

THE LOCAL INDEX DENSITY OF THE PERTURBED
DE RHAM COMPLEX

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Abstract. A perturbation of the de Rham complex was introduced by Witten for an exact 1-form Θ and later extended by Novikov for a closed 1-form on a Riemannian manifold M . We use invariance theory to show that the perturbed index density is independent of Θ ; this result was established previously by J. A. Álvarez López, Y. A. Kordyukov and E. Leichtnam (2020) using other methods. We also show the higher order heat trace asymptotics of the perturbed de Rham complex exhibit nontrivial dependence on Θ . We establish similar results for manifolds with boundary imposing suitable boundary conditions and give an equivariant version for the local Lefschetz trace density. In the setting of the Dolbeault complex, one requires Θ to be a $\bar{\partial}$ closed 1-form to define a local index density; we show in contrast to the de Rham complex, that this exhibits a nontrivial dependence on Θ even in the setting of Riemann surfaces.

Keywords: Witten deformation; local index density; de Rham complex; Dolbeault complex; equivariant index density

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1. INTRODUCTION

1.1. Motivation. Let M be a compact manifold without boundary of dimension m . If h is a smooth function on M , let $\text{ext}(dh)$ be exterior multiplication by dh . For s a real constant, Witten in [34] defined a perturbation of the de Rham differential by setting $d_{sh} = d + s \text{ext}(dh)$. Since d_{sh} is gauge equivalent to d , the associated Betti numbers are unchanged. Let g be a Riemannian metric on M ; the de Rham codifferential is then given by $\delta_{sh} = \delta + s \text{int}(dh)$, where $\text{int}(dh)$ is interior multiplication by dh . The perturbed Laplacian then takes the form $\Delta_{sh} = d_{sh}\delta_{sh} + \delta_{sh}d_{sh}$.

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Note that Δ_{sh} in general is not gauge equivalent to Δ and thus can have a different spectrum. When h is a Morse function, Witten used this family of elliptic complexes to give a beautiful analytical proof of the Morse inequalities, by analyzing the spectrum of Δ_{sh} as $s \rightarrow \infty$. We also refer to related work by Helffer and Sjöstrand (see [20]), and by Bismut and Zhang (see [5]).

More generally, let Θ be a real closed 1-form on M . By replacing dh by Θ , Novikov in [25], [26] defined perturbed operators $d_{s\Theta}$, $\delta_{s\Theta}$ and $\Delta_{s\Theta}$ and used these operators to estimate the zeros of Θ if Θ is of Morse type. Because $d_{s\Theta}$ may not be gauge equivalent to d , the twisted Betti numbers defined by $d_{s\Theta}$ in general depend on s and on Θ . However, the twisted Betti numbers are constant on the complement of a finite set S of values of s , where the dimensions may jump. The twisted Betti numbers for $s \in \mathbb{R} \setminus S$ are called the Novikov numbers of $[\Theta]$; they are used in the Novikov version of the Morse inequalities. We also refer to related work of Braverman and Farber (see [6]) and of Pazhitnov (see [30]). This remains an active research area; see, for example, [7], [8], [19], [24].

Since we shall not be examining asymptotic limits, we set $s = 1$ henceforth. We shall examine the local index density for the Novikov Laplacian Δ_Θ . We shall show that the twisted index density vanishes if m is odd, while if m is even, the local index density is the Euler form and, in particular, does not depend on Θ . We obtain that the heat trace invariants of smaller order are trivial, and the heat trace invariants of higher order exhibit a nontrivial dependence on Θ . Our initial motivation for this result is its application in certain trace formula for foliated flows, studied by the first author, Kordyukov and Leichtnam, see [1]. We use methods of invariance theory; we note that a previous proof of this result was given in [1] by analyzing Getzler's proof of the local index theorem (see [4]); this result also follows from arguments of [5].

We shall establish two extensions of this result that are new. First, we will examine the twisted index density for a manifold with boundary, where we impose suitable boundary conditions. Second, we will establish an equivariant version for maps giving a Lefschetz trace formula for d_Θ . In both instances, we show that the resulting formulas are independent of Θ . It is natural to conjecture that this must always be the case. We show this is not in fact true by examining a Novikov type perturbation of the Dolbeault complex for Riemann surfaces, using a $(0, 1)$ -form Θ with $\bar{\partial}\Theta = 0$. We show that in this setting, the local index density exhibits a nontrivial dependence on Θ .

1.2. Notational conventions. Let $\mathcal{M} = (M, g)$ be a compact smooth Riemannian manifold of dimension m without boundary. Subsequently, we shall relax this condition and permit M to have smooth boundary, but for the moment we work in the context of closed manifolds. Let V be a smooth vector bundle over M . We shall

usually assume that V is equipped with a smooth Hermitian fiber metric and a local frame. We let $C^\infty(V)$ denote the space of smooth sections to V .

1.3. Operators of Laplace type. Let $\vec{x} = (x^1, \dots, x^m)$ be a system of local coordinates on M . Adopt the *Einstein convention* and sum over repeated indices. A second order partial differential operator D on $C^\infty(V)$ is of *Laplace type* if

$$(1.1) \quad D = -\{g^{ij} \text{id } \partial_{x^i} \partial_{x^j} + A^k \partial_{x^k} + B\},$$

where the coefficients A^k and B are linear endomorphisms of the bundle V and the leading symbol of D is scalar and is given by the metric tensor; this condition is independent of the coordinate system and the local frame for V , which are chosen. Let $dx = \sqrt{\det(g_{ij})} dx^1 \dots dx^m$ denote the Riemannian measure on M .

The following result follows from work of Seeley, see [31]. Seeley worked with complex powers of the Laplacian, but one can use the Mellin transform to relate his results to results concerning the heat equation and obtain thereby the following result.

Theorem 1.1. *Let D be an operator of Laplace type over a compact Riemannian manifold M without boundary. The heat operator e^{-tD} is of trace class. There exist local invariants $a_n(x, D)$, which vanish if n is odd, such that*

$$(1.2) \quad \text{Tr}\{e^{-tD}\} \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M a_n(x, D) dx \quad \text{as } t \downarrow 0.$$

We will discuss these invariants in further detail in Section 3.

1.4. Elliptic complexes of Dirac type. Let $\{V^0, \dots, V^l\}$ be a finite collection of smooth vector bundles over M which are equipped with Hermitian fiber metrics. We suppose given first order partial differential operators $\alpha^p: C^\infty(V^p) \rightarrow C^\infty(V^{p+1})$ satisfying $\alpha^{p+1}\alpha^p = 0$ for $0 \leq p < l$. We say that

$$\mathcal{V} := \{\alpha^p: C^\infty(V^p) \rightarrow C^\infty(V^{p+1})\}_{0 \leq p \leq l-1}$$

is an *elliptic complex of Dirac type* if the associated second order operators of the complex, $D_{\mathcal{V}}^p := (\alpha^p)^* \alpha^p + \alpha^{p-1} (\alpha^{p-1})^*$, are of Laplace type. In this context, define the *associated cohomology groups* of the elliptic complex by setting

$$H^p(\mathcal{V}) := \frac{\ker\{\alpha^p: C^\infty(V^p) \rightarrow C^\infty(V^{p+1})\}}{\text{image}\{\alpha^{p-1}: C^\infty(V^{p-1}) \rightarrow C^\infty(V^p)\}}.$$

The Hodge decomposition theorem then identifies $H^p(\mathcal{V}) = \ker(D_{\mathcal{V}}^p)$. These vector spaces are finite dimensional and we define

$$\text{index}\{\mathcal{V}\} := \sum_{p=0}^l (-1)^p \dim\{H^p(\mathcal{V})\} = \sum_{p=0}^l (-1)^p \dim\{\ker(D_{\mathcal{V}}^p)\}.$$

The *associated heat trace invariants* of the elliptic complex are defined by

$$(1.3) \quad a_n(x, \mathcal{V}) := \sum_{p=0}^l (-1)^p a_n(x, D_{\mathcal{V}}^p).$$

A cancellation argument due to Bott shows that

$$(1.4) \quad \sum_{p=0}^l (-1)^p \dim\{\ker(D_{\mathcal{V}}^p)\} = \sum_{p=0}^l (-1)^p \text{Tr}\{e^{-tD_{\mathcal{V}}^p}\}$$

is independent of t . Consequently, equations (1.2), (1.3) and (1.4) yield

$$(1.5) \quad \int_M a_n(x, \mathcal{V}) \, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \text{index}(\mathcal{V}) & \text{if } n = m. \end{cases}$$

The invariant $a_m(x, \mathcal{V})$ is called the *local index density*. This local formalism is crucial for the geometrical index theorem for manifolds with boundary of Atiyah, Patodi and Singer, see [3]. If m is odd, then $a_m(x, \mathcal{V}) = 0$, so the index vanishes and we shall therefore usually restrict ourselves to the case where m is even.

1.5. The Chern-Gauss-Bonnet theorem. Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame for the tangent bundle TM and let $\{e^1, \dots, e^m\}$ be the dual orthonormal frame for the cotangent bundle T^*M . If $I = (i_1, \dots, i_m)$ and $J = (j_1, \dots, j_m)$ are ordered collections of m indices, we define

$$\sigma(I, J) := g(e^{i_1} \wedge \dots \wedge e^{i_m}, e^{j_1} \wedge \dots \wedge e^{j_m}).$$

This vanishes unless I and J are collections of distinct indices; in this setting $\sigma(I, J)$ is the sign of the permutation taking I to J . Let R_{ijkl} be the curvature tensor of \mathcal{M} ; we adopt the sign convention that $R_{1221} = +1$ for the unit sphere in \mathbb{R}^3 . If $m = 2\bar{m}$ even, we define the *Euler form* by setting

$$(1.6) \quad \mathcal{E}_m(x, g) := \frac{(-1)^{\bar{m}}}{8^{\bar{m}} \pi^{\bar{m}} \bar{m}!} \sum_{\substack{|I|=m \\ |J|=m}} \sigma(I, J) R_{i_1 i_2 j_1 j_2} \dots R_{i_{m-1} i_m j_{m-1} j_m}(x).$$

If m is odd, we set $\mathcal{E}_m = 0$. Let τ be the scalar curvature, let $\|\varrho\|^2$ be the square of the norm of the Ricci tensor, and let $\|R\|^2$ be the square of the norm of the full curvature tensor of \mathcal{M} . Then

$$\mathcal{E}_2 = (4\pi)^{-1}\tau \quad \text{and} \quad \mathcal{E}_4 = (32\pi^2)^{-1}\{\tau^2 - 4\|\varrho\|^2 + \|R\|^2\}.$$

We have the following result of Chern (see [9]) which gives a formula for the Euler-Poincaré characteristic $\chi(M) := \sum_p (-1)^p \dim\{H^p(M; \mathbb{R})\}$ in terms of curvature.

Theorem 1.2. *Let \mathcal{M} be a smooth compact even dimensional Riemannian manifold without boundary. Then*

$$\chi(M) = \int_M \mathcal{E}_m(x, g) \, dx.$$

1.6. The de Rham complex. The classic example of an elliptic complex of Dirac type is given by the *de Rham complex*

$$\mathcal{D}(\mathcal{M}) := \{d^p: C^\infty(\Lambda^p(M)) \rightarrow C^\infty(\Lambda^{p+1}(M))\}_{0 \leq p \leq m-1},$$

where $\Lambda^p(M) := \Lambda^p(T^*M)$ is the bundle of exterior p forms and d^p is exterior differentiation. The Hodge-de Rham theorem identifies $H^p(\mathcal{D}(\mathcal{M}))$ with the topological cohomology groups $H^p(M; \mathbb{R})$ and consequently

$$\text{index}(\mathcal{D}(\mathcal{M})) = \sum_p (-1)^p \dim\{H^p(M; \mathbb{R})\} = \chi(M).$$

Equation (1.5) gives a local formula

$$\chi(M) = \int_M a_m(x, \mathcal{D}(\mathcal{M})) \, dx.$$

McKean and Singer in [23] conjectured that the local index density $a_m(x, \mathcal{D}(\mathcal{M}))$ could be identified with the Euler form $\mathcal{E}_m(x, g)$ and verified this conjecture in dimensions $m = 2$ and $m = 4$. Their conjecture was subsequently established by Patodi (see [29]) in all dimensions; the following result of Patodi gives a heat equation proof of Theorem 1.2.

Theorem 1.3. *Let M be a compact Riemannian manifold without boundary. Then $a_m(x, \mathcal{D}(\mathcal{M})) = \mathcal{E}_m(x, g)$.*

Different proofs of Theorem 1.3 were given subsequently by Atiyah, Bott and Patodi (see [2]) using the spin-c complex and by Gilkey (see [11]) using invariance theory. Again, the subject has a lengthy history and we refer to [14] for further details.

