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FINITE GROUPS WITH TWO ROWS WHICH DIFFER IN ONLY ONE ENTRY IN CHARACTER TABLES

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Abstract. Let G be a finite group. If G has two rows which differ in only one entry in the character table, we call G an RD1-group. We investigate the character tables of RD1-groups and get some necessary and sufficient conditions about RD1-groups.

Keywords: finite group; irreducible character; character table

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1. Introduction

It is interesting to investigate the structure of a finite group by its character table. In [3], Gagola proved that if the character table of a group G has a row (corresponding to an irreducible character) with precisely two nonzero entries, then G has a unique minimal normal subgroup N which is necessarily an elementary abelian p-group for a prime p. In [1], Bianchi and Herzog considered finite groups with two columns in its character table which differ in only one entry. It turns out that such groups exist and they are exactly the finite groups with a nontrivial intersection of the kernels of all but one irreducible characters or equivalently, finite groups with an irreducible character vanishing on all but two conjugacy classes which were investigated in [3].

In this paper, we consider finite groups with two rows differing in only one entry in character tables. Such groups will be called *RD1-groups* (Rows of the character table Differing by 1 entry).

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To eliminate trivialities, we shall assume that if G is an RD1-group, then $|G| > 2$. There are many RD1-groups such as the Symmetry group $\mathrm{Sym}(3)$ and the Dihedral group D_{2n}, where n is odd.

Let G be a finite group. The character table of G will be denoted by $CT(G)$. The set of all irreducible characters of a group G will be denoted by $\mathrm{Irr}(G)$. Suppose $k = |\mathrm{Irr}(G)|$, the conjugacy classes of G are denoted by $\mathscr{K}_1 = \{1\}, \mathscr{K}_2, \ldots, \mathscr{K}_k$, and the orders of conjugacy classes satisfy $1 = |\mathscr{K}_1| \leqslant |\mathscr{K}_2| \leqslant \ldots \leqslant |\mathscr{K}_k|$. Let $c_1 = 1$, $c_2, \ldots, c_k$ be representatives of $\mathscr{K}_1 = \{1\}$, $\mathscr{K}_2, \ldots, \mathscr{K}_k$, respectively.

Let G be an RD1-group and suppose that the irreducible characters χ and ψ have different values only on conjugacy class $\mathscr{K}_j$. Thus, $\chi(c_i) = \psi(c_i)$ for $i \neq j$ and $\chi(c_j) \neq \psi(c_j)$, where c_i are representatives of $\mathscr{K}_i$. In other words, $\chi(g) = \psi(g)$ for all $g \in G - \mathscr{K}_j$ and $\chi(g) \neq \psi(g)$ for all $g \in \mathscr{K}_j$.

The structure of this paper is as follows. In Section 2, the basic properties of RD1-groups will be determined. In Section 3, we shall give some necessary and sufficient conditions about RD1-groups.

2. Basic properties of RD1-groups

We keep the foregoing notation and suppose that the irreducible characters χ and ψ have different values only on conjugacy class $\mathscr{K}_j$. First of all, we prove that χ or ψ must be the principal character 1_G.

Lemma 2.1. *Let G be an RD1-group. Then χ or ψ is equal to the principal character 1_G.*

P r o o f. Suppose $\chi, \psi \in \mathrm{Irr}(G) - \{1_G\}$, by the first orthogonality relation, we have

$$\begin{cases} 0 = [\chi, 1_G] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \chi(g) = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} \chi(g) + |\mathscr{K}_j| \chi(c_j) \Bigg], \\ 0 = [\psi, 1_G] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \psi(g) = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} \psi(g) + |\mathscr{K}_j| \psi(c_j) \Bigg], \end{cases}$$

where c_j is the representative of conjugacy class $\mathscr{K}_j$. Since $\chi(c_i) = \psi(c_i)$ for $i \neq j$, we have $\chi(g) = \psi(g)$ for $g \in G - \mathscr{K}_j$, which implies that $|\mathscr{K}_j| \chi(c_j) = |\mathscr{K}_j| \psi(c_j)$ and $\chi(c_j) = \psi(c_j)$, a contradiction with our assumption. Therefore χ or ψ is equal to 1_G. $\square$

Without loss of generality, we now assume that 1_G and χ have different values on $\mathscr{K}_j$ and have the same values on $\mathscr{K}_i$, $i \neq j$. Thus $\chi(c_i) = 1$ for $i \neq j$ and $\chi(c_j) \neq 1$.

