Czechoslovak Mathematical Journal, 71 (146) (2021), 641–654

RADIAL MINKOWSKI ADDITIVE OPERATORS

Lewen Ji, Nanchang

Received August 12, 2019. Published online April 14, 2021.

Abstract. We give some characterizations for radial Minkowski additive operators and prove a new characterization of balls. Finally, we show the property of radial Minkowski homomorphism.

Keywords: characterization; radial Minkowski additive operator; radial Minkowski homomorphism

MSC 2020: 52A40, 52A20

1. Introduction

The classical Brunn-Minkowski theory originated from Brunn's doctoral thesis in 1887 and Minkowski's work in 1901; the real valued valuations are at the center of Brunn-Minkowski theory. In the 1930s, Blaschke started a systematic investigation, and then Hadwiger obtained the famous Hadwiger's characterization theorem. The Hadwiger's characterization theorem provides the connection between rigid motion invariant set functions and symmetric polynomials (see [4]) for further results and generalizations, see [1], [2], [3], [5], [20], [29]. The following Minkowski endomorphism was introduced by Schneider in [21].

Definition 1.1 ([21]). The map $\Phi\colon \mathcal{K}^n \to \mathcal{K}^n$ is called a *Minkowski endomorphism* if it satisfies the following conditions:

(i) Φ is continuous with respect to Hausdorff metric;

(ii) Φ is Minkowski additive,

$$\Phi(K+L) = \Phi(K) + \Phi(L) \quad \text{for all } K, L \in \mathcal{K}^n;$$

The research has been supported by the National Natural Science Foundation of China (No. 71762001), the Science and Technology Project of Education Department of Jiangxi Province (No. GJJ180414) and East China University of Technology Research Foundation for Advanced Talents (No. DHBK2018050).

DOI: 10.21136/CMJ.2021.0366-19

(iii) Φ is $SO(n)$ equivariant:

$$\Phi(TK) = T\Phi(K) \quad \text{for all } T \in SO(n) \text{ and for all } K \in \mathcal{K}^n,$$

where $\mathcal{K}^n$ is the the family of convex bodies (nonempty, compact, convex sets in $\mathbb{R}^n$). A convex body K is uniquely determined by its support function $h_K(u) = \max\{\langle x, u\rangle\colon x \in K\}$ (see [22]), here $\langle x, u\rangle$ is the inner product of x and u.

The author in [21] established a complete classification of Minkowski endomorphism in the case when $n = 2$. Subsequently, Kiderlen in [14] proved the important convolution representation (see Theorem 1.1) with respect to Minkowski endomorphism. For more results regarding Minkowski endomorphism, see [6], [7], [11], [21], [31].

Theorem 1.1 ([14]). *Let Φ be a Minkowski endomorphism. Then there exists a unique zonal distribution $\mu \in C_o^{-\infty}(S^{n-1})$ of order at most 2, such that*

$$h_{\Phi(K)} = h_K * \mu,$$

where $C_o^{-\infty}(S^{n-1})$ is the space of distributions vanishing on the restriction of linear functions to the sphere and $h_K * \mu$ denotes the spherical convolution of the support function with the distribution μ. Moreover, Φ is uniformly continuous if and only if μ is a signed Borel measure.

In 1975, Lutwak in [18] introduced the dual Brunn-Minkowski theory, which played a very important role in the solution of the Busemann-Petty problem, see [8], [9], [10], [30]. In the dual theory, convex bodies are replaced by star bodies, support functions are replaced by radial functions, and the addition of star bodies is the radial sum defined by $K \widetilde{+} L = \{x \widetilde{+} y\colon x \in K, y \in L\}$, where $x \widetilde{+} y$ is defined to be the usual vector sum of the points x and y, if both of them are contained in a line through origin, and 0 otherwise. Valuations on convex bodies had been extended to star bodies; Klain in [16] obtained the classification theorem for star-shaped bodies which is a dual analogue of Hadwiger's characterization theorem of the Minkowski mixed volumes. For more results, see [12], [13], [15], [17], [19], [26], [27], [28].