1.7. The Witten deformation. So far, our discussion is entirely classical. We now turn to more recent history. Let $\text{ext}(\Theta): \omega \rightarrow \Theta \wedge \omega$ denote exterior multiplication and let $\text{int}(\Theta)$ be the dual, interior multiplication. Witten in [34] introduced the deformed exterior derivative setting $d_h := d + \text{ext}(dh)$, i.e., $d_h \omega = d\omega + dh \wedge \omega$; the adjoint is then $\delta_h := \delta + \text{int}(dh)$. This defines a deformed de Rham complex $\mathcal{D}(\mathcal{M})_h$. Because the deformed differential $d_h \omega = e^{-h} d(e^h \omega)$ is gauge equivalent to the exterior derivative d , it yields isomorphic cohomology groups. Witten then introduced a parameter t and examined the behaviour of d_{th} as $t \rightarrow \infty$. In this paper, we consider a generalization of this deformation. Let Θ be a closed 1-form on M . Set

$$d_\Theta := d + \text{ext}(\Theta) \quad \text{and} \quad \delta_\Theta := \delta + \text{int}(\Theta).$$

We may then compute

$$\begin{aligned} d_\Theta^2 \omega &= d(d\omega + \Theta \wedge \omega) + \Theta \wedge (d\omega + \Theta \wedge \omega) \\ &= d^2 \omega + d\Theta \wedge \omega - \Theta \wedge d\omega + \Theta \wedge d\omega + \Theta \wedge \Theta \wedge \omega = 0. \end{aligned}$$

Since we have introduced a lower order perturbation, the associated second order operators $\Delta_\Theta^p := d_\Theta^{p-1} \delta_\Theta^{p-1} + \delta_\Theta^p d_\Theta^p$ are still of Laplace type, so

$$\mathcal{D}(\mathcal{M})_\Theta := \{d_\Theta^p: C^\infty(\Lambda^p) \rightarrow C^\infty(\Lambda^{p+1})\}_{0 \leq p \leq m-1}$$

is an elliptic complex of Dirac type. We will establish the following result in Section 2, which gives some of the general properties of these cohomology groups. Many of these properties already appear in earlier papers on the topic (see [6], [27], [30]); we have provided proofs in this paper for the sake of completeness as our discussion is somewhat different than that given previously.

Let $\beta_p(M, \mathcal{D}(\mathcal{M})_\Theta) := \dim\{H^p(M, \mathcal{D}(\mathcal{M})_\Theta)\}$ be the twisted Betti numbers.

Lemma 1.4. *Let \mathcal{M} be a compact connected Riemannian manifold.*

- (1) *If $[\Theta_1] = [\Theta_2]$ in $H^1(M; \mathbb{R})$, then $\beta_p(\mathcal{D}(\mathcal{M})_{\Theta_1}) = \beta_p(\mathcal{D}(\mathcal{M})_{\Theta_2})$ for all p .*
- (2) *If $[\Theta] \neq 0$ in $H^1(M; \mathbb{R})$, then $\beta_0(M, \Theta) = 0$.*
- (3) *If M is orientable, then $\beta_p(M, \Theta) = \beta_{m-p}(M, -\Theta)$.*

- (4) Let Θ_i be closed 1-forms on compact Riemannian manifolds $\mathcal{M}_i = (M_i, g_i)$. Let $\mathcal{M} = (M_1 \times M_2, g_1 + g_2)$ be the product Riemannian manifold and let $\Theta(x^1, x^2) := \Theta_1(x^1) + \Theta_2(x^2)$ on M . Then

$$\beta_n(\mathcal{D}(\mathcal{M})_\Theta) = \sum_{p+q=n} \beta_p(\mathcal{D}(M_1)_{\Theta_1})\beta_q(\mathcal{D}(M_2)_{\Theta_2}).$$

- (5) Let M_g be the g -hole torus. If $[\Theta] \neq 0$ in $H^1(M; \mathbb{R})$, then $\beta_0(M_g) = 1$, $\beta_1(M_g) = 2g$, $\beta_2(M_g) = 1$, $\beta_0(\mathcal{D}(\mathcal{M})_\Theta) = 0$, $\beta_1(\mathcal{D}(\mathcal{M})_\Theta) = 2g - 2$, and $\beta_2(\mathcal{D}(\mathcal{M})_\Theta) = 0$.

The index of an elliptic complex is unchanged by perturbations; consequently, $\text{index}(\mathcal{D}(\mathcal{M})_\Theta) = \chi(M)$ for any Θ despite the fact that the cohomology groups can change. More is true. It is perhaps somewhat surprising that the local index density is unchanged by a Witten deformation as the following result, which we will establish in Section 4, shows.

Theorem 1.5. *Adopt the notation established above.*

- (1) $a_n(x, \mathcal{D}(\mathcal{M})_\Theta) = 0$ for $n < m$.
- (2) If m is even, then $a_m(x, \mathcal{D}(\mathcal{M})_\Theta) = \mathcal{E}_m(x, g)$ is independent of Θ .

The vanishing of a_n for $n < m$ and the independence of Θ if $n = m$ given in Theorem 1.5 is sharp. Let $\mathbb{T}^m = \mathbb{R}^m / (2\pi\mathbb{Z})^m$ be the flat cubical torus and let $0 \in \mathbb{T}^m$ be the basepoint. Since \mathbb{T}^m is flat, $a_n(x, \mathcal{D}(\mathbb{T}^m)) = 0$ for $n > 0$. We will establish the following result in Section 3.3.

Lemma 1.6. *If n is even and $n > m$, then the local formula $a_n(x, \mathcal{D}_\Theta)$ exhibits nontrivial dependence upon Θ .*

We have assumed that Θ is real in considering the Witten deformation. Should Θ be purely imaginary, we can use gauge invariance to show that the local index density is unchanged. We argue as follows. Locally, we can choose a real function h so that $\sqrt{-1}dh = \Theta$. We consider a locally defined unitary deformation to define $d_\Theta := e^{-\sqrt{-1}h}de^{\sqrt{-1}h}$; the adjoint is then given by $\delta_\Theta = \delta - \sqrt{-1}\text{int}(dh)$ and the associated Laplacian is $\Delta_\Theta = e^{-\sqrt{-1}h}\Delta e^{\sqrt{-1}h}$. Since Δ_Θ and Δ differ by a locally defined unitary gauge transformation, $a_k(x, \Delta_\Theta) = a_k(x, \Delta)$. Consequently, after taking the super-trace, the index density (and in fact all invariants $a_k(x, \mathcal{D}(\mathcal{M})_\Theta)$) are unchanged.

1.8. The Dolbeault complex. Theorem 1.5 shows the local index density is not affected by the perturbing function h in the case of the de Rham complex. It is

natural to conjecture, therefore, that this might always be the case. This is, however, not the case as we see in the following. If \mathcal{M} is a holomorphic manifold of dimension $m = 2\bar{m}$ which is equipped with a Hermitian inner product, let

$$\mathcal{C}(\mathcal{M}) := \{\sqrt{2}\bar{\partial}^q: C^\infty(\Lambda^{0,q}(\mathcal{M})) \rightarrow C^\infty(\Lambda^{0,q+1}(\mathcal{M}))\}$$

be the Dolbeault complex; the factor of $\sqrt{2}$ is introduced to ensure the resulting second order operators are of Laplace type and is inessential. Let $\text{Td}_{\bar{m}}$ be the Todd form; we refer to Hirzebruch, see [21] for details. The analogue of Theorem 1.2 in this situation is the classical Riemann-Roch formula:

Theorem 1.7. *Adopt the notation established above. Then*

$$\text{Index}(\mathcal{C}(\mathcal{M})) = \int_M \text{Td}_{\bar{m}}(x) dx.$$

Patodi in [28] generalized Theorem 1.3 to the complex setting by showing that the local index density agrees with the Todd form if the underlying geometry is Kähler; we also refer to subsequent work by Atiyah, Bott and Patodi (see [2]) and Gilkey (see [10]); again, the literature is extensive. The following result of Patodi gives a heat equation proof of Theorem 1.7 in the Kähler setting.

Theorem 1.8. *If M is a Kähler manifold, then $a_{2\bar{m}}(x, \mathcal{C}(\mathcal{M})) = \text{Td}_{\bar{m}}$.*

The assumption that M is Kähler is essential; Gilkey, Nikčević and Pohjanpelto in [16] showed that Theorem 1.8 fails in general if \mathcal{M} is not Kähler, i.e., the local index density is generically not given by the Todd form in the non-Kähler setting.

Let Θ be 1-form of type $(0, 1)$ with $\bar{\partial}\Theta = 0$. We again introduce the Witten deformation defining an elliptic complex of Dirac type

$$(1.7) \quad \mathcal{C}(\mathcal{M})_\Theta := \{\sqrt{2}(\bar{\partial}^q + \text{ext}(\Theta)): C^\infty(\Lambda^{0,q}(\mathcal{M})) \rightarrow C^\infty(\Lambda^{0,q+1}(\mathcal{M}))\}.$$

However, even in the 2-dimensional setting (which is always Kähler), the local index density does not agree with the Todd form in the perturbed setting. We will establish the following result in Section 3.4 illustrating this; we present this example to show that assertion 2 of Theorem 1.5 does not follow from some universal principle. Let $\Re(\Theta)$ be the real part of the 1-form Θ .

Lemma 1.9. *If \mathcal{M} is a Riemann surface, then $a_2(x, \mathcal{C}(\mathcal{M})_\Theta) = \tau/8\pi - \delta(\Re(\Theta))/\pi$.*

We plan to examine the deformed Dolbeault complex more generally in the Kähler context in further detail in a subsequent paper and identify more precisely the local index density for the Witten deformation in arbitrary dimensions.

1.9. The signature complex. Let M be an oriented $4k$ -dimensional manifold. Let $d + \delta: C^\infty(\Lambda^\pm(M)) \rightarrow C^\infty(\Lambda^\mp(M))$ be the Hirzebruch signature complex. We then have $d_\Theta + \delta_\Theta = d + \delta + (\text{ext} + \text{int})(\Theta)$. Now $(\text{ext} - \text{int})(\Theta): \Lambda^\pm \rightarrow \Lambda^\mp$ but $(\text{ext} + \text{int})(\Theta)$ does not have this property if $\Theta \neq 0$. So $d_\Theta + \delta_\Theta$ does not induce a map on the signature complex; it is not possible to deform the signature complex in this fashion. Similarly, the spin complex cannot be deformed in this fashion. The de Rham and Dolbeault complexes are \mathbb{Z} graded and this seems to be at the essence of the Witten deformation; the signature and spin complexes, on the other hand, are \mathbb{Z}_2 graded and this makes all the difference.

1.10. Heat trace asymptotics for manifolds with boundary. Let D be an operator of Laplace type acting on $C^\infty(V)$ over a compact Riemannian manifold with boundary. We assume the boundary ∂M to be nonempty; we must impose suitable boundary conditions to ensure D is elliptic. There is a natural connection ∇ induced by D on V (see Lemma 3.1 below). Let $\vec{\nu}$ be the inward geodesic unit normal near the boundary. We assume given an orthogonal direct sum decomposition $V = V_D \oplus V_N$ of two smooth complementary vector bundles over ∂M . Denote the corresponding orthogonal projections by π_D and π_N . Assume given a smooth endomorphism S of V_N over ∂M . If $\varphi \in C^\infty(V)$, set

$$(1.8) \quad \mathcal{B}\varphi := \{\pi_D\varphi \oplus \pi_N(\nabla_{\vec{\nu}}\varphi + S\varphi)\}|_{\partial M}.$$

In other words, we take Dirichlet boundary conditions on V_D and Robin boundary conditions on V_N . We let

$$\text{Domain}(D, \mathcal{B}) := \{\varphi \in C^\infty(V) : \mathcal{B}\varphi = 0\}.$$

We refer to Greiner (see [18]) and to Seeley (see [32]) for the proof of the following result which extends Theorem 1.1 to the setting at hand. The interior invariants $a_n(x, D)$ agree with those of Theorem 1.1 and do not reflect the boundary condition; the boundary invariants $a_l^{\text{bd}}(y, D, \mathcal{B})$ which are defined for $y \in \partial M$ are new. Again, we will discuss these invariants in more detail in Section 2. We let dy denote the Riemannian measure of the boundary.