If $j = 1$, it can be seen that $\chi(1) = 1 - |G|$ since $0 = [\chi, 1_G]$, a contradiction. Thus $j \neq 1$. And χ and 1_G have the same value on conjugacy class $\mathscr{K}_1 = \{1\}$, which means $\chi(1) = 1_G(1) = 1$.

Note that the orders of conjugacy classes satisfy $1 = |\mathscr{K}_1| \leqslant |\mathscr{K}_2| \leqslant \ldots \leqslant |\mathscr{K}_k|$ and $c_1 = 1$, $c_2, \ldots, c_k$ are representatives of $\mathscr{K}_1 = \{1\}$, $\mathscr{K}_2, \ldots, \mathscr{K}_k$, respectively. And χ and 1_G have different values only on the conjugacy class $\mathscr{K}_j$.

Lemma 2.2. *Let G be an RD1-group. Then $j = k$, $\chi(c_k) = -1$, the row of $CT(G)$ corresponding to χ is $(1, 1, \ldots, 1, -1)$, the kth column of $CT(G)$ is $(1, -1, 0, \ldots, 0)^\top$. Moreover, $|\mathscr{K}_k| > |\mathscr{K}_{k-1}| \geqslant 1$, $|G| = 2|\mathscr{K}_k|$.*

Proof. Suppose that $\chi(g) = 1$ for all $g \in G - \mathscr{K}_j$ and $\chi(g) \neq 1$ for all $g \in \mathscr{K}_j$. By the first orthogonality relation, we have

$$\begin{cases} 0 = [\chi, 1_G] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \chi(g) = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} \chi(g) + |\mathscr{K}_j| \chi(c_j) \Bigg], \\ 1 = [\chi, \chi] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |\chi(g)|^2 = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} |\chi(g)|^2 + |\mathscr{K}_j| |\chi(c_j)|^2 \Bigg], \end{cases}$$

where c_j is the representative of the conjugacy class $\mathscr{K}_j$. Notice that $\chi(g) = 1$ for all $g \in G - \mathscr{K}_j$. We have

$$\begin{cases} \chi(c_j) = \dfrac{-|G| + |\mathscr{K}_j|}{|\mathscr{K}_j|} < 0, \\ |\chi(c_j)| = 1. \end{cases}$$

It can be seen that $\chi(c_j) \in \mathbb{Z}$ since $\chi(c_j) \in \mathbb{Q}$ and $\chi(c_j)$ is an algebraic integer. Hence $\chi(c_j) = -1$ as $\chi(c_j) < 0$. It follows that $|G| = 2|\mathscr{K}_j|$.

If $k = 2$, then $|G| = 1_G(1)^2 + \chi(1)^2 = 1 + 1 = 2$, a contradiction. So $k > 2$.

For any $\varphi \in \mathrm{Irr}(G) - \{1_G, \chi\}$, by the first orthogonality relation, we have

$$\begin{cases} 0 = [\varphi, 1_G] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \varphi(g) = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} \varphi(g) + |\mathscr{K}_j| \varphi(c_j) \Bigg], \\ 0 = [\varphi, \chi] = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \varphi(g) \overline{\chi(g)} = \dfrac{1}{|G|} \Bigg[\sum_{g \in G - \mathscr{K}_j} \varphi(g) \times 1 + |\mathscr{K}_j| \varphi(c_j) \times (-1) \Bigg], \end{cases}$$

where c_j is the representative of conjugacy class $\mathscr{K}_j$. It follows that $\varphi(c_j) = 0$.

If the first row of $CT(G)$ corresponds to 1_G and the second row of $CT(G)$ corresponds to χ, then the jth column of $CT(G)$ is $(1, -1, 0, \ldots, 0)^\top$.

By the second orthogonality relation, we have $|C_G(c_j)| = 2$.

We claim that the conjugacy class $\mathscr{K}_j$ is the only one such that $|C_G(c_j)| = 2$. Assume that there exists a conjugacy class $\mathscr{K}_t \neq \mathscr{K}_j$ such that $|C_G(c_t)| = 2$, where c_t is the representative element of $\mathscr{K}_t$. It can be seen that

$$\begin{aligned}|C_G(c_t)| = 2 &= \sum_{\psi\in\mathrm{Irr}(G)} |\psi(c_t)|^2 = |1_G(c_t)|^2 + |\chi(c_t)|^2 + \sum_{\psi\in\Delta} |\psi(c_t)|^2 \\ &= 1 + 1 + \sum_{\psi\in\Delta} |\psi(c_t)|^2,\end{aligned}$$

where $\Delta = \mathrm{Irr}(G) - \{1_G, \chi\}$.