In this paper, we consider the radial Minkowski additive. Let $\mathcal{S}^n$ be the set of star bodies in $\mathbb{R}^n$, Φ be a map from $\mathcal{S}^n$ into some abelian group. Map Φ is called *radial Minkowski additive map* if

$$\Phi(K \widetilde{+} L) = \Phi(K) \widetilde{+} \Phi(L),$$

where $K, L \in \mathcal{S}^n$. Since $(K \cap L) \widetilde{+} (K \cup L) = K \widetilde{+} L$, the radial Minkowski additive map Φ satisfies the relation

$$\Phi(K \cap L) \widetilde{+} \Phi(K \cup L) = \Phi(K) \widetilde{+} \Phi(L);$$

this property is called *radial valuation*. A map Φ is called *p-homogeneous* if for $K \in \mathcal{S}^n$ and $\lambda \geqslant 0$,

$$\Phi(\lambda K) = \lambda^p \Phi(K).$$

From the geometric point of view, the radial Minkowski additive maps enjoying an invariance property are of particular interest. Under extra continuity assumptions, we get the following uniqueness result proved in Section 3 (see Theorem 3.1).

Theorem 1.2. *Let $n \geqslant 2$. If the map $\Phi\colon \mathcal{S}^n \to \mathbb{R}$ is linear (Φ is radial Minkowski additive and 1-homogeneous), invariant under proper rotations and continuous at the unit ball B_n, then Φ is a constant multiple of the dual mean width.*

Remark 1.1. Let Φ be a radial Minkowski additive map from $\mathcal{S}^n$ into $\mathbb{R}$ or $\mathbb{R}^n$. For $K \in \mathcal{S}^n$ and the definition of radial function, we have $2K = K \widetilde{+} K$, hence $\Phi(2K) = 2\Phi(K)$, by induction, this gives $\Phi(kK) = k\Phi(K)$ for $k \in \mathbb{N}$. Moveover, we obtain $k\Phi(K) = \Phi(kK) = \Phi(m(k/m)K) = m\Phi((k/m)K)$ for $k, m \in \mathbb{N}$, hence $\Phi(qK) = q\Phi(K)$ for rational $q > 0$. If Φ is continuous on $\mathcal{S}^n$, we then conclude that $\Phi(\Gamma K) = \Gamma\Phi(K)$ for real $\Gamma \geqslant 0$. Thus, the assumption 'linear and continuous at B_n' in Theorem 1.2 is weaker than the assumption 'radial Minkowski additive and continuous at $\mathcal{S}^n$'.

In Section 4, we generalize the result of Kiderlen (see [14]); in order to prove the result, we introduce the radial Minkowski endomorphism.

Definition 1.2. The map $\Phi\colon \mathcal{S}^n \to \mathcal{S}^n$ is called a *radial Minkowski endomorphism* if it satisfies the following conditions:

(i) Φ is continuous with respect to radial metric;

(ii) Φ is radial Minkowski additive,

$$\Phi(K \widetilde{+} L) = \Phi(K) \widetilde{+} \Phi(L) \quad \text{for all } K, L \in \mathcal{S}^n;$$

(iii) Φ is $SO(n)$ equivariant:

$$\Phi(TK) = T\Phi(K) \quad \text{for all } T \in SO(n).$$

Theorem 1.3. *$\Phi\colon \mathcal{S}^n \to \mathcal{S}^n$ is a radial Minkowski homomorphism if and only if there is a nonnegative measure $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ such that $\varrho(\Phi K, \cdot) = \varrho(K, \cdot) * \mu$.*

2. Notation and background material

A set $S \subset \mathbb{R}^n$ is called *starshaped* with respect to o if it is not empty and $[o, x] \subset S$ for all $x \in S$. For a compact starshaped set K, the radial function is defined by [22]

$$\varrho_K(u) = \varrho(K, u) = \max\{\lambda \geqslant 0\colon\ \lambda u \in K\},$$

where $u \in S^{n-1}$. We say that K is a star body if K is a compact starshaped set with a positive continuous radial function, the set of all star bodies in $\mathbb{R}^n$ is denoted by $\mathcal{S}^n$, and $\mathcal{S}^n_e$ is the subset of $\mathcal{S}^n$ that contains the origin-symmetric star bodies. From the definition of radial function we have for $\lambda > 0$ and $T \in SO(n)$,

$$\varrho(\lambda K, u) = \lambda \varrho(K, u), \quad \varrho(TK, u) = \varrho(K, T^{-1}u).$$