Theorem 1.10. *Let D be an operator of Laplace type over a compact smooth Riemannian manifold with smooth boundary. Let \mathcal{B} be the boundary conditions of*

equation (1.8). The heat operator $e^{-tD_{\mathcal{B}}}$ is of trace class. There exist local invariants $a_l^{\text{bd}}(y, D, \mathcal{B})$ defined on the boundary so there is a complete asymptotic series as $t \downarrow 0$ of the form

$$\text{Tr}\{e^{-tD_{\mathcal{B}}}\} \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M a_n(x, D) dx + \sum_{l=0}^{\infty} t^{(l-(m-1))/2} \int_{\partial M} a_l^{\text{bd}}(y, D, \mathcal{B}) dy.$$

1.11. The Chern-Gauss-Bonnet theorem for manifolds with boundary. Let M be a compact Riemannian manifold with smooth boundary ∂M . Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame for $TM|_{\partial M}$ so that e_m is the inward unit normal. We define the *second fundamental form* by setting

$$L_{ab} := (\nabla_{e_a} e_b, e_m) \quad \text{for } 1 \leq a, b \leq m-1.$$

We have $\text{vol}(S^{2j-1}) = 2\pi^j/(j-1)!$ and $\text{vol}(S^{2j}) = j!\pi^j 2^{2j+1}/(2j)!$. Let indices a_i and b_j range from 1 through $m-1$. For $0 \leq 2k \leq m-1$, set

$$(1.9) \quad Q_{k,m}(y, g) := \frac{g(e_{a_1} \wedge \dots \wedge e_{a_{m-1}}, e_{b_1} \wedge \dots \wedge e_{b_{m-1}})}{(-8\pi)^k k! (m-1-2k)! \text{vol}(S^{m-1-2k})} \\ \times R(a_1, a_2, b_1, b_2) \dots R(a_{2k-1}, a_{2k}, b_{2k-1}, b_{2k}) \\ \times L(a_{2k+1}, b_{2k+1}) \dots L(a_{m-1}, b_{m-1})(y),$$

where we do not have any R terms if $2k = 0$ and we do not have any L terms if $2k = m-1$. The Chern-Gauss-Bonnet Theorem (see [9]) for manifolds with boundary then becomes:

Theorem 1.11. *Let M be a compact Riemannian manifold of dimension m with smooth boundary ∂M . Then*

$$\chi(M) = \int_M \mathcal{E}_m(x, g) dx + \int_{\partial M} \sum_{0 \leq 2k \leq m-1} Q_{k,m}(y, g) dy.$$

In low dimensions, this takes the form

$$\chi(M^2) = \frac{1}{4\pi} \int_{M^2} \tau dx + \frac{1}{2\pi} \int_{\partial M} L_{aa} dy, \\ \chi(M^3) = \frac{1}{8\pi} \int_{\partial M^3} (R_{a_1 a_2 a_2 a_1} + L_{a_1 a_1} L_{a_2 a_2} - L_{a_1 a_2} L_{a_1 a_2}) dy, \\ \chi(M^4) = \frac{1}{32\pi^2} \int_{M^4} (\tau^2 - 4\|\varrho\|^2 + \|R\|^2) dx \\ + \frac{1}{24\pi^2} \int_{\partial M^4} \{3\tau L_{aa} + 6R_{amam} L_{bb} + 6R_{acbc} L_{ab} \\ + 2L_{aa} L_{bb} L_{cc} - 6L_{ab} L_{ab} L_{cc} + 4L_{ab} L_{bc} L_{ac}\} dy.$$

1.12. The de Rham complex for manifolds with boundary. Let M be a compact Riemannian manifold with smooth boundary. Let $\vec{\nu}$ be the inward unit geodesic normal defined near the boundary; let $\vec{\nu}^*$ be the associated dual 1-form. Decompose

$$\Lambda(T^*M)|_{\partial M} = \Lambda(T^*\partial M) \oplus \vec{\nu}^* \wedge \Lambda(T^*\partial M).$$

Let π_1 be orthogonal projection on $\Lambda(T^*\partial M)$ and π_2 be orthogonal projection on $\vec{\nu}^* \wedge \Lambda(T^*\partial M)$. If i is the natural inclusion of ∂M into M , then $\pi_1\omega = i^*\omega$.

Relative boundary conditions. We define

$$C_R^\infty(\Lambda M) := \{\omega \in C^\infty(\Lambda M) : i^*\omega = 0\}.$$

Since $i^*d = di^*$, we have an elliptic complex

$$\mathcal{D}(\mathcal{M})_R := \{d : C_R^\infty(\Lambda^p(M)) \rightarrow C_R^\infty(\Lambda^{p+1}(M))\}.$$

As before, we define

$$H^p(\mathcal{D}(\mathcal{M})_R) := \frac{\ker\{d^p : C_R^\infty(\Lambda^p(M)) \rightarrow C_R^\infty(\Lambda^{p+1}(M))\}}{\text{image}\{d^{p-1} : C_R^\infty(\Lambda^{p-1}(M)) \rightarrow C_R^\infty(\Lambda^p(M))\}}.$$

The associated boundary conditions for Δ^p are of the form given in equation (1.8):

$$\mathcal{B}_R(\omega) = i^*\omega \oplus i^*(\delta\omega).$$

Let $H^p(M, \partial M; \mathbb{R})$ be the relative cohomology groups in algebraic topology. Then

$$H^p(\mathcal{D}(\mathcal{M})_R) = \ker(\Delta_{\mathcal{B}_R}^p) = H^p(M, \partial M; \mathbb{R}).$$

We will discuss this subsequently in more detail in Section 4. If M is oriented, then let \star be the Hodge operator; \star defines an isomorphism from the relative cohomology groups $H^p(M, \partial M; \mathbb{R})$ to the absolute cohomology groups $H^{m-p}(M; \mathbb{R})$. Thus

$$\chi(M) = (-1)^m \chi(M, \partial M).$$

Example 1.12. Let $M = [0, \pi]$. The Laplacian with relative boundary conditions defines Dirichlet boundary conditions on $C^\infty(\Lambda^0([0, \pi]))$ and Neumann boundary conditions on $C^\infty(\Lambda^1([0, \pi]))$. Then $\{\sin(nx)\}_{n \geq 1}$ is a spectral resolution of the Laplacian on functions and $\{\cos(nx) dx\}_{n \geq 0}$ is a spectral resolution of the Laplacian on 1-forms. Then $\ker(d) = \{0\}$ and $\text{coker}(d) = \text{Span}\{dx\}$. And these are exactly the relative cohomology groups of the interval

$$H^p([0, \pi], \partial[0, \pi]; \mathbb{R}) = \begin{cases} 0 & \text{if } p = 0, \\ \mathbb{R} & \text{if } p = 1. \end{cases}$$

Set $a_l^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_R) = \sum_p (-1)^p a_l^{\text{bd}}(t, \Delta^p, \mathcal{B}_R)$. One then obtains a local formula for the relative Poincaré characteristic:

$$(-1)^m \chi(M) = \chi(M, \partial M) = \int_M a_m(x, \mathcal{D}(\mathcal{M})) dx + \int_{\partial M} a_{m-1}^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_R) dy.$$

The interior invariant $a_m(x, \mathcal{D}(\mathcal{M}))$ vanishes if m is odd; the boundary invariants $a_{m-1}^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_R)$ are generically nonzero. We have previously identified the invariants $a_m(x, \mathcal{D}(\mathcal{M}))$ with the geometrical invariants $\mathcal{E}_m(x, g)$ (where we set $\mathcal{E}_m = 0$ if m is odd). Gilkey in [14] gave a heat equation proof of the Chern-Gauss-Bonnet Theorem by identifying the associated boundary invariants with the boundary integrands of the Chern-Gauss-Bonnet Theorem.

Theorem 1.13. *Adopt the notation given above. Then*

$$a_{m-1}^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_R) = (-1)^m \sum_{0 \leq 2k \leq m-1} Q_{k,m}(y, g).$$

We have $i^*(\Theta \wedge \omega) = d(i^*h) \wedge i^*\omega$ so $d_{\Theta} : C_R^\infty(\Lambda^p(M)) \rightarrow C_R^\infty(\Lambda^{p+1}(M))$ and we obtain a deformed elliptic complex $\mathcal{D}(\mathcal{M})_{R,\Theta}$ with associated second order boundary conditions for the Laplacian

$$\mathcal{B}_{R,\Theta}(\omega) = i^*\omega \oplus i^*(\delta_{\Theta}\omega).$$

Theorem 1.5 extends to this setting to become:

Theorem 1.14. *Adopt the notation established above.*

- (1) $a_l^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_{R,\Theta}) = 0$ for $l < m - 1$.
- (2) $a_{m-1}^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_{R,\Theta}) = (-1)^m \sum_{0 \leq l \leq m-1} Q_{l,m}(y, g)$ is independent of Θ .

Absolute boundary conditions. We suppose M is orientable and define

$$C_A^\infty(\Lambda M) := \{\omega \in C^\infty(\Lambda M) : i^*(\star\omega) = 0\}.$$

Since $\delta = \pm \star d \star$, we have a dual elliptic complex

$$\begin{aligned} C_A^\infty(\Lambda M) &:= \{\omega \in C^\infty(\Lambda M) : i^*(\star\omega) = 0\}, \\ \mathcal{D}(\mathcal{M})_A &:= \{\delta : C_A^\infty(\Lambda^p M) \rightarrow C_A^\infty(\Lambda^{p-1}(M))\}. \end{aligned}$$

The cohomology groups $H^p(\mathcal{D}(\mathcal{M})_A)$ are defined analogously; set $\mathcal{B}_A(\omega) := \mathcal{B}_R(\star\omega)$. We then have

$$H^p(\mathcal{D}(\mathcal{M})_A) = \ker(\Delta_{\mathcal{B}_A}^p) = H^p(M; \mathbb{R}).$$

Theorem 1.13 extends immediately to this situation to yield

$$(1.10) \quad a_{m-1}^{\text{bd}}(y, \mathcal{D}(\mathcal{M})_A) = \sum_{0 \leq l \leq m-1} Q_{l,m}(y, g).$$

We note that all the theory is local; thus the existence of a global orientation is irrelevant; absolute boundary conditions can be defined invariantly and equation (1.10) continues to hold. We use δ_Θ to define the Witten deformation in this instance and the appropriate results go thru.

1.13. Lefschetz fixed-point formulas. Let \mathcal{M} be a compact Riemannian manifold without boundary. Let $\mathcal{T}: M \rightarrow M$ be a smooth map, not necessarily an isometry. The pull-back $\mathcal{T}^*: C^\infty(\Lambda^p M) \rightarrow C^\infty(\Lambda^p M)$ satisfies $\mathcal{T}^* \circ d = d \circ \mathcal{T}^*$. Assume $\mathcal{T}^* \Theta = \Theta$, so $\mathcal{T}^* \circ \text{ext}(\Theta) = \text{ext}(\Theta) \circ \mathcal{T}^*$. Consequently, \mathcal{T}^* induces a chain map of \mathcal{D}_Θ and defines an action \mathcal{T}^* on the cohomology groups $H^p(\mathcal{D}_\Theta)$ with associated *Lefschetz number*

$$\mathcal{L}(\mathcal{T}, \Theta) := \sum_{p=0}^m (-1)^p \text{Tr}\{\mathcal{T}^*: H^p(\mathcal{D}_\Theta) \rightarrow H^p(\mathcal{D}_\Theta)\}.$$

The cancellation argument of Bott then yields

$$\mathcal{L}(\mathcal{T}, \mathcal{D}_\Theta) = \sum_{p=0}^m (-1)^p \text{Tr}\{\mathcal{T}^* e^{-t\Delta_\Theta^p}\}.$$

We assume that the fixed point set of \mathcal{T} consists of the finite disjoint union of smooth submanifolds N_μ of dimension m_μ ; this is automatic, of course, if \mathcal{T} is an isometry. Define the *normal bundle* $\nu(N_\mu) := \{T(M)/T(N_\mu)\}|_{N_\mu}$. Since $d\mathcal{T}$ preserves $T(N_\mu)$, there is an induced map $d\mathcal{T}_\nu$ on the normal bundle; we assume as a non-degeneracy condition that $\det(d\mathcal{T}_\nu - \text{id}) \neq 0$, i.e., that there are no infinitesimal directions left fixed by $d\mathcal{T}_\nu$. We note that if \mathcal{T} is an isometry, this non-degeneracy condition is automatically satisfied. The following result follows from results proved independently by Gilkey in [13] and Lee in [22].