So $\psi(c_t) = 0$ for all $\psi \in \Delta$. Thus, the tth column of $CT(G)$ corresponding to $\mathscr{K}_t$ is $(1, 1, 0, \ldots, 0)^\top$, which implies that $t \neq 1$. Since $j \neq 1$, we have $\chi(1) = 1_G(1) = 1$. By the second orthogonality relation of 1st and tth column, it follows that $0 = 1+1 = 2$, a contradiction. So the conjugacy class $\mathscr{K}_j$ is the only one such that $|C_G(c_j)| = 2$.

If $|C_G(a)| = 1$ for $a \in G$, then $a = 1$ and $\{1\} = C_G(1) = G$, a contradiction. So $|C_G(a)| \neq 1$. Thus, $|C_G(c_i)| > 2$ for $i \neq j$ and $|\mathscr{K}_j|$ is maximal in $\{|\mathscr{K}_1|, |\mathscr{K}_2|, \ldots, |\mathscr{K}_k|\}$. Since $|\mathscr{K}_1| \leqslant |\mathscr{K}_2| \leqslant \ldots \leqslant |\mathscr{K}_k|$, it follows that $j = k$ and $|\mathscr{K}_k| > |\mathscr{K}_{k-1}| \geqslant 1$. By replacing j by k in the previous statements, we obtain the required results. $\square$

Let $\ker\chi$ be the kernel of the irreducible character χ and $Z(\chi)$ be the center of χ. We have the following properties about the RD1-group.

Lemma 2.3. *Let G be an RD1-group, then*

(i) $\chi^2 = 1_G$, $\ker\chi = G - \mathscr{K}_k$, $Z(\chi) = G$ *and* $G' \leqslant G - \mathscr{K}_k$.

(ii) *The column which has exactly two nonzeros in the character table is unique.*

(iii) *G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel. Moreover, G is solvable.*

Proof. (i) By Lemma 2.2, we have $\chi(g) = -1$ for $g \in \mathscr{K}_k$ and $\chi(g) = 1$ for $g \in G - \mathscr{K}_k$. It follows that $\chi^2 = 1_G$, $\ker\chi = G - \mathscr{K}_k$ and $Z(\chi) = G$. Since $\chi(1) = 1$, we have $G' \leqslant \ker\chi = G - \mathscr{K}_k$.

(ii) By Lemma 2.2, the kth column of $CT(G)$ is $(1, -1, 0, \ldots, 0)^\top$. Suppose that there exists conjugacy class $\mathscr{K}_t \neq \mathscr{K}_k$ which has exactly two nonzeros in the character table of G. Since $1_G(g) = 1$ and $\chi(g) \neq 0$ for all $g \in G$, it follows that $1_G(c_t) \neq 0$ and $\chi(c_t) \neq 0$ for $c_t \in \mathscr{K}_t$. By our assumption, $\psi(c_t) = 0$ for all $\psi \in \mathrm{Irr}(G) - \{1_G, \chi\}$ and $t \neq k$. By the second orthogonality relation, we have $0 = \overline{1_G(1)}1_G(c_t) + \overline{\chi(1)}\chi(c_t) + 0 = 1 \times 1 + 1 \times 1 = 2$, a contradiction. Therefore the column which has exactly two nonzeros in the character table is unique.

(iii) By Lemma 2.2, we know that G is nonabelian and $|C_G(c_k)| = 2$, where c_k is the representative of the conjugacy class $\mathscr{K}_k$. Therefore G is a Frobenius group with a complement H of order 2 and an abelian odd-order kernel K, see [2], Lemma 2.3. Since $|G/K| = |H| = 2$, it follows that G/K is abelian and G/K is solvable. Noticing that K is a subgroup with odd order, we get that G is solvable. $\square$

Theorem 2.1. *The group G is an RD1-group if and only if G has a conjugacy class which has exactly two nonzeros in the character table.*

Proof. Suppose G is an RD1-group, by Lemma 2.2, the group G has a conjugacy class which has exactly two nonzeros in the character table.