The radial linear combination of star bodies is defined by

$$\varrho(\lambda K \widetilde{+} \mu L, \cdot) = \lambda \varrho(K, \cdot) + \mu \varrho(L, \cdot) \quad \text{for any } K, L \in \mathcal{S}^n.$$

The dual analog form for the Hausdorff metric, the radial metric, is defined by

$$\delta(K, L) = \inf\{\varepsilon > 0\colon\ K \subset L \widetilde{+} \varepsilon B_n,\ L \subset K \widetilde{+} \varepsilon B_n\},$$

where B_n is the Euclidean unit ball of $\mathbb{R}^n$. It is easy to prove that

$$\delta(K, L) = \sup_{u \in S^{n-1}} |\varrho(K, u) - \varrho(L, u)| = \|\varrho_K - \varrho_L\|_\infty.$$

Note that if $K, L \in \mathcal{S}^n$, then $K \cap L$, $K \cup L \in \mathcal{S}^n$ and

$$\varrho_{K \cap L}(\cdot) = \min\{\varrho_K(\cdot), \varrho_L(\cdot)\}, \quad \varrho_{K \cup L}(\cdot) = \max\{\varrho_K(\cdot), \varrho_L(\cdot)\}.$$

Lutwak in [18] introduced the notion of the dual mean width $\widetilde{M}(K)$,

$$\widetilde{M}(K) = \frac{2}{n\omega_n} \int_{S^{n-1}} \varrho(K, u) \, \mathrm{d}u,$$

and the dual Steiner point $\tilde{s}(K)$,

$$\tilde{s}(K) = \frac{1}{\omega_n} \int_{S^{n-1}} \varrho(K, u) u \, \mathrm{d}u,$$

where $K \in \mathcal{S}^n$. From the definition of dual mean width and dual Steiner point, we obtain

$$\widetilde{M}(K \widetilde{+} L) = \widetilde{M}(K) + \widetilde{M}(L), \quad \tilde{s}(K \widetilde{+} L) = \tilde{s}(K) + \tilde{s}(L).$$

In the following, we define the quantity

$$\tilde{s}_q(K) = \frac{1}{\omega_n} \int_{S^{n-1}} \varrho(K,u)^q u \, \mathrm{d}u,$$

where $K \in \mathcal{S}^n$ and $q \neq 0$. We then obtain that the quantity $\tilde{s}_q(\cdot)$ satisfies the radial valuation property.

Proposition 2.1. *Let $K, L \in \mathcal{S}^n$. If $q \neq 0$, then*

$$\tilde{s}_q(K) + \tilde{s}_q(L) = \tilde{s}_q(K \cap L) + \tilde{s}_q(K \cup L).$$

Proof. Assume that $K, L \in \mathcal{S}^n$ and consider the disjoint partition of $S^{n-1} = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where

$$\Omega_0 = \{u \in S^{n-1} \colon \varrho(K,u) = \varrho(L,u)\}, \quad \Omega_1 = \{u \in S^{n-1} \colon \varrho(K,u) < \varrho(L,u)\},$$
$$\Omega_2 = \{u \in S^{n-1} \colon \varrho(K,u) > \varrho(L,u)\}.$$

Since $K \cap L$, $K \cup L \in \mathcal{S}^n$ for $\mathcal{H}^{n-1}$-almost all $u \in \Omega_0$ we have

$$\varrho(K,u) = \varrho(L,u) = \varrho(K \cup L, u) = \varrho(K \cap L, u).$$

For $\mathcal{H}^{n-1}$-almost all $u \in \Omega_1$ we have

$$\varrho(K,u) = \varrho(K \cap L, u), \quad \varrho(L,u) = \varrho(K \cup L, u).$$

For $\mathcal{H}^{n-1}$-almost all $u \in \Omega_2$ we have

$$\varrho(L,u) = \varrho(K \cap L, u), \quad \varrho(K,u) = \varrho(K \cup L, u).$$