Lemma 1.15. *Assume that \mathcal{T} is non-degenerate and that $\mathcal{T}^* \Theta = \Theta$. There exist local invariants $a_n(x_\mu, \mathcal{T}, \mathcal{D}_\Theta)$, which are called the local Lefschetz density, such that*

$$\mathcal{L}(\mathcal{T}, \mathcal{D}_\Theta) \sim \sum_{\mu} \sum_{n=0}^{\infty} t^{(n-m_\mu)/2} \int_{N_\mu} a_{n,m_\mu}(x_\mu, \mathcal{T}, \mathcal{D}_\Theta) dx_\mu.$$

We define $\nu(N)$ abstractly as the quotient of $T(M)|_N$ by the subbundle $T(N)$. Let $d\mathcal{T}: TM|_N \rightarrow TM|_N$ be the tangent map. Since \mathcal{T} preserves N , it preserves $T(N)$ and thus there is an induced map $d\mathcal{T}_\nu$ of $\nu(N)$ to itself. We have assumed that $\det(d\mathcal{T}_\nu - \text{id}) \neq 0$. Thus, we may identify $\nu(N)$ with the span of the generalized eigenvectors of $d\mathcal{T}$ corresponding to eigenvalues other than $+1$ to regard $\nu(N)$ as a subbundle of $T(M)|_N$. This defines a canonical decomposition

$$(1.11) \quad TM|_N = TN \oplus \nu(N).$$

Gilkey and Lee generalized Theorem 1.3 as follows:

Theorem 1.16. *Assume that $\mathcal{T}: M \rightarrow M$ is non-degenerate, that the submanifolds N_μ are totally geodesic, and that the decomposition of equation (1.11) is orthogonal. Then*

- (1) $a_{n,m_\mu}(x_\mu, \mathcal{T}, \mathcal{D}) = 0$ on N_μ if $n < m_\mu$;
- (2) $a_{m_\mu,m_\mu}(x_\mu, \mathcal{T}, \mathcal{D}) = \text{sign}(\det(\text{id} - d\mathcal{T}_\nu))\mathcal{E}_{m_\mu}(x, g_\mu)$.

This gives a heat equation proof of the classical formula

$$\mathcal{L}(\mathcal{T}, \mathcal{D}) = \sum_{\mu} \text{sign}(\det(\text{id} - d\mathcal{T}_\nu))\chi(N_\nu).$$

In fact, the assumption that the submanifolds N_μ are totally geodesic in Theorem 1.16 is unnecessary as the following more general result shows.

Theorem 1.17. *Assume that $\mathcal{T}: M \rightarrow M$ is non-degenerate, that the decomposition of equation (1.11) is orthogonal, and that $\mathcal{T}^*\Theta = \Theta$. Then*

- (1) $a_{n,m_\mu}(x_\mu, \mathcal{T}, \mathcal{D}_\Theta) = 0$ on N_μ if $n < m_\mu$;
- (2) $a_{m_\mu,m_\mu}(x_\mu, \mathcal{T}, \mathcal{D}_\Theta) = \text{sign}(\det(\text{id} - d\mathcal{T}_\nu))\mathcal{E}_{m_\mu}(x, g_\mu)$.

This then shows that $\mathcal{L}(\mathcal{T}, \mathcal{D}_\Theta)$ is independent of Θ ; this is perhaps by no means obvious, because the cohomology groups $H^p(\mathcal{D}_\Theta)$ need not be independent of Θ .

2. THE COHOMOLOGY GROUPS OF THE PERTURBED DE RHAM COMPLEX

In this section, we establish the basic properties of the cohomology groups $\mathcal{D}(\mathcal{M})_\Theta$ by proving Lemma 1.4.

2.1. The proof of Lemma 1.4 (1): Cohomological invariance. Let \mathcal{M} be a compact Riemannian manifold without boundary of dimension m . Suppose that

$[\Theta_1] = [\Theta_2]$ in $H^1(M; \mathbb{R})$, i.e., that $\Theta_2 - \Theta_1 = dh$ for some $h \in C^\infty(M)$. Let $\Psi_h(\omega) := e^h \omega$. Then

$$d_{\Theta_1}(\Psi_h(\omega)) = d(e^h \omega) + e^h \Theta_1 \wedge \omega = e^h(d\omega + dh \wedge \omega + \Theta_1 \wedge \omega) = \Psi_h(d_{\Theta_2}(\omega)).$$

Thus, Ψ_h is a chain map intertwining $\mathcal{D}(\mathcal{M})_{\Theta_1}$ and $\mathcal{D}(\mathcal{M})_{\Theta_2}$ and consequently Ψ_h intertwines the two cohomology groups. Assertion 1 of Lemma 1.4 follows. \square

2.2. The proof of Lemma 1.4 (2): The cohomology group H^0 . This result was first established by Braverman and Farber (see [6], Corollary 1.6); they used a spectral sequence result to show the triviality of the 0th Novikov number if $[\Theta] \neq 0$. We give a different derivation to keep our treatment as self-contained as possible. Suppose that there exists $0 \neq f \in H^0(\mathcal{D}(\mathcal{M})_\Theta)$, where $[\Theta] \neq 0$ in $H^1(M; \mathbb{R})$. We argue for a contradiction. By replacing f by $-f$ if need be, we can assume f is positive somewhere. Let $\{\mathcal{O}_n\}$ be an open cover of M by small geodesic balls. Such balls are geodesically convex and hence contractable. Since $d\Theta = 0$, we can express $\Theta = dh_n$ on \mathcal{O}_n . On \mathcal{O}_n , $d(e^{h_n} f) = e^{h_n} \{df + f dh_n\} = e^{h_n} \{df + f\Theta\} = 0$. Thus $e^{h_n} f = c_n$ is constant or, equivalently, $f = c_n e^{-h_n}$. Thus either f is positive on \mathcal{O}_n , or f is negative on \mathcal{O}_n , or f vanishes identically on \mathcal{O}_n . Since M is assumed connected and f is positive at some point, we conclude f is positive globally so we can write $f = e^h$ for $h = \ln(f)$. We then have $df + f\Theta = e^h \{dh + \Theta\} = 0$ and consequently $\Theta = d(-h)$ is trivial in $H^1(M; \mathbb{R})$, contrary to our assumption. \square

2.3. The proof of Lemma 1.4 (3): Poincaré duality. We generalize the usual proof of Poincaré duality to the setting at hand; we refer, for example, to Gilkey (see [14]) for further details concerning Clifford algebras. Let $\text{Clif}(M)$ be the Clifford algebra of M . Let orn be the orientation form. Then Clifford multiplication $c(\text{orn})$ defines an isometry from $\Lambda^p(M)$ to $\Lambda^{m-p}(M)$ with $c(\text{orn})^2 = (-1)^{m(m+1)/2} \text{id}$. We have

$$c(\text{orn}) \circ d = (-1)^{m-1} \delta \circ c(\text{orn}), \quad c(\text{orn}) \circ \delta = (-1)^{m-1} d \circ c(\text{orn}).$$

Thus $c(\text{orn})$ induces an isomorphism from $\ker(\Delta_{\mathcal{D}(\mathcal{M})}^p)$ to $\ker(\Delta_{\mathcal{D}(\mathcal{M})}^{m-p})$ which is called *Poincaré duality*. Clifford multiplication by a covector is defined by setting $c(\xi) = \text{ext}(\xi) - \text{int}(\xi)$;

$$c(\xi) \circ c(\text{orn}) = (-1)^{m-1} c(\text{orn}) \circ c(\xi),$$

which we disentangle to obtain

$$\text{ext}(\xi)c(\text{orn}) = (-1)^m c(\text{orn}) \text{int}(\xi) \quad \text{and} \quad \text{int}(\xi)c(\text{orn}) = (-1)^m c(\text{orn}) \text{ext}(\xi).$$

Consequently,

$$\begin{aligned} d_{\Theta} \circ c(\text{orn}) &= (d + \text{ext}(\Theta)) \circ c(\text{orn}) = (-1)^{m-1} c(\text{orn}) \circ (\delta - \text{int}(\Theta)) \\ &= (-1)^{m-1} c(\text{orn}) \circ \delta_{-\Theta}, \\ \delta_{\Theta} \circ c(\text{orn}) &= (\delta + \text{int}(\Theta)) \circ c(\text{orn}) = (-1)^{m-1} c(\text{orn}) \circ (d - \text{ext}(\Theta)) \\ &= (-1)^{m-1} c(\text{orn}) \circ d_{-\Theta}. \end{aligned}$$

We can now conclude $c(\text{orn}): \ker(d_{\Theta} + \delta_{\Theta}) \rightarrow \ker(d_{-\Theta} + \delta_{-\Theta})$ and assertion 3 of Lemma 1.4 follows. \square

Example 2.1. The following simple example will be useful in the proof that we shall give subsequently in Section 3.3 of Lemma 1.6. Let $M = S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle with the usual flat metric. Let $\Theta = \theta dx$. We identify 1 with dx to express $d_{\Theta} = \partial_x + \theta$ and $\delta_{\Theta} = -\partial_x + \theta$. Thus

$$(2.1) \quad \Delta_{\Theta}^0 = \delta_{\Theta} d_{\Theta} = -\partial_x^2 - \partial_x \theta + \theta^2 \quad \text{and} \quad \Delta_{\Theta}^1 = d_{\Theta} \delta_{\Theta} = -\partial_x^2 + \partial_x \theta + \theta^2.$$

Thus, interchanging 1 and dx does not interchange the two Laplacians; we must also change the sign of θ .

2.4. The proof of Lemma 1.4 (4): The Künneth formula. Let \mathcal{V}_i be elliptic complexes over compact Riemannian manifolds $\mathcal{M}_i = (M_i, g_i)$ of dimension m_i without boundary. Let $\mathcal{M} = (M_1 \times M_2, g_1 + g_2)$ be the product Riemannian manifold. Let $\mathcal{V}_i := \{\alpha_i^n: C^\infty(V_i^n) \rightarrow C^\infty(V_i^{n+1})\}$ be elliptic complexes of Dirac type over M_i . Define $\mathcal{V} := \mathcal{V}_1 \otimes \mathcal{V}_2$ over \mathcal{M} by setting

$$(2.2) \quad V^n = \bigoplus_{p+q=n} V_1^p \otimes V_2^q \quad \text{and} \quad \alpha^n = \bigoplus_{p+q=n} \alpha_1^p \otimes \text{id}^q + (-1)^p \text{id}^p \otimes \alpha_2^q.$$

It is then a simple algebraic exercise to show:

Lemma 2.2. *Let \mathcal{V}_i be elliptic complexes of Dirac type over Riemannian manifolds \mathcal{M}_i for $i = 1, 2$. Use equation (2.2) to define $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ over $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$.*

- (1) \mathcal{V} is an elliptic complex of Dirac type.
- (2) $\Delta_{\mathcal{V}}^n = \bigoplus_{p+q=n} \{\Delta_{\mathcal{V}_1}^p \otimes \text{id} + \text{id} \otimes \Delta_{\mathcal{V}_2}^q\}$.
- (3) $H^n(\mathcal{V}) = \bigoplus_{p+q=n} H^p(\mathcal{V}_1) \otimes H^q(\mathcal{V}_2)$.
- (4) $\text{Tr}_{L^2}\{e^{-t\Delta_{\mathcal{V}}^n}\} = \sum_{p+q=n} \text{Tr}_{L^2}\{e^{-t\Delta_{\mathcal{V}_1}^p}\} \text{Tr}_{L^2}\{e^{-t\Delta_{\mathcal{V}_2}^q}\}$.
- (5) $\text{index}(\mathcal{V}) = \text{index}(\mathcal{V}_1) \text{index}(\mathcal{V}_2)$.
- (6) $a_n((x_1, x_2), \mathcal{V}) = \sum_{p+q=n} a_p(x_1, \mathcal{V}_1) a_q(x_2, \mathcal{V}_2)$.

The de Rham complex decomposes in this fashion; given closed 1-forms Θ_i over M_i , we have $\mathcal{D}(\mathcal{M})_{\Theta_1+\Theta_2} = \mathcal{D}(\mathcal{M})_{\Theta_1} \otimes \mathcal{D}(\mathcal{M})_{\Theta_2}$. Assertion 4 of Lemma 1.4 now follows. \square

2.5. The proof of Lemma 1.4 (5): An example. Let \mathcal{M}_g be the g -hole torus. Then $\chi(\mathcal{M}_g) = 2 - 2g$. Choose a closed 1-form Θ which is nonzero in $H^1(M; \mathbb{R})$. Then $-\Theta$ is also nonzero in $H^1(M; \mathbb{R})$. Thus, by assertion (2.3), $H^0(M_g, \mathcal{D}(\mathcal{M})_{\Theta}) = 0$ and $H^2(M_g, \mathcal{D}(\mathcal{M})_{\Theta}) \approx H^0(M_g, \mathcal{D}(\mathcal{M})_{-\Theta}) = 0$. Since the index is unchanged by lower order perturbations,

$$2 - 2g = \chi(M) = \chi(\mathcal{D}(\mathcal{M})_{\Theta}) = \beta_0(\mathcal{D}(\mathcal{M})_{\Theta}) + \beta_2(\mathcal{D}(\mathcal{M})_{\Theta}) - \beta_1(\mathcal{D}(\mathcal{M})_{\Theta}).$$

Consequently, $\beta_1(\mathcal{D}(\mathcal{M})_{\Theta}) = 2g - 2$. \square

3. LOCAL INVARIANTS OF THE HEAT TRACE

Since the heat trace asymptotics are at the heart of our analysis, it is worth reviewing some well-known results concerning them. Although they have a long history, the first modern treatment of these invariants in a quite general setting is given by Seeley (see [31]); we also refer to [15] and the accompanying bibliography for further discussion as the literature in question is a vast one.

