, Z)|_{\mathcal{T}_{eCe}(M)} = 0$ implies $Z \in \mathcal{F}_{eCe}(M)$.

(a) Let an eCe-comodule $Z \in \operatorname{Cogen}(M)$, then there is an eCe-comodule monomorphism $\theta\colon Z \to M^L$ for some L. Since S is fully faithful, there is the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\theta"] \arrow[d, "\cong"] & M^L \arrow[d, "\cong"] \\ S(Z) \arrow[r, "\varphi"] & S(M^L). \end{tikzcd}$$

Consequently, the C-comodule homomorphism $\varphi\colon\ S(Z) \to S(M^L)$ is injective if and only if θ is injective if and only if $\operatorname{Ext}^1_{eCe}(Z,M) = 0$ if and only if $\operatorname{Ext}^1_C(S(Z),S(M))$ $= 0$ by Lemma 3.8 (1), i.e., $S(Z) \in \mathcal{F}_{eCe}(S(M))$.

(b) In order to prove that $(\mathcal{T}_C(S(M)), \mathcal{F}_C(S(M)))$ is a torsion pair in $\mathcal{M}^C$, we need to check the following three conditions:

(b_1) $\operatorname{Hom}_C(S(Y),S(Z)) = 0$ for all $S(Y) \in \mathcal{T}_C(S(M))$ and $S(Z) \in \mathcal{F}_C(S(M))$;

(b_2) $\operatorname{Hom}_C(S(Y),-)|_{\mathcal{F}_C(S(M))} = 0$ implies $S(Y) \in \mathcal{T}_C(S(M))$;

(b_3) $\operatorname{Hom}_C(-,S(Z))|_{\mathcal{T}_C(S(M))} = 0$ implies $S(Z) \in \mathcal{F}_C(S(M))$.

Firstly, we prove (b_1). Since S is fully faithful, by 2' we get $\operatorname{Hom}_C(S(Y),S(Z)) \cong \operatorname{Hom}_{eCe}(Y,Z) = 0$ for any $Y \in \mathcal{T}_{eCe}(M), Z \in \mathcal{F}_{eCe}(M)$, where

$$\begin{aligned} S(Y) \in \mathcal{T}_C(S(M)) &= \{S(X) \in \mathcal{M}^C;\ \operatorname{Hom}_C(S(Y),S(M)) = 0\},\\ S(Z) \in \mathcal{F}_C(S(M)) &= \{S(X) \in \mathcal{M}^C;\ \operatorname{Ext}_C(S(Z),S(M)) = 0\}. \end{aligned}$$

Next, we prove (b_2). Since M is a tilting right eCe-comodule, it follows that $\mathcal{F}_{eCe}(M) = \operatorname{Cogen}(M)$. Since S is fully faithful, we get the commutative diagram, see Diagram 4. This shows that $\operatorname{Hom}_C(S(Y),-)|_{\mathcal{F}_C(S(M))} = 0$ if and only if $S(Y) \in \mathcal{T}_C(S(M))$. Now, we prove ($b_3$). It follows from ($2'''$) that $\operatorname{Hom}_{eCe}(-,Z)|_{\mathcal{T}_{eCe}(M)} = 0$ implies $Z \in \mathcal{F}_{eCe}(M)$, i.e., $\operatorname{Ext}_{eCe}(Z,M) = 0$. Since S is fully faithful, we have $\operatorname{Hom}_C(S(-),S(Z))|_{\mathcal{T}_C(S(M))} = 0$. By Lemma 3.6 (1), we obtain $S(\operatorname{Ext}_C(Z,M)) = \operatorname{Ext}_C(S(Z),S(M)) = 0$. As a consequence, $\operatorname{Hom}_C(-,S(Z))|_{\mathcal{T}_C(S(M))} = 0$ implies $\operatorname{Ext}_C(S(Z),S(M)) = 0$, i.e., $S(Z) \in \mathcal{F}_C(S(M))$. $\square$

Corollary 7.5. *Assume that C is a basic coalgebra, an idempotent $e \in C^*$ defines a perfect localization and $X \cong Ce$ is a quasi-finite injective cogenerator. If M is a tilting right eCe-comodule, then the C-comodule $S(M)$ satisfies the statements:*

(a) *$\mathcal{F}_C(S(M)) = \operatorname{Cogen}(S(M))$;*

(b) *$(\mathcal{T}_C(S(M)), \mathcal{F}_C(S(M)))$ is a torsion pair in $\mathcal{M}^C$.*

Proof. The necessity is obtained from Theorem 7.4. Sufficiency: By Lemma 7.3 and our assumption, we get that

(a) $\mathcal{F}_{eCe}(TS(M)) = \operatorname{Cogen}(TS(M))$;

(b) $(\mathcal{T}_{eCe}(TS(M)), \mathcal{F}_{eCe}(TS(M)))$ is a torsion pair in $\mathcal{M}^{eCe}$.