It follows that

$$\begin{aligned}\int_{S^{n-1}} \varrho(K,u)^q u \, \mathrm{d}u &= \int_{\Omega_0} \varrho(K,u)^q u \, \mathrm{d}u + \int_{\Omega_1} \varrho(K,u)^q u \, \mathrm{d}u + \int_{\Omega_2} \varrho(K,u)^q u \, \mathrm{d}u \\ &= \int_{\Omega_0} \varrho(K \cap L,u)^q u \, \mathrm{d}u + \int_{\Omega_1} \varrho(K \cap L,u)^q u \, \mathrm{d}u \\ &\quad + \int_{\Omega_2} \varrho(K \cup L,u)^q u \, \mathrm{d}u\end{aligned}$$

and

$$\begin{aligned}\int_{S^{n-1}} \varrho(L,u)^q u \, \mathrm{d}u &= \int_{\Omega_0} \varrho(L,u)^q u \, \mathrm{d}u + \int_{\Omega_1} \varrho(L,u)^q u \, \mathrm{d}u + \int_{\Omega_2} \varrho(L,u)^q u \, \mathrm{d}u \\ &= \int_{\Omega_0} \varrho(K \cup L,u)^q u \, \mathrm{d}u + \int_{\Omega_1} \varrho(K \cup L,u)^q u \, \mathrm{d}u \\ &\quad + \int_{\Omega_2} \varrho(K \cap L,u)^q u \, \mathrm{d}u.\end{aligned}$$

Since

$$\int_{S^{n-1}} \varrho(K\cap L,u)^q u\,\mathrm{d}u = \int_{\Omega_0} \varrho(K\cap L,u)^q u\,\mathrm{d}u + \int_{\Omega_1} \varrho(K\cap L,u)^q u\,\mathrm{d}u + \int_{\Omega_2} \varrho(K\cap L,u)^q u\,\mathrm{d}u,$$

$$\int_{S^{n-1}} \varrho(K\cup L,u)^q u\,\mathrm{d}u = \int_{\Omega_0} \varrho(K\cup L,u)^q u\,\mathrm{d}u + \int_{\Omega_1} \varrho(K\cup L,u)^q u\,\mathrm{d}u + \int_{\Omega_2} \varrho(K\cup L,u)^q u\,\mathrm{d}u,$$

we obtain

$$\int_{S^{n-1}} \varrho(K,u)^q u\,\mathrm{d}u + \int_{S^{n-1}} \varrho(L,u)^q u\,\mathrm{d}u = \int_{S^{n-1}} \varrho(K\cap L,u)^q u\,\mathrm{d}u + \int_{S^{n-1}} \varrho(K\cup L,u)^q u\,\mathrm{d}u.$$

The desired result is obtained. □

From Proposition 2.1 we obtain that the dual mean width and dual Steiner point satisfy the radial valuation property, see [22].

3. A new characterization of balls

In this section, in order to prove Theorem 1.2, we first introduce the radial rotation mean. If there exists a positive number m and rotations $\varrho_1,\varrho_2,\dots,\varrho_m\in SO(n)$ such that

$$K_r = \frac{1}{m}\varrho_1 K \tilde{+} \frac{1}{m}\varrho_2 K \tilde{+} \dots \tilde{+} \frac{1}{m}\varrho_m K,$$

then we say that K_r is a *radial rotation mean* of K. In view of the notion of radial rotation mean, we show the following result; our proof is based on the techniques in [22], Theorem 3.3.2.

Lemma 3.1. *For every convex body $K\in\mathcal{S}^n$ with $\dim K>0$ there is a sequence of radial rotation means of K converging to a ball.*

Proof. For $L\in\mathcal{S}^n$, let $d(L):=\min\{\lambda>0\colon L\subset\lambda B_n\}$, which is continuous. Since λB_n containing K contains all radial rotation means of K, the family $\mathcal{R}(K)$ of radial rotation means of K is bounded. Hence, the function d attains a minimum $d_0>0$ on the compact set $\mathrm{cl}\mathcal{R}(K)$, i.e., $d_0=d(L)$, $L\in\mathrm{cl}\mathcal{R}(K)$. We next prove $L=d_0B_n$; otherwise, we assume that $L\neq d_0B_n$, which implies that there exists a vector $u_0 \in S^{n-1}$ with $\varrho(L, u_0) < d_0$. Thus, we can find a suitable neighbourhood U of u_0 on S^{n-1} such that $\varrho(L, u) < d_0$ for every $u \in U$. Since S^{n-1} is compact, there exists finitely many rotations $\varrho_1, \varrho_2, \ldots, \varrho_m \in SO(n)$ such that $\bigcup_{i=1}^{m} \varrho_i U = S^{n-1}$. Let

$$\bar{L} = \frac{1}{m}\varrho_1 L \widetilde{+} \frac{1}{m}\varrho_2 L \widetilde{+} \ldots \widetilde{+} \frac{1}{m}\varrho_m L.$$