3.1. A covariant formalism. We adopt the notation of equation (1.1). If ∇ is a connection on V and if φ is a smooth section to V , let $\varphi_{;ij}$ be the components of $\nabla^2\varphi$. If E is an auxiliary endomorphism of V , we can define

$$D(g, \nabla, E)f := -(g^{ij} f_{;ij} + Ef).$$

It is then immediate that $D(g, \nabla, E)$ is an operator of Laplace type on V . The following observation (see [15], Lemma 4.1.1) shows that any operator of Laplace type can be put in this form. Let Γ_{ij}^k be the Christoffel symbols of the Levi-Civita connection.

Lemma 3.1. *Let $D = -\{g^{ij} \text{id } \partial_{x^i} \partial_{x^j} + A^k \partial_{x^k} + B\}$ be an operator of Laplace type on V . Then there exists a unique connection $\nabla = \nabla(D)$ on V and a unique endomorphism $E = E(D)$ of V such that $D = D(g, \nabla, E)$. If ω is the connection 1-form of ∇ , then*

$$\omega_i = \frac{1}{2} g_{ij} (A^j + g^{kl} \Gamma_{kl}^j \text{id}), \quad E = B - g^{ij} (\partial_{x^i} \omega_j + \omega_i \omega_j - \omega_k \Gamma_{ij}^k \text{id}).$$

Example 3.2. If $\Delta^p = (d\delta + \delta d)^p$ is the Laplacian of the de Rham complex acting on p -forms, then the associated connection ∇ is the Levi-Civita connection and the associated endomorphism E is given by the Weitzenböck formulas; in particular, it is linear in the curvature tensor. The formalism of Lemma 3.1 is the Bochner formalism that expresses the true Laplacian in terms of the rough Laplacian and curvature.

3.2. The weight. We assign weight 2 to the endomorphism E , to the components R_{ijkl} of the curvature tensor of the Levi-Civita connection, and to the components of the curvature Ω_{ij} of the auxiliary connection ∇ on V . We increase the weight by 1 for every explicit covariant derivative which appears. We sum over repeated indices relative to a local orthonormal frame for TM .

Lemma 3.3. *Let $D = D(g, \nabla, E)$ be an operator of Laplace type on a compact Riemannian manifold M .*

- (1) $a_n(x, D)$ is a polynomial of total weight n in the covariant derivatives of the components of the tensors $\{E, R_{ijkl}, \Omega_{ij}\}$.
- (2) $a_0(x, D) = (4\pi)^{-m/2} \text{Tr}\{\text{id}\}$.
- (3) $a_2(x, D) = (4\pi)^{-m/2} \frac{1}{6} \text{Tr}\{6E + \tau \text{id}\}$.
- (4) $a_4(x, D) = (4\pi)^{-m/2} \frac{1}{360} \text{Tr}\{60E_{,kk} + 60\tau E + 180E^2 + 12\tau_{,kk} \text{id} + 5\tau^2 \text{id} - 2\|\varrho\|^2 \text{id} + 2\|R\|^2 \text{id} + 30\Omega_{ij}\Omega_{ij}\}$.
- (5) $a_{2n}(x, D) = ((4\pi)^{m/2} 2^{n+1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1))^{-1} \times \text{Tr}\{(8n+4)E + 2n\tau \text{id}\}_{;k_1 k_1 \dots k_{n-1} k_{n-1}} + \text{lower order terms}$.

Proof. Seeley in [31] gave an explicit recursive formalism for computing the local invariants $a_n(x, D)$. It follows from this formalism that a_n vanishes for n odd and that these invariants are homogeneous of weight n if n is even. We use Lemma 3.1 to express the symbol of D in terms of $\{g, \nabla, E\}$. We refer to [15] and the references cited as the subject has an extensive history; we note that similarly explicit formulas exist for a_6 and a_8 ; as they become much more complicated, we shall not exhibit them in the interests of brevity. \square

3.3. The proof of Lemma 1.6. In proving Lemma 1.6, it suffices to show that the local formula $a_n(x, \mathcal{D}_\Theta) - a_n(x, D)$ does not vanish identically. We first study the case $m = 1$.

Lemma 3.4. *Fix a basepoint x_0 in the circle $\mathcal{M} := (\mathbb{R}/2\pi\mathbb{Z}, dx^2)$. Let $\Theta = \theta dx$.*

- (1) $a_2(x, \mathcal{D}(\mathcal{M})_\Theta) = -1/(\sqrt{\pi})\delta\Theta$.
- (2) *If $k > 1$, then $a_{2k}(x_0, \mathcal{D}(\mathcal{M})_\Theta) = c_k \theta^{(2n-1)} + \text{terms involving derivatives of order less than } 2k - 1$ for some nonzero constant c_k .*

Proof. Let $\Theta = \theta dx$. We use equation (2.1) to express $\Delta_\Theta^0 = -(\partial_x^2 + \partial_x \theta - \theta^2)$ and $\Delta_\Theta^1 = -(\partial_x^2 - \partial_x \theta - \theta^2)$. The formalism of Lemma 3.1 then yields

$$E(\Delta_\Theta^0) = \partial_x \theta - \theta^2 \quad \text{and} \quad E(\Delta_\Theta^1) = -\partial_x \theta - \theta^2,$$

since there are no first order terms. The scalar curvature τ of the circle vanishes. Consequently, by Lemma 3.3,

$$a_2(x, \Delta_\Theta^0) = \frac{1}{\sqrt{4\pi}}(\partial_x \theta - \theta^2) \quad \text{and} \quad a_2(x, \Delta_\Theta^1) = \frac{1}{\sqrt{4\pi}}(-\partial_x \theta - \theta^2).$$

Consequently,

$$a_2(x, \mathcal{D}(\mathcal{M})_\Theta) = \frac{1}{\sqrt{\pi}}\partial_x \theta = -\frac{1}{\sqrt{\pi}}\delta\Theta.$$

This proves assertion (1). By Lemma 3.3 (5), we obtain $a_{2k}(x, \mathcal{D}(\mathcal{M})_\Theta) = c_k \theta^{(2k-1)}$ + lower order terms, where $c_k \neq 0$. \square

Case 1: Suppose $m = 2l + 1$ is odd. Let $x \in S^1$ and $\xi \in S^{2l}$. We shall consider a specific example to show the local formula $a_{2k}(x, \mathcal{D}_\Theta) - a_{2k}(x, \mathcal{D})$ does not vanish identically and thus is generically nonzero. We take $\mathcal{M}_1 := (S^1, dx^2)$, and $\mathcal{M}_2 := (S^{2l}, g_0)$, where g_0 is the round metric. Let

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 = (S^1 \times S^{2l}, dx^2 + g_0) \quad \text{and} \quad \Theta = \theta dx.$$

Let $2k \geq 2 + 2l > \dim\{S^1 \times S^{2l}\}$. We apply Lemma 2.2 (6) and the calculation performed above to $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ to see

$$(3.1) \quad a_{2k}((x, \xi), \mathcal{D}(\mathcal{M})_\Theta) = \sum_{2p+2q=2k} a_{2p}(x, \mathcal{D}(\mathcal{M})_\Theta) a_{2q}(\xi, \mathcal{D}(\mathcal{M})).$$

Since (S^{2l}, g_0) is homogeneous, $a_k(\xi, \mathcal{D}_{S^{2l}})$ is constant. We therefore use equation (1.5) and the fact that $\chi(S^{2l}) = 2$ to see that

$$(3.2) \quad a_{2a}(\xi, \mathcal{D}_{S^{2l}}) = \begin{cases} 0 & \text{if } 2a \neq 2l, \\ 2 \operatorname{vol}(S^{2l-1})^{-1} & \text{if } 2a = 2l. \end{cases}$$

We use equations (3.1) and (3.2) to conclude therefore

$$a_{2k}((x, \xi), \mathcal{M}_\Theta) = 2 \operatorname{vol}(S^{2l})^{-1} a_{2k-2l}(x, \mathcal{D}(S^1)_\Theta).$$

Lemma 3.4 now shows $a_{2k}((x_0, \xi), \mathcal{M}_\Theta)$ exhibits nontrivial dependence on Θ ; the same follows for any Riemannian manifold of dimension $2l + 1$.

Case 2: Suppose $m = 2l + 2$ is even. We first suppose $l = 0$ and $m = 2$. We take $\mathbb{T}^2 := (S^1, dx) \times (S^1, dy)$ to be the product flat cubical torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$. Let $\Theta = \theta_1 dx + \theta_2 dy$. We apply Lemma 2.2 and the computations performed above to see that the coefficient of $\theta_1^{(2a-1)}\theta_2^{(2b-1)}$ in $a_{2a+2b}((x, y), \mathcal{D}(\mathbb{T}^2)_\Theta)$ is nonzero provided that $2a \geq 2$ and $2b \geq 2$. Hence, a_{2n} exhibits nontrivial dependence on Θ if $2n \geq 4$. Taking products as in the odd dimensional case with S^{2l} then yields that a_{2n} exhibits nontrivial dependence on Θ if $2n \geq 4 + 2l = \dim\{\mathcal{M}\} + 2$. This completes the proof of Lemma 1.6. \square

3.4. The Dolbeault complex on the torus. We use essentially the same argument to study the Dolbeault complex on the torus and establish Lemma 1.9 that we used to study the de Rham complex on the circle. The major additional complexity is notational; we must use a Hermitian inner product rather than an orthogonal inner product and be careful as a result with taking complex conjugation. Let $M = \mathbb{C}/\mathbb{Z}^2$ be the square torus with the standard Euclidean metric and complex structure. Then $\Lambda^{0,0}(M) = \text{Span}\{1\}$ and $\Lambda^{0,1}(M) = \text{Span}\{d\bar{z}\}$. Since $d\bar{z}$ has length $\sqrt{2}$, we use instead 1 and $(1/\sqrt{2})d\bar{z}$ as our basis. Let $\Theta = \theta d\bar{z}$. We adopt the notation of equation (1.7) to define the deformed Dolbeault complex $\mathcal{C}(\mathcal{M})_\Theta$. Since we are in dimension 2, $\mathcal{C}(\mathcal{M})_\Theta$ becomes

$$A: C^\infty(\Lambda^{0,0}(M)) \rightarrow C^\infty(\Lambda^{0,1}(M)),$$

where

$$Af := \sqrt{2}(\partial_{\bar{z}}f + \theta f) d\bar{z} = 2(\partial_{\bar{z}}f + \theta)(d\bar{z}/\sqrt{2}).$$

We have that $A = 2(\partial_{\bar{z}} + \theta)$ and $A^* = 2(-\partial_z + \bar{\theta})$. Consequently,

$$\begin{aligned} \Delta^{(0,0)} &= A^*A = -4\{\partial_z\partial_{\bar{z}} + \theta\partial_z - \bar{\theta}\partial_{\bar{z}} - |\theta|^2 + \partial_z\theta\}, \\ \Delta^{(0,1)} &= AA^* = -4\{\partial_z\partial_{\bar{z}} + \theta\partial_z - \bar{\theta}\partial_{\bar{z}} - |\theta|^2 - \partial_{\bar{z}}\bar{\theta}\}. \end{aligned}$$

We have $-4\partial_z\partial_{\bar{z}} = -(\partial_x^2 + \partial_y^2)$, so these are operators of Laplace type. We use the formalism of Lemma 3.1 to compute $E(\Delta^{(0,0)})$ and $E(\Delta^{(0,1)})$. All the terms but $\partial_z\theta$ and $\partial_{\bar{z}}\bar{\theta}$ cancel out when we take the supertrace, so by Lemma 3.3

$$a_2(z, \mathcal{C}_\Theta) = \frac{4}{4\pi}\{\partial_{\bar{z}}\bar{\theta} + \partial_z\theta\}.$$

We have $\delta\Theta = \delta(\theta dx - \sqrt{-1}\theta dy) = -\{\partial_x\theta - \sqrt{-1}\partial_y\theta\} = -2\partial_z\theta$. Consequently,

$$-\frac{4}{4\pi}\delta(\Re(\Theta)) = -\frac{2}{4\pi}\delta(\Theta + \bar{\Theta}) = \frac{4}{4\pi}(\partial_{\bar{z}}\bar{\theta} + \partial_z\theta).$$

More generally, if M is a Riemann surface, the curvature of the metric on M will enter. Since we can always choose holomorphic coordinates, so the 1-jets of the metric vanish at the point in question (every Riemann surface is Kähler), there are no cross terms, so

$$a_2(x, \mathcal{C}_\Theta) = -\frac{\delta(\Re(\Theta))}{\pi} + c\tau,$$

where c is a universal constant and τ is the scalar curvature. Taking $\mathcal{M} = S^2$ with the standard structures yields the index is 1 and hence $c = 1/(8\pi)$. \square

3.5. Heat trace asymptotics for manifolds with boundary. Let M be a compact Riemannian manifold with smooth nonempty boundary ∂M . Let D be an operator of Laplace type on M ; we impose the boundary conditions given in equation (1.8). We refer to Gilkey (see [15]) for the proof of the following result; it is based on the work of many authors; we shall content ourselves by giving formulas for a_0 , a_1 and a_2 . Formulas are available for a_3 and a_4 . We let indices a, b range from 1 to $m-1$ and index an orthonormal frame for the tangent bundle; the index ‘ m ’ is the inward unit geodesic normal. We let ‘ \cdot ’ indicate tangential covariant differentiation.