Since $TS \simeq 1_{\mathcal{M}^C/\mathcal{T}}$, we have $TS(M) \cong M$. Consequently,

(a) $\mathcal{F}_{eCe}(M) = \operatorname{Cogen}(M)$;

(b) $(\mathcal{T}_{eCe}(M), \mathcal{F}_{eCe}(M)$ is a torsion pair in $\mathcal{M}^{eCe}$. $\square$

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$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C(M^X, M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_C(M^X, E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_C(M^X, E/M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{eCe}(T(M^X), TM) \arrow[r] \arrow[d, "\|"] & \mathrm{Hom}_{eCe}(T(M^X), TE) \arrow[r] \arrow[d, "\|"] & \mathrm{Hom}_{eCe}(T(M^X), T(E/M)) \arrow[r] \arrow[d, "\|"] & \mathrm{Ext}_{eCe}(T(M^X), TM) \arrow[d, "\|"] \\
0 \arrow[r] & \mathrm{Hom}_{eCe}((TM)^X, TM) \arrow[r] & \mathrm{Hom}_{eCe}((TM)^X, TE) \arrow[r] & \mathrm{Hom}_{eCe}((TM)^X, T(E/M)) \arrow[r] & \mathrm{Ext}_{eCe}((TM)^X, TM),
\end{tikzcd}
$$

Diagram 1.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{eCe}(M^X, M) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(M^X, E) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(M^X, E/M) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_C(S(M^X), SM) \arrow[r] \arrow[d, "\|"] & \mathrm{Hom}_C(S(M^X), SE) \arrow[r] \arrow[d, "\|"] & \mathrm{Hom}_C(S(M^X), S(E/M)) \arrow[r] \arrow[d, "\|"] & \mathrm{Ext}_C(S(M^X), SM) \arrow[d, "\|"] \\
0 \arrow[r] & \mathrm{Hom}_C((SM)^X, SM) \arrow[r] & \mathrm{Hom}_C((SM)^X, SE) \arrow[r] & \mathrm{Hom}_C((SM)^X, S(E/M)) \arrow[r] & \mathrm{Ext}_C((SM)^X, SM),
\end{tikzcd}
$$

Diagram 2.

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_C((SM)^X, SM) \arrow[r] \arrow[d, "T"] & \mathrm{Hom}_C((SM)^X, SE) \arrow[r] \arrow[d, "T"] & \mathrm{Hom}_C((SM)^X, S(E/M)) \arrow[r] \arrow[d, "T"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{eCe}((TSM)^X, TSM) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}((TSM)^X, TSE) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}((TSM)^X, TS(E/M)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}_{eCe}((TSM)^X, TSM) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{eCe}(M^X, M) \arrow[r] & \mathrm{Hom}_{eCe}(M^X, E) \arrow[r] & \mathrm{Hom}_{eCe}(M^X, E/M) \arrow[r] & \mathrm{Ext}_{eCe}(M^X, M),
\end{tikzcd}
$$

Diagram 3.

$$
\begin{tikzcd}
\mathrm{Hom}_{eCe}(Y, -)|_{\mathcal{F}_{eCe}(M)} = 0 \arrow[r, Leftrightarrow] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(Y, -)|_{\mathrm{Prod}\,(M)} = 0 \arrow[r, Leftrightarrow] \arrow[d, "\cong"] & \mathrm{Hom}_{eCe}(Y, M) = 0 \arrow[d, "\cong"] \\
\mathrm{Hom}_C(S(Y), -)|_{\mathcal{F}_C(S(M))} = 0 \arrow[r, Leftrightarrow] & \mathrm{Hom}_C(S(Y), -)|_{\mathrm{Prod}\,(S(M))} = 0 \arrow[r, Leftrightarrow] & \mathrm{Hom}_C(S(Y), S(M)) = 0.
\end{tikzcd}
$$

Diagram 4.

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