There is a number $i \in \{1, 2\ldots, m\}$ with $u \in \varrho_i U$, hence $\varrho_i^{-1} u \in U$, which implies $\varrho(L, \varrho_i^{-1} u) < d_0$, hence

$$\varrho(\bar{L}, u) = \frac{1}{m}\sum_{i=1}^{m} \varrho(\varrho_i L, u) = \frac{1}{m}\sum_{i=1}^{m} \varrho(L, \varrho_i^{-1} u) < d_0.$$

By the continuity of $\varrho(\bar{L}, \cdot)$, this yields $d(\bar{L}) < d_0$. There is a sequence $\{K_j\}$ in $\mathcal{R}(K)$ converging to L, and we deduce that

$$\overline{K}_j = \frac{1}{m}\varrho_1 K_j \widetilde{+} \frac{1}{m}\varrho_2 K_j \widetilde{+} \ldots \widetilde{+} \frac{1}{m}\varrho_m K_j \to \bar{L}$$

for $j \to \infty$; this gives $d(\overline{K}_j) < d_0$ for large j. Since $\overline{K}_j \in \mathcal{R}(K)$, this contradicts the minimality of d_0. Therefore L is a ball, which proves the theorem. $\square$

From Lemma 3.1 we deduce:

Theorem 3.1. *Let $n \geqslant 2$. If the map $\Phi\colon \mathcal{S}^n \to \mathbb{R}$ is radial Minkowski additive, invariant under proper rotations and continuous at the unit ball B_n, then Φ is a constant multiple of the dual mean width.*

Proof. For $x \in \mathbb{R}^n$, we can choose $\varrho \in SO(n)$ with $\varrho x = -x$, hence $\Phi(o) = \Phi(\{x\} \widetilde{+} \varrho\{x\}) = 2\Phi(\{x\})$. Thus $\Phi(o) = 0$ gives $\Phi(\{x\}) = 0$ in general. Let $K \in \mathcal{S}^n$ and $\dim K > 0$. If $K_r = 1/m(\lambda_1 \varrho_1 K \widetilde{+} \ldots \widetilde{+} \lambda_m \varrho_m K)$ with rational numbers $\lambda_1, \ldots, \lambda_m > 0$ and rotations $\varrho_1, \ldots, \varrho_m \in SO(n)$, then $\Phi(K_r) = (\lambda_1 + \ldots + \lambda_m)\Phi(K)$ by the properties of Φ. Since the dual mean width has the same properties as Φ, we conclude that $\Phi(K)/\widetilde{M}(K) = \Phi(K_r)/\widetilde{M}(K_r)$. It follows from Lemma 3.1 that the integer m, the rotations ϱ_i and the rational numbers λ_i can be chosen so that (K_r, B_n) is smaller than a given number $\varepsilon > 0$. Since Φ is continuous at B_n, we deduce that $\Phi(K)/\widetilde{M}(K) = \frac{1}{2}\Phi(B_n)$. $\square$

With the help of Lemma 3.1 and dual Brunn-Minkowski functional of degree α, we consider the stationary domains under prescribed dual mean width. We say that $K \in \mathcal{S}^n$ is stationary for a functional $F\colon \mathcal{S}^n \to \mathbb{R}^+$ if

$$\frac{\mathrm{d}}{\mathrm{d}t} F((1-t)K \widetilde{+} tL)\Big|_{t=0^+} = 0 \quad \text{for all } L \in \mathcal{S}^n.$$

We now give the definition of dual Brunn-Minkowski functional.

Definition 3.1. $F\colon \mathcal{S}^n \to \mathbb{R}^+$ is called a *dual Brunn-Minkowski functional of degree* α if it satisfies the following condition:

(i) F is rigid motion invariant: if $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is rigid motion, then

$$F(\varphi(K)) = F(K);$$

(ii) F is radial Hausdorff continuous: if $K_m \to K$ in radial metric, then

$$F(K_m) \to F(K) \quad \text{as } m \to \infty;$$

(iii) F is differentiable, i.e., there exists

$$\frac{\mathrm{d}}{\mathrm{dt}} F((1-t)K \widetilde{+} tL)\big|_{t=0^+};$$

(iv) F is α-homogeneous for $\alpha \neq 0$:

$$F(tK) = t^\alpha F(K) \quad \text{for } t \in \mathbb{R}^+;$$

(v) F satisfies the dual Brunn-Minkowski inequality:

$$F((1-t)K \widetilde{+} tL))^\alpha \leqslant (1-t)F(K)^\alpha + tF(L)^\alpha,$$

and equality holds if and only if K and L are dilates.