Theorem 3.5. *Adopt the notation given above. Let $\psi = \pi_N - \pi_D$.*

- (1) $a_0^{\text{bd}}(y, D, \mathcal{B}) = \frac{1}{4}(4\pi)^{-(m-1)/2} \text{Tr}\{\psi\}$.
- (2) $a_1^{\text{bd}}(y, D, \mathcal{B}) = \frac{1}{6}(4\pi)^{-m/2} \text{Tr}\{2L_{aa} + 12S\}$.
- (3) $a_2^{\text{bd}}(y, D, \mathcal{B}) = \frac{1}{384}(4\pi)^{-(m-1)/2} \text{Tr}\{96\psi E + 16\psi\tau + 8\psi R_{amam} + (13\pi_N - 7\pi_D)L_{aa}L_{bb} + (2\pi_N + 10\pi_D)L_{ab}L_{ab} + 96SL_{aa} + 192S^2 - 12\psi_{:a}\psi_{:a}\}$.

4. AN AXIOMATIC CHARACTERIZATION OF THE EULER FORM

In this section, we will follow the original treatment of [11] as it seems the most simple; this is a coordinate formalism. We note, however, that there does exist a covariant formalism based on Weyl’s (see [33]) second theorem of invariants; we refer to Gilkey, Park and Sekigawa (see [17]) for details.

4.1. Local invariants. Let $\mathcal{P}_{m,n}$ be the finite dimensional vector space of all invariants of weight n (see Section 3.2) in the components of the covariant derivatives of the curvature tensor of the Levi-Civita connection and the components of the covariant derivatives of the closed 1-form Θ ; the structure group here is $O(m)$ and these spaces are trivial for n odd. Weyl’s First Theorem of Orthogonal Invariants is applicable, and a spanning set can be constructed using contractions of indices.

Let $\{e_1, \dots, e_m\}$ be a local orthonormal frame for TM . We assign weight 1 to the components Θ_i of the 1-form Θ and increase the weight by 1 for every explicit covariant derivative which appears. We have, for example,

$$\begin{aligned} \mathcal{P}_{m,0} &= \text{Span}\{1\}, \\ \mathcal{P}_{m,2} &= \text{Span}\{\Theta_i\Theta_i, \delta\Theta = -\Theta_{i;i}, \tau = R_{ijji}\}, \\ \mathcal{P}_{m,4} &= \text{Span}\{\Theta_{i;ijj}, \Theta_{i;j}\Theta_{i;j}, \Theta_{i;i}\Theta_{j;j}, \Theta_{i;j}\Theta_i\Theta_j, \Theta_i\Theta_i\Theta_j\Theta_j, \Theta_i\tau_{;i}, \\ &\quad \Theta_i\Theta_j\varrho_{ij}, \Theta_i\Theta_i\tau, \tau_{;ii}, \tau^2, \|\varrho\|^2, \|R\|^2\}. \end{aligned}$$

The elements given above are a basis for $\mathcal{P}_{m,2}$ if $m \geq 2$; however, we must impose the relation $\tau = 0$ if $m = 1$. Similarly, we must impose the relation $\tau^2 - 4\|\varrho\|^2 + \|R\|^2 = 0$ to obtain a basis for $\mathcal{P}_{m,4}$ if $m = 3$.

4.2. The restriction map. We define the restriction r of such a contraction of indices from dimension m to dimension $m - 1$ by restricting the range of summation from 1 to m to from 1 to $m - 1$.

Lemma 4.1. *The map r from $\mathcal{P}_{m,n}$ onto $\mathcal{P}_{m,n-1}$ is well defined.*

Proof. Weyl's theorem of invariants as described above will yield trivially that r is surjective once we know r is well defined. However, the components of the covariant derivatives of the curvature tensor and of Θ are not algebraically independent; there are relations given by the various curvature identities. For example, we have

$$R_{ijji} = -R_{ijij} \quad \text{and} \quad \Theta_{i;kl}\tau_{;ikl} = \Theta_{i;lk}\tau_{;ikl} + R_{ijkl}\Theta_l\tau_{;ikl}.$$

Thus, there is no natural basis to choose for $\mathcal{P}_{m,n}$ and it is not a priori obvious that restricting the range of summation is independent of the representation of a given invariant in terms of summations of indices.

To get around this difficulty, we adopt a different formalism which is not covariant. Fix m for the moment. Fix a point x_0 of M and choose a system of local coordinates $X := (x^1, \dots, x^m)$ on M centered at x_0 . Let $g_{ij/k} := \partial_{x^k}\{g_{ij}\}$. We normalize the choice of local coordinates so that $g_{ij}(X, g)(x_0) = \delta_{ij}$ is the Kronecker symbol and so that $g_{ij/k}(X, g)(x_0) = 0$. Apart from that we impose no further normalizations. Let $\alpha = (\alpha(1), \dots, \alpha(m))$ be a multi-index, where $\alpha(i)$ are nonnegative integers. Set $|\alpha| = \alpha(1) + \dots + \alpha(m)$ and $\partial_x^\alpha = \partial_{x^1}^{\alpha(1)} \dots \partial_{x^m}^{\alpha(m)}$. We expand $\Theta = \Theta_i dx^i$ and $ds^2 = g_{ij} dx^i dx^j$. We introduce formal variables $\Theta_{i/\beta}$ for $\partial_x^\beta \Theta_i$ and $g_{ij/\alpha}$ for $\partial_x^\alpha g_{ij}$, where α and β are multi-indexes. We set $g_{ij/\alpha} = g_{ji/\alpha}$ but otherwise introduce no relations other than

$$g_{ij} = \delta_{ij}, \quad g_{ij/\alpha} = 0 \quad \text{for } |\alpha| = 1 \text{ and } g_{ij/\alpha} = g_{ji/\alpha}.$$

Let $\tilde{\mathcal{P}}_m$ be the polynomial algebra in the formal variables $\{g_{ij/\alpha}, \Theta_{i/\beta}\}$. If $P \in \tilde{\mathcal{P}}_m$ and if X is a system of local coordinates centered at x_0 satisfying the normalizations $g_{ij}(X, g)(x_0) = \delta_{ij}$ and $g_{ij/\alpha}(X, g)(x_0) = 0$ if $|\alpha| = 1$, then we can evaluate $P(X, g, \Theta)(x_0)$; we say that P is *invariant* if $P(X, g, \Theta)(x_0) = P(g, \Theta)(x_0)$ is independent of the particular local coordinate system X which was chosen. To be consistent with the notation established in Section 3.2, we define

$$(4.1) \quad \text{order}\{\Theta_{i/\beta}\} := |\beta| + 1 \quad \text{and} \quad \text{order}\{g_{ij/\alpha}\} := |\alpha|.$$

For example, the curvature tensor R_{ijkl} has order 2 when expressed in terms of the derivatives of the metric; it is linear in the 2-jets and quadratic in the 1-jets of the metric. Let $\tilde{\mathcal{P}}_{m,n}$ be the finite dimensional vector space of all invariant polynomials in these variables which are homogeneous of total order n .

By expressing covariant differentiation in terms of the Christoffel symbols and then using the Koszul formula, we can express the covariant derivatives of the curvature and of Θ in terms of ordinary partial derivatives. Conversely, of course, we were to restrict to geodesic coordinates, then we could express the partial derivatives of the curvature and of Θ covariantly. Thus, we can identify $\mathcal{P}_{m,n}$ and $\tilde{\mathcal{P}}_{m,n}$.

The variables $\Theta_{i/\beta}$ and $g_{ij/\alpha}$ are algebraically independent; Taylor's Theorem shows that there are no "hidden" relations such as there are with the covariant derivatives of the curvature tensor and of Θ . To count the number of times an index μ appears in a variable, we set

$$\text{deg}_\mu(\Theta_{i/\beta}) = \delta_{i\mu} + \beta(\mu) \quad \text{and} \quad \text{deg}_\mu(g_{ij/\alpha}) = \delta_{i\mu} + \delta_{j\mu} + \alpha(\mu).$$

The restriction map $\tilde{r}: \tilde{\mathcal{P}}_{m,n} \rightarrow \tilde{\mathcal{P}}_{m-1,n}$ is then well defined and induced by the algebraic map defined on the generators by

$$\tilde{r}(\Theta_{i/\beta}) = \begin{cases} \Theta_{i/\beta} & \text{if } \text{deg}_m(\Theta_{i/\beta}) = 0, \\ 0 & \text{if } \text{deg}_m(\Theta_{i/\beta}) > 0, \end{cases} \quad \tilde{r}(g_{ij/\alpha}) = \begin{cases} g_{ij/\alpha} & \text{if } \text{deg}_m(g_{ij/\alpha}) = 0, \\ 0 & \text{if } \text{deg}_m(g_{ij/\alpha}) > 0. \end{cases}$$

Since the map \tilde{r} on $\tilde{\mathcal{P}}_{m,n}$ is well defined and since \tilde{r} agrees with r under the identification of $\tilde{\mathcal{P}}_{m,n}$ with $\mathcal{P}_{m,n}$, we obtain that r is well defined as well. \square

4.3. The kernel of the restriction map. To simplify the notation, we shall identify $r = \tilde{r}$ and $\tilde{\mathcal{P}}_{m,n} = \mathcal{P}_{m,n}$ henceforth. Let $\mathcal{Q}_{m,n}$ be the linear subspace of $\mathcal{P}_{m,n}$ generated by the $g_{ij/\alpha}$ variables, i.e., where we set $\Theta = 0$. We refer to [11] for the proof of the following result characterizing the Euler form defined in equation (1.6); it was central to the heat equation proof of the Chern-Gauss-Bonnet theorem given there.

Lemma 4.2. *Let n be even.*

- (1) *If $n < m$, then $\ker(r: \mathcal{Q}_{m,n} \rightarrow \mathcal{Q}_{m-1,n}) = \{0\}$.*
- (2) *If $n = m$, then $\ker(r: \mathcal{Q}_{m,n} \rightarrow \mathcal{Q}_{m-1,n}) = \text{Span}\{\mathcal{E}_m\}$.*

Example 4.3. In algebraic topology, the Euler class associates to each m dimensional oriented vector bundle V an element of $H^m(M)$; this class is “unstable“ in the sense that the class vanishes if V decomposes in the form $V = W \oplus 1$, i.e., if V admits a global non-vanishing section. This is, of course, closely related to the fact that $\chi(M) = 0$ if M admits a global non-vanishing vector field. Lemma 4.2 is the reflection of this “instability” on the algebraic level; the Euler form vanishes if the Riemannian metric is locally an isometric product with a flat factor.

We generalize Lemma 4.2 to the setting at hand.

Lemma 4.4.