Conversely, suppose that the group G has a conjugacy class C which has exactly two nonzeros in the character table. Assume that $c \in C$. Since $1_G(c) = 1$, by our assumptions, there exists some $\chi \in \mathrm{Irr}(G) - \{1_G\}$ such that $\chi(c) \neq 0$ and $\psi(c) = 0$ for $\psi \in \mathrm{Irr}(G) - \{1_G, \chi\}$.

By the second orthogonality relation, it follows that $0 = 1 + \chi(c)\chi(1)$ and $\chi(c) = -1/\chi(1)$, where $c \in C$. Since $\chi(c)$ is an algebraic integer and $\chi(c) = -1/\chi(1) \in \mathbb{Q}$, we have $\chi(c) \in \mathbb{Z}$, $\chi(1) = 1$ and $\chi(c) = -1$.

For any conjugacy class $D \neq C$, we have $0 = 1 + \chi(d)\overline{\chi(c)} + 0$ and $\chi(d) = 1$, where $d \in D$. So $\chi(c) = -1 \neq 1_G(c)$ and $\chi(d) = 1 = 1_G(1)$, which implies that χ and 1_G have different values only on conjugacy class C. Therefore G is an RD1-group. $\square$

3. Characterizations on RD1-groups

Before proving Theorem 3.1, we introduce some character properties about Frobenius groups. We denote by $c(M)$ the number of conjugacy classes of M.

Lemma 3.1 ([4], Theorem 9.1.15). *Suppose that G is a Frobenius group with complement H and kernel M.*

(i) *If $1_M \neq \varphi \in \mathrm{Irr}(M)$, then $\varphi^G \in \mathrm{Irr}(G)$.*

(ii) $\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \{\varphi^G \colon 1_M \neq \varphi \in \mathrm{Irr}(M)\}$.

Lemma 3.2 ([4], Theorem 9.1.16). *Suppose that G is a Frobenius group with complement H and kernel M, so G has $[c(M)-1]/|H|$ distinct irreducible characters of the form φ^G, where $\varphi \in \mathrm{Irr}(M)$. (Here, $c(M)$ is the number of conjugacy classes of M.)*

Lemma 3.3 ([4], Corollary 9.1.17). *If G is a Frobenius group with complement H and kernel M, then $c(G) = c(H) + [c(M) - 1]/|H|$.*

Lemma 3.4. *Suppose that G is a Frobenius group with a complement H of order 2 and an abelian odd-order kernel N. Then $N = G'$.*

P r o o f. Since $|G/N| = |H| = 2$, G/N is abelian, which implies that $G' \leqslant N \leqslant G$. For any $1_N \neq \varphi \in \mathrm{Irr}(N)$, by Lemma 3.1, $\varphi^G \in \mathrm{Irr}(G)$. Since N is abelian, we have $\varphi(1) = 1$ and $\varphi^G(1) = |G:N|\varphi(1) = 2 \times 1 = 2$. Since $|H| = 2$, by Lemma 3.1, G has two linear irreducible characters. Thus $|G:G'| = 2$, which implies that $N = G'$. $\square$

Inspired by James and Liebeck in [6], we have the following theorem.

Theorem 3.1. *The group G is an RD1-group if and only if G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel G'.*

P r o o f. Suppose G is an RD1-group. By Lemmas 2.3 and 3.4, the group G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel G'.

Conversely, suppose G is a Frobenius group with a complement H of order 2 and an abelian odd-order n kernel G'. We have $G = G'H$, $G' \trianglelefteq G$, $H \cap G' = 1$, $|H| = 2$, $|G'| = n$, $|G| = 2n$.

By Lemma 3.1, we have $\mathrm{Irr}(G) = \mathrm{Irr}(H) \cup \{\varphi^G\colon 1_{G'} \neq \varphi \in \mathrm{Irr}(G')\}$. By Lemma 3.2 and G' being abelian, we see that the number of irreducible characters of G of the form φ^G is equal to $[c(G') - 1]/|H| = \frac{1}{2}(n-1)$. Therefore by Lemma 3.3, G has $2 + \frac{1}{2}(n-1) = \frac{1}{2}(n+3)$ conjugacy classes, which implies that G has $\frac{1}{2}(n+3)$ irreducible characters.

For any $1_{G'} \neq \varphi \in \mathrm{Irr}(G')$, since G' is abelian, we have $\varphi^G(1) = |G:G'|\varphi(1) = 2 \times 1 = 2$. So G has $\frac{1}{2}(n-1)$ irreducible characters of degree 2 and two linear irreducible characters.

Since G' is normal in G, $G' - \{1\}$ is a union of some conjugacy classes of G. Suppose $G' - \{1\} = x_1^G \cup x_2^G \cup \ldots \cup x_t^G$ with $x_i^G \neq x_j^G$ for $i \neq j$, where $x_i \in G'$, $x_i^G = \{x_i^g\colon g \in G\}$.

For any $1 \neq x \in G'$ we claim that $x^G = x^H$, where $x^H = \{x^h\colon h \in H\}$. Obviously, $x^H \subseteq x^G$. Conversely, for any $g \in G = G'H$ we have $g = nh$ for some $n \in G'$, $h \in H$. Since $x^g = x^{nh} = (n^{-1}xn)^h$ and G' is abelian, we have $x^g = x^h \in x^H$. So $x^G = x^H$.

Consider the action of H on G' by conjugation. For any $1 \neq x \in G'$ we have $|x^H| = |H:H_x|$, where $H_x = \{h \in H\colon x^h = x\} = H \cap C_G(x) = C_H(x)$. Since $C_H(x) \leqslant C_G(x) \leqslant G'$, the last inequality by the property of the Frobenius group (see [5], page 121), it follows that $C_H(x) \leqslant G' \cap H = 1$, which implies that $H_x = C_H(x) = \{1\}$. So $|x^H| = |H:H_x| = 2$. Since $1 \neq x_i \in G'$, $|x_i^G| = |x_i^H| = 2$ for $i = 1, 2, \ldots, t$. Hence, we have $t = \frac{1}{2}(n-1)$.

Up to now, we get the identity conjugacy class $\{1\}$ and $\frac{1}{2}(n-1)$ conjugacy classes x_i^G, $i = 1, 2, \ldots, t$ of size 2. Notice that G has $\frac{1}{2}(n+3)$ conjugacy classes. There is one remaining conjugacy class of size n as $\frac{1}{2}(n+3) - 1 - \frac{1}{2}(n-1) = 1$ and $2n - 1 - \frac{1}{2}(n-1) \times 2 = n$. Suppose $H = \langle b \rangle$ and $o(b) = 2$. Since $b \notin G'$, we denote the remaining conjugacy class by b^G. Now the conjugacy classes of G are $\{1\}, x_1^G, \ldots, x_t^G, b^G$.

Since G' is normal in G and $G/G' \cong H$, by Lemma 3.1, we see that the linear irreducible characters of G are obtained by lifting the irreducible characters of G/G' to G. Those irreducible characters are given by 1_G and χ_2, where $\chi_2(g) = \widehat{\chi}_2(gG')$, $\widehat{\chi}_2 \in \mathrm{Irr}(G/G') - \{1_{G/G'}\}$. Since $\widehat{\chi}_2(gG') = 1$ for $g \in G'$ and $\widehat{\chi}_2(gG') = -1$ for $g \notin G'$, we have

$$\chi_2(g) = \begin{cases} 1, & g \in G', \\ -1, & g \notin G', \end{cases}$$

which implies that χ_2 takes value 1 on the conjugacy classes $\{1\}$, $x_1^G, \ldots, x_t^G$ and χ_2 takes value -1 on the conjugacy class b^G. Therefore the rows of the character table of G corresponding to 1_G and χ_2 differ in only one entry. So G is an RD1-group. $\square$

In the following, we consider the zero points of nonlinear irreducible characters of G and the restriction of nonlinear irreducible characters of G on G'.

Theorem 3.2. *If G is an RD1-group, then:*

(a) *Every nonlinear irreducible character vanishes only on one conjugacy class. Furthermore, they vanish on the same conjugacy class.*

(b) *For every nonlinear irreducible character $\chi \in \mathrm{Irr}(G)$, $\chi_{G'} = \lambda + \theta$, where $\lambda \neq \theta \in \mathrm{Lin}(G')$ are conjugate in G.*

Proof. Since G is an RD1-group, G is a Frobenius group with a complement of order 2 and an abelian odd-order kernel G'. By using the notation of Theorem 3.1, we have $G = G'H$, $H = \langle b \rangle$, $o(b) = 2$, $|G'| = n$, $H \cap G' = 1$, $G' = 1 \cup x_1^G \cup x_2^G \cup \ldots \cup x_t^G$ and $G = G' \cup b^G$.