- (1) *If $2k < m$, then $\ker(r: \mathcal{P}_{m,2k} \rightarrow \mathcal{P}_{m-1,2k}) = \{0\}$.*
- (2) *If $2k = m$, then $\ker(r: \mathcal{P}_{m,2k} \rightarrow \mathcal{P}_{m-1,2k}) = \text{Span}\{\mathcal{E}_m\}$.*

Proof. Let $0 \neq P \in \mathcal{P}_{m,2k}$. Assume that $r(P) = 0$. Let

$$A = g_{i_1 j_1 / \alpha_1} \cdots g_{i_l j_l / \alpha_l} \Theta_{k_1 / \beta_1} \cdots \Theta_{k_p / \beta_p}$$

be a monomial of P . There need not be any $g_{ij/\alpha}$ variables and in this instance, we take $l = 0$. Similarly, there need not be any $\Theta_{k/\beta}$ variables and in this instance, we take $p = 0$. Then $r(A) = 0$, so $\deg_m(A) \neq 0$. Since this is true for every monomial of P , we can permute the coordinate indices to conclude $\deg_i(A) \neq 0$ for $1 \leq i \leq m$. By replacing x^i by $-x^i$, we conclude $\deg_i(A)$ is even and hence $\deg_i(A) \geq 2$ for $1 \leq i \leq m$. We normalized the coordinate system, so $g_{ij}(x_0) = \delta_{ij}$ and $g_{ij/k}(x_0) = 0$. Consequently, $2 \leq |\alpha_i|$ for all i . Using the definition of order given in equation (4.1), we have

$$(4.2) \quad 2l \leq \sum_{i=1}^l |\alpha_i| \quad \text{and} \quad 2k = \text{order}(A) = \sum_{i=1}^l |\alpha_i| + \sum_{j=1}^p (1 + |\beta_j|).$$

Since $\deg_\mu(A) \geq 2$ for $1 \leq \mu \leq m$, we can apply equation (4.2) to estimate

$$(4.3) \quad \begin{aligned} 2m &\leq \sum_{\mu=1}^m \deg_\mu(A) = 2l + \sum_{i=1}^l |\alpha_i| + \sum_{j=1}^p (1 + |\beta_j|) \\ &\leq \sum_{i=1}^l 2|\alpha_i| + 2 \sum_{j=1}^p (1 + |\beta_j|) = 4k. \end{aligned}$$

We conclude therefore that $2k \geq m$ or, equivalently, $r(P) = 0$ and $2k < m$ implies $P = 0$. This proves assertion 1. If $m = 2k$, then all the inequalities must have been equalities. Consequently,

$$\sum_{j=1}^p (1 + |\beta_p|) = 2 \sum_{j=1}^p (1 + |\beta_p|)$$

and thus $p = 0$. This implies that $P \in \mathcal{Q}_{m,n}$; assertion (2) now follows from Lemma 4.2. \square

4.4. An extension to manifolds with boundary. We generalize Lemma 4.2 to the setting of manifolds with boundary to obtain an axiomatic characterization of the invariants $Q_{k,m}$ given in equation (1.9). Our first step is to normalize the coordinates suitably.

Lemma 4.5. *Fix a point y_0 of the boundary. Let indices a, b and c range from 1 through $m - 1$. We can choose local coordinates $\vec{y} = (y^1, \dots, y^m)$ centered at y_0 , where $y^m \geq 0$, so that the conditions are satisfied.*

- (1) $\partial M = \{\vec{y}: y^m = 0\}$ and ∂_{y^m} is the inward unit normal on the boundary.
- (2) $g_{ab}(y_0) = \delta_{ab}$ and $g_{ab/c}(y_0) = 0$.
- (3) $g_{mm} \equiv 1$, $g_{am} \equiv 0$, and $L_{ab}(y_0) = -\frac{1}{2}g_{ab/m}(y_0)$.

Proof. Choose coordinates (y^1, \dots, y^{m-1}) on the boundary which are centered at y_0 so that $g_{ab}(y_0) = \delta_{ab}$ and so that $g_{ab/c}(y_0) = 0$. Let ν be the inward unit normal vector field on the boundary. Define local coordinates (y^1, \dots, y^m) near the boundary by

$$(y^1, \dots, y^m) \rightarrow \exp_{(y^1, \dots, y^{m-1})}(y^m \nu(y^1, \dots, y^{m-1})) \quad \text{for } y^m \geq 0.$$

It is then immediate that $M = \{\vec{y}: y^m \geq 0\}$ and $\partial M = \{\vec{y}: y^m = 0\}$. These coordinates are characterized by the fact that $t \rightarrow (y^1, \dots, y^{m-1}, t)$ are unit speed geodesics in M starting at $(y^1, \dots, y^{m-1}, 0)$ normal to the boundary when $t = 0$. The normalizations of the Lemma now follow. \square

The index m is distinguished. Let indices $\{a, b, c, d\}$ range from 1 through $m-1$ and index the coordinate frame $\{\partial_{y^1}, \dots, \partial_{y^{m-1}}\}$ for the tangent bundle of the boundary. Let $\mathcal{P}_{m,n}^{\text{bd}}$ be the finite dimensional vector space of all invariants in the components of the covariant derivatives of the auxiliary closed 1-form Θ , of the covariant derivatives of the curvature tensor of the Levi-Civita connection, of the components of the curvature tensor of the Levi-Civita connection, and of the components of the tangential

covariant derivatives (with respect to the Levi-Civita connection of the boundary) of the second fundamental form; the structure group is $O(m-1)$ and these spaces are nonzero even if n is odd (see equation (4.4) below). The spaces $\tilde{\mathcal{P}}_{m,n}^{\text{bd}}$ are defined similarly by using the variables $g_{ij/\alpha}$ and $\Theta_{i/\beta}$ subject to the normalizations of Lemma 4.5 and we may identify, as before, $\mathcal{P}_{m,n}^{\text{bd}}$ with $\tilde{\mathcal{P}}_{m,n}^{\text{bd}}$. We construct invariants by summing tangential indices. We have, for example,

$$(4.4) \quad \mathcal{P}_{m,1}^{\text{bd}} = \text{Span}\{L_{aa}, \Theta_m\}.$$

There is a small amount of technical fuss involved in defining the restriction map $r: \mathcal{P}_{m,n}^{\text{bd}} \rightarrow \mathcal{P}_{m-1,n}^{\text{bd}}$. We have used the last index m for the normal coordinate. So we set $r(A) = 0$ if A involves the index '1' and otherwise define $r(A)$ by shifting each index $i \rightarrow i-1$ for $i > 1$. Thus, for example,

$$r(\Theta_{1/3}) = 0, \quad r(\Theta_{2/3}) = \Theta_{1/2}, \quad r(g_{12/34}) = 0, \quad r(g_{23/34}) = g_{12/23}$$

and the like. Geometrically, the restriction $r(A)$ applied to a structure (M_1, \cdot) is simply A applied to the Cartesian product structure $(S^1 \times M_1, \cdot)$, where the structures are flat in the S^1 direction. The map r is well defined and surjective. We define $\mathcal{Q}_{m,n}^{\text{bd}} \subset \mathcal{P}_{m,n}^{\text{bd}}$ as before by setting $\Theta = 0$. Lemma 4.4 extends to this situation; we refer to Gilkey (see [12]) for the proof of the following result.

Lemma 4.6. *Let n be arbitrary.*

- (1) *If $n < m-1$, then $\ker\{r: \mathcal{Q}_{m,n}^{\text{bd}} \rightarrow \mathcal{Q}_{m-1,n}^{\text{bd}}\} = \{0\}$.*
- (2) *If $n = m-1$, then $\ker\{r: \mathcal{Q}_{m,n}^{\text{bd}} \rightarrow \mathcal{Q}_{m-1,n}^{\text{bd}}\} = \text{Span}\{Q_{k,m}\}$.*

The appropriate generalization of Lemma 4.4 to this setting becomes:

Lemma 4.7. *Let n be arbitrary.*

- (1) *If $n < m-1$, then $\ker\{r: \mathcal{P}_{m,n}^{\text{bd}} \rightarrow \mathcal{P}_{m-1,n}^{\text{bd}}\} = \{0\}$.*
- (2) *If $n = m-1$, then $\ker\{r: \mathcal{P}_{m,n}^{\text{bd}} \rightarrow \mathcal{P}_{m-1,n}^{\text{bd}}\} = \text{Span}\{Q_{k,m}\}$.*

Proof. We give essentially the same proof as that given to prove Lemma 4.4. Suppose that $0 \neq P \in \mathcal{P}_{m,n}^{\text{bd}}$ satisfies $r(P) = 0$.

$$A = g_{a_1 b_1 / \alpha_1} \cdots g_{a_l b_l / \alpha_l} \Theta_{k_1 / \beta_1} \cdots \Theta_{k_p / \beta_p} g_{c_1 d_1 / m} \cdots g_{c_q d_q / m}$$

be a monomial of P . The α_μ must satisfy $|\alpha_\mu| \geq 2$, since the components of the second fundamental form $g_{ab/m}$ are the only (possibly) nonzero first derivatives of

the metric tensor at y_0 . Since $r(P) = 0$, we have $\deg_a(A) \geq 2$ for $1 \leq a \leq m - 1$. Equation (4.2) generalizes to become

$$2l \leq \sum_{i=1}^l |\alpha_i|, \quad n = \text{order}(A) = \sum_{i=1}^l |\alpha_i| + \sum_{j=1}^p (1 + |\beta_j|) + q.$$

Since we are not including the index m in the sum, some of the equalities of equation (4.3) may become inequalities and we obtain

$$\begin{aligned} 2(m-1) &\leq \sum_{\mu=2}^m \deg_\mu(A) \leq 2l + \sum_{i=1}^l |\alpha_i| + \sum_{j=1}^p (1 + |\beta_j|) + 2q \\ &\leq \sum_{i=1}^l 2|\alpha_i| + 2 \sum_{j=1}^p (1 + |\beta_j|) + 2q = 2n. \end{aligned}$$

Consequently, if $n < m - 1$, $P = 0$ while if $n = m - 1$, then all of the inequalities must have been equalities. We conclude as before that $p = 0$, so $P \in \mathcal{Q}_{m,m-1}^{\text{bd}}$ and the desired conclusion follows from Lemma 4.6. \square

4.5. An equivariant extension. Let \mathcal{M} be a compact Riemannian manifold without boundary. Let \mathcal{T} be a smooth map of M to itself with $\mathcal{T}^*\Theta = \Theta$. Let N be one component of the fixed point set of \mathcal{T} . We assume the Riemannian metric is chosen, so that the splitting of equation (1.11) is orthogonal. Let $s := \dim(N)$. Let indices a, b and c range from 1 through s , let indices u, v and w range from $s + 1$ through m , and let indices i, j and k range from 1 through m .

Lemma 4.8. *Let N be a component of the fixed point set of \mathcal{T} of dimension s . Assume $\det(I - d\mathcal{T}_\nu) \neq 0$ on N . Let g be a Riemannian metric so $T(N) \perp_g \nu(N)$. Fix a point P of N . There exist local coordinates (\vec{x}, \vec{y}) defined near P , where $\vec{x} = (x^1, \dots, x^s)$ and $\vec{y} = (y^{s+1}, \dots, y^m)$, such that*

- (1) $g(\partial_{x^a}, \partial_{x^b})(P) = \delta_{ab}$ and $\partial_{x^c} g(\partial_{x^a}, \partial_{x^b})(P) = 0$.
- (2) $N = \{(\vec{x}, \vec{y}) : \vec{y} = 0\}$ and $\nu(N) = \text{Span}\{\partial_{y^{s+1}}, \dots, \partial_{y^m}\}|_N$.
- (3) $g(\partial_{x^a}, \partial_{y^u})|_N = 0$ and $g(\partial_{y^u}, \partial_{y^v})|_N = \delta_{uv}$.
- (4) $d\mathcal{T}|_N = \begin{pmatrix} \text{id} & 0 \\ 0 & d\mathcal{T}_\nu \end{pmatrix}$.
- (5) Expand $\mathcal{T}(\vec{x}, \vec{y}) = (\mathcal{T}^1(\vec{x}, \vec{y}), \dots, \mathcal{T}^m(\vec{x}, \vec{y}))$. Then

$$\partial_{x^b} \mathcal{T}^a(\vec{x}, 0) = \delta_b^a, \quad \partial_{y^u} \mathcal{T}^a(\vec{x}, 0) = 0 \quad \text{and} \quad \partial_{x^b} \mathcal{T}^u(\vec{x}, 0) = 0.$$

Proof. Choose local coordinates $\vec{x} = (x^1, \dots, x^s)$ for N near P so that the normalizations of assertion 1 hold. Let $\{e_{s+1}, \dots, e_m\}$ be a local orthonormal frame for $\nu(N)$ near P . The map $(\vec{x}, \vec{y}) \rightarrow \exp_{\vec{x}}(y^u e_u)$ defines local coordinates on M near P so that the remaining assertions hold. \square