Since $(x_jb)^2 = x_jbx_jb = x_jb^{-1}x_jb = x_jx_j^b \in G'$, G' is abelian and $b(x_jb)^2 = (bx_jb)x_jb = x_j(bx_jb)b = (x_jb)^2b$, which implies that $(x_jb)^2 \in C_G(b) \leqslant H$ (see [5], page 121). So $(x_jb)^2 \in G' \cap H = 1$ and $x_j^b = x_j^{-1} \in G'$. It can be seen that $x_j^{-1} \neq x_j$ and $|x_j^G| = 2$. Otherwise, 2 divides $|G'| = n$, a contradiction. Since the representatives x_j, x_j^{-1} for the classes of G' are contained in x_j^G, by the formula on [5], page 64, it can be obtained that

$$\varphi^G(x_j) = |C_G(x_j)|\Big[\frac{\varphi(x_j)}{|C_{G'}(x_j)|} + \frac{\varphi(x_j^{-1})}{|C_{G'}(x_j^{-1})|}\Big],$$

where $1_{G'} \neq \varphi \in \mathrm{Irr}(G')$. Since $x_j^b = x_j^{-1}$, it implies $|C_{G'}(x_j^{-1})| = |C_{G'}(x_j^b)| = |C_{G'}(x_j)^b| = |C_{G'}(x_j)|$. Note that G' is abelian, we have $C_{G'}(x_j) = G'$. Then $|C_G(x_j)| = |G|/|x_j^G| = \frac{1}{2}|G| = |G'|$ and $|C_{G'}(x_j^{-1})| = |C_{G'}(x_j)| = |C_G(x_j)|$. So $\varphi^G(x_j) = \varphi(x_j) + \overline{\varphi(x_j)}$.

If $\varphi^G(x_j) = 0$, which means $\varphi(x_j) + \overline{\varphi(x_j)} = 0$, we may assume that $\varphi(x_j) = c\mathrm{i}$ for some $c \in \mathbb{R}$ and $\mathrm{i}^2 = -1$. Since $\varphi \in \mathrm{Irr}(G')$ and G' is abelian, it follows that $\varphi(x_j)^n = \varphi(x_j^n) = \varphi(1) = 1$. However, we have $\varphi(x_j)^n = (c\mathrm{i})^n \neq 1$ as n is odd, a contradiction. So $\varphi^G(x_j) \neq 0$.

Notice that $b^G \cap G' = \emptyset$. We have $\varphi^G(b) = 0$. Since $G' = 1 \cup x_1^G \cup x_2^G \cup \ldots \cup x_t^G$ and $G = G' \cup b^G$, every nonlinear irreducible character vanishes on only one conjugacy class b^G. So the proof of (a) is completed.

For every nonlinear irreducible character $\chi \in \mathrm{Irr}(G)$, by Theorem 3.1, we know that $\chi(1) = 2$. Suppose $e = [\chi_{G'}, \lambda_1] \neq 0$ and $\lambda_1 \in \mathrm{Irr}(G') = \mathrm{Lin}(G')$. By Clifford's Theorem, we have

$$\chi_{G'} = e(\lambda_1 + \lambda_2 + \ldots + \lambda_l),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_l$ are distinct conjugates of λ_1 in G. Since $\chi(1) = 2 = el\lambda_1(1) = el$, we have $e = 2$, $l = 1$ or $e = 1$, $l = 2$. Anyway, $\chi_{G'}$ is reducible. So $\chi_{G'} = \lambda_1 + \lambda_2$ (see [5], Corollary 6.19), and the proof of (b) is completed. $\square$

References

[1] *M. Bianchi, M. Herzog*: Finite groups with non-trivial intersections of kernels of all but one irreducible characters. Int. J. Group Theory *7* (2018), 63–80. zbl MR doi

[2] *D. Chillag*: On zeros of characters of finite groups. Proc. Am. Math. Soc. *127* (1999), 977–983. zbl MR doi

[3] *S. M. Gagola, Jr.*: Characters vanishing on all but two conjugacy classes. Pac. J. Math. *109* (1983), 363–385. zbl MR doi

[4] *L. C. Grove*: Groups and Characters. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs and Tracts. John Wiley & Sons, New York, 1997. zbl MR doi

[5] *I. M. Isaacs*: Character Theory of Finite Groups. Academic Press, New York, 1976. zbl MR doi

[6] *G. James, M. Liebeck*: Representations and Characters of Groups. Cambridge University Press, New York, 2001. zbl MR doi

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