We normalize the coordinate system henceforth so that Lemma 4.8 holds but we impose no other conditions. Modulo a normalizing constant of $-\frac{1}{2}$, the $g_{ab/u}$ variables give the components of the second fundamental form. Let ${}_{/}\alpha_1, \alpha_2 := \partial_x^{\alpha_1} \partial_y^{\alpha_2}$, where α_i are suitable multi-indices. Taking into account the normalizations of Lemma 4.8 we introduce variables for the remaining derivatives of the metric, of Θ , and of \mathcal{T} (taking into account the fact that one index is up) and define the order by setting:

- (1) $g_{ab/u}$; $\text{order}(g_{ab/u}) = 1$. These variables are tensorial.
- (2) $g_{uv/w}$; $\text{order}(g_{uv/w}) = 1$.
- (3) $g_{au/v}$; $\text{order}(g_{au/v}) = 1$.
- (4) $g_{ij/\alpha_1, \alpha_2}$ for $|\alpha_1| + |\alpha_2| \geq 2$; $\text{order}(g_{ij/\alpha_1, \alpha_2}) = |\alpha_1| + |\alpha_2|$.
- (5) $\Theta_{i/\alpha_1, \alpha_2}$; $\text{order}(\Theta_{i/\alpha_1, \alpha_2}) = 1 + |\alpha_1| + |\alpha_2|$.
- (6) $\mathcal{T}_v^u := \partial_{y^v} \mathcal{T}^u$; $\text{order}(\mathcal{T}_v^u) = 0$. These variables are tensorial and give $d\mathcal{T}_\nu$.
- (7) $\mathcal{T}_{/\alpha_1, \alpha_2}^a$ for $|\alpha_2| \geq 2$; $\text{order}(\mathcal{T}_{/\alpha_1, \alpha_2}^a) = |\alpha_1| + |\alpha_2| - 1$.
- (8) $\mathcal{T}_{/\alpha_1, \alpha_2}^u$ for $|\alpha_1| + |\alpha_2| \geq 2$ and $|\alpha_2| \geq 1$; $\text{order}(\mathcal{T}_{/\alpha_1, \alpha_2}^u) = |\alpha_1| + |\alpha_2| - 1$.

A word of explanation is in order. We have ruled out $g_{ab/c}$ since we normalized the coordinate system, so this vanishes at P ; hence this variable is not present. We have also ruled out $g_{au/b}$ and $g_{uv/a}$ since $g_{au} \equiv 0$ and $g_{uv} \equiv \delta_{uv}$ on N . In considering the $\mathcal{T}_{/\alpha_1, \alpha_2}^a$ variables, we require $|\alpha_2| \geq 2$ since $\mathcal{T}^a(x, 0) \equiv x^a$ and $\partial_{y^u} \mathcal{T}^a(x, 0) \equiv 0$ on N . Finally, in considering $\mathcal{T}_{/\alpha_1, \alpha_2}^u$, we assume $|\alpha_1| + |\alpha_2| \geq 2$ and $|\alpha_2| \geq 1$ since we have already introduced the \mathcal{T}_v^u variables and since $\partial_{x^a} \mathcal{T}^u \equiv 0$ on N . As was done previously, deg_μ counts the number of times an index μ appears in the variables given above.

Lemma 4.9.

- (1) $\sum_{c=1}^s \text{deg}_c(g_{ab/u}) = 2 \text{order}(g_{ab/u})$.
- (2) $\sum_{c=1}^s \text{deg}_c(g_{uv/w}) < 2 \text{order}(g_{uv/w})$ and $\sum_{c=1}^s \text{deg}_c(g_{au/v}) < 2 \text{order}(g_{au/v})$.
- (3) $\sum_{c=1}^s \text{deg}_c(g_{ij/\alpha_1, \alpha_2}) \leq 2 \text{order}(g_{ij/\alpha_1, \alpha_2})$. *If equality holds, then this variable contains no normal indices and $|\alpha_1| = 2$.*
- (4) $\sum_{c=1}^s \text{deg}_c(\Theta_{i/\alpha_1, \alpha_2}) < 2 \text{order}(\Theta_{i/\alpha_1, \alpha_2})$.
- (5) $\sum_{c=1}^s \text{deg}_c(\mathcal{T}_{/\alpha_1, \alpha_2}^a) < 2 \text{order}(\mathcal{T}_{/\alpha_1, \alpha_2}^a)$ and $\sum_{c=1}^s \text{deg}_c(\mathcal{T}_{/\alpha_1, \alpha_2}^u) < 2 \text{order}(\mathcal{T}_{/\alpha_1, \alpha_2}^u)$.

Proof. We note $\text{ord}(g_{ij/k}) = 1$. We prove assertions 1 and 2 by computing:

$$\begin{aligned}\sum_{c=1}^s \deg_c(g_{ab/u}) &= 2 = 2 \text{ order}(g_{ab/u}), \\ \sum_{c=1}^s \deg_c(g_{uv/w}) &= 0 < 2 = 2 \text{ order}(g_{ab/u}), \\ \sum_{c=1}^s \deg_c(g_{av/w}) &= 1 < 2 = 2 \text{ order}(g_{av/w}).\end{aligned}$$

Since $|\alpha_1| + |\alpha_2| \geq 2$, we may compute

$$\sum_{c=1}^s \deg_c(g_{ij/\alpha_1, \alpha_2}) \leq 2 + |\alpha_1| \leq |\alpha_1| + |\alpha_2| + |\alpha_1| \leq 2(|\alpha_1| + |\alpha_2|) = 2 \text{ order}(g_{ij/\alpha_1, \alpha_2}).$$

If equality holds, i and j are tangential indices, $|\alpha_1| = 2$ and $|\alpha_2| = 0$. Assertion 3 follows. To prove assertion 4, we compute:

$$\sum_{c=1}^s \deg_c(\Theta_{i/\alpha_1, \alpha_2}) \leq 1 + |\alpha_1| < 2 + 2|\alpha_1| + 2|\alpha_2| = 2 \text{ order}(\Theta_{i/\alpha_1, \alpha_2}).$$

We use the normalizations of (7) and (8) above to complete the proof by computing:

$$\begin{aligned}\sum_{c=1}^s \deg_c(\mathcal{T}_{\alpha_1, \alpha_2}^a) &= |\alpha_1| + 1 \leq |\alpha_1| + |\alpha_2| - 1 < 2(|\alpha_1| + |\alpha_2| - 1) = 2 \text{ order}(\mathcal{T}_{\alpha_1, \alpha_2}^a), \\ \sum_{c=1}^s \deg_c(\mathcal{T}_{\alpha_1, \alpha_2}^u) &= |\alpha_1| \leq |\alpha_1| + |\alpha_2| - 1 < 2(|\alpha_1| + |\alpha_2| - 1) = 2 \text{ order}(\mathcal{T}_{\alpha_1, \alpha_2}^u).\end{aligned}$$

□

We let $\mathfrak{P}_{n,m,s}$ be the space of all invariants in these variables of total order n with coefficients which are smooth functions of the \mathcal{T}_u^v variables with the proviso that $\det(\text{id} - \mathcal{T}_v^u) \neq 0$. We define $r: \mathfrak{P}_{n,m,s} \rightarrow \mathfrak{P}_{n,m,s-1}$ as before.

Lemma 4.10. *Let $0 \neq P \in \ker\{r: \mathfrak{P}_{m,n,s} \rightarrow \mathfrak{P}_{m,n,s-1}\}$. Then $n \geq s$. If $n = s$, then $P = P(d\mathcal{T}_\nu, g_{ab/cd}, g_{ab/u})$. If $\int_N P(d\mathcal{T}, g)(x) dx$ is independent of g modulo the requirement that $T(M)|_N \perp_g \nu(N)$, then $P = P(d\mathcal{T}_\nu, g_{ab/cd})$.*

Proof. Again, we count. Let $0 \neq P \in \mathfrak{P}_{m,n,s}$ satisfy $r(P) = 0$. Let A be a monomial of P . Then $\deg_a(A) \geq 2$ for $1 \leq a \leq s$. We apply Lemma 4.9 to estimate

$$2s \leq \sum_{a=1}^s \deg_a(A) \leq 2 \text{ order}(A) = 2n$$

and consequently $n \geq s$. In the limiting case, none of the inequalities in these equations could be strict. This implies, in particular, that $\Theta_{i/\alpha_1, \alpha_2}$ and $\mathcal{T}_{/\alpha_1, \alpha_2}^i$ variables do not appear. Furthermore, when considering the $g_{ij/\alpha_1, \alpha_2}$ and $g_{ij/k}$ variables, P depends on the variables given.

We work purely locally. Suppose $0 \neq P \in \ker\{r: \mathfrak{P}_{m,n,n} \rightarrow \mathfrak{P}_{m,n,n-1}\}$ but that we may decompose $P = g_{ab/u}^k A_0 + \dots$ for $k > 0$, where $A_0(x_0, 0, g) \neq 0$ and we have deleted the lower terms in the variable $g_{ab/u}$. We suppose A_0 does not vanish as a local formula and perturb the metric g so $c := A_0(x_0, 0, g) \neq 0$. Let $\varphi(x, y) = \varphi_1(x)\varphi_2(y)$, where $\varphi_1(x)$ is a mesa function which is identically 1 near the point $(x_0, 0)$ and has small compact support and let φ_2 be a similar mesa function identically 1 near $y = 0$. Let $dx^a \odot dx^b$ denote the symmetric tensor product. We let $g_\varepsilon := g + \varepsilon\varphi_1(x)\varphi_2(y)cy^u dx^a \odot dx^b$. By Lemma 4.8, the only variable which is effected by the perturbation is $g_{ab/u, \varepsilon}(x, 0) = g_{ab} + \varepsilon c\varphi_1(x)$ and thus $P(x, g_\varepsilon) = \varepsilon^k c^k \varphi_1(x)A_0(x, g) + \dots$, where we have omitted lower order terms in ε . Since the integral is independent of ε , we have

$$0 = \int_N A_0(d\mathcal{T}, g)(x)\varphi_1(x)\{A_0(d\mathcal{T}, g)(x_0)\}^k dx.$$

Since by hypothesis $A_0(d\mathcal{T}, g)(x_0) \neq 0$ and since $\varphi_1(x)$ is nonnegative, has small support, and is nonzero at x_0 , this is a contradiction. \square

5. THE LOCAL INDEX DENSITY

There is a geometric definition of the restriction map r which is useful. Let $\mathcal{M}_1 = (M_1, g_1, \Theta_1)$ be a Riemannian manifold of dimension $m-1$ without boundary, equipped with a closed 1-form Θ_1 . Let $\mathcal{M}_2 = (S^1, dx^2, \Theta_2 = 0)$ be the circle with the usual metric and vanishing 1-form Θ_2 . We form

$$\mathcal{M} := \mathcal{M}_1 \times \mathcal{M}_2 = (M_1 \times S^1, g + dx^2, \Theta_1).$$

Let $P \in \mathcal{P}_{m,n}$. It is then immediate that

$$r(P)(\mathcal{M}_1)(x_1) = P(\mathcal{M}_1 \times \mathcal{M}_2)(x_1, x_2)$$

for any point x_2 of the circle; the particular point in question does not matter since \mathcal{M}_2 is homogeneous. Let $a_{m,n}(x, \mathcal{D}_\Theta)$ be the invariants of the heat equation for the perturbed de Rham complex. Since $(S^1, dx^2, 0)$ is flat, $a_{m,n}$ vanishes on the circle. Consequently, we may use assertion 6 of Lemma 2.2 to conclude that

$a_{m,n}((x_1, x_2), \mathcal{D}_\Theta) = 0$ and consequently $a_{m,n}(\cdot, \mathcal{D}_\Theta) \in \mathcal{P}_{m,n}$ satisfies $r(a_{m,n}) = 0$. We therefore deduce that $a_{m,n}(\cdot, \mathcal{D}_\Theta) = 0$ if $n < m$ by assertion 1 of Lemma 4.4. This establishes the first assertion of Theorem 1.5. In the limiting case, where $n = m$, we have $a_{m,m}$ is independent of Θ by assertion 2 of Lemma 4.4. We may therefore take $\Theta = 0$ and derive the second assertion of Theorem 1.5 from Theorem 1.3. This completes the proof of Theorem 1.5. The derivation of Theorem 1.13 from Theorem 1.11 using Lemma 4.7 is analogous and is therefore omitted; the factor of $(-1)^m$ arises from the fact that we are using Poincaré duality to interchange $H^p(M; \mathbb{R})$ (which is defined by absolute boundary conditions) and $H^{m-p}(M, \partial M; \mathbb{R})$ (which is defined by relative boundary conditions) so that $\chi(M, \partial M) = (-1)^m \chi(M)$. Similarly, the derivation of Theorem 1.17 from Theorem 1.16 follows directly from Lemma 4.10. \square

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