Theorem 3.2. *Let $F\colon \mathcal{S}^n \to \mathbb{R}^+$ be a dual Brunn-Minkowski functional of degree α. If $K \in \mathcal{S}^n$ is a stationary domain for the functional*

$$\Phi(K) = \frac{F(K)^\alpha}{\widetilde{M}(K)},$$

then K is a ball.

P r o o f. By Lemma 3.1, we know that there exists a sequence of dual rotation means of K which converges to a ball in radial Hausdorff distance, and F is continuous. Let B_r^n be the ball of radius r. We have

$$F(B_r^n)^\alpha = \lim_{m\to\infty} F\Big(\frac{1}{m}\varrho_1 K \widetilde{+} \frac{1}{m}\varrho_2 K \widetilde{+} \ldots \widetilde{+} \frac{1}{m}\varrho_m K\Big)^\alpha.$$

Using the dual Brunn-Minkowski inequality (v) and (iv) in Definition 3.1, we obtain

$$F(B_r^n)^\alpha \leqslant F(K)^\alpha.$$

However, the dual mean width of K, $\widetilde{M}(K)$ satisfies (i), (ii), (iv) and is linear with respect to radial sum, thus, the maximizers of the quotient functional $\Phi(K)$ over $\mathcal{S}^n$

are balls. We next shall prove that balls are unique. Otherwise, we assume K_1 is a maximizer of Φ, different from a ball, and let $\widetilde{M}(B_r^n) = \widetilde{M}(K_1)$. Since K_1 and B_r^n are not dilates, we get

$$\Phi\Big(\frac{K_1\widetilde{+}B_r^n}{2}\Big) < \frac{1}{2}\Phi(K_1) + \frac{1}{2}\Phi(B_r^n) = \Phi(B_r^n),$$

against the assumption that B_r^n.

On the other hand, put

$$\Psi_1(t) = F((1-t)K\widetilde{+}tL)^\alpha, \quad \Psi_2(t) = \widetilde{M}((1-t)K\widetilde{+}tL)^\alpha, \quad \Psi(t) = \frac{\Psi_1(t)}{\Psi_2(t)}.$$

It is clear that $\Psi_1(t)$ is concave on $[0,1]$ and $\Psi_2(t)$ is linear. Therefore for $t \in [0,1]$,

$$\Psi(t) \geqslant \frac{\Psi_1(0) + \Psi'_{1,+}(0)t}{\Psi_2(0) + \Psi'_{2,+}(0)t},$$

where $\Psi'_{1,+}$ means the right derivatives of Ψ_1 at 0. Assume that K is stationary for Φ. This gives

$$\frac{\Psi_1(0) + \Psi'_{1,+}(0)t}{\Psi_2(0) + \Psi'_{2,+}(0)t} = \frac{\Psi_1(0)}{\Psi_2(0)}.$$

It follows that

$$\Psi(t) \geqslant \Psi(0), \quad t \in [0,1].$$

By the arbitrariness of L, we deduce that K is a maximizer for Φ over $\mathcal{S}^n$. This completes the proof. □

4. Radial Minkowski homomorphism

In the following, we give the proof of Theorem 1.3; some basic facts on spherical harmonics are needed, see [25]. Let $SO(n)$ and S^{n-1} be equipped with the invariant probability measures. Let $C(SO(n))$ and $C(S^{n-1})$ be the sets of continuous functions on $SO(n)$ and S^{n-1}, respectively. Similarly, $M(SO(n)), M(S^{n-1})$ denote their dual spaces of signed finite Borel measures. Assume $f \in C(S^{n-1})$ and $\mu \in M(S^{n-1})$. We have $Tf(u) = f(T^{-1}u)$, $T \in SO(n)$, thus, we say that the group $SO(n)$ is left translation invariant. The convolution $\mu * f \in C(S^{n-1})$ of a measure $\mu \in M(SO(n))$ and a function $f \in C(S^{n-1})$ is defined by

$$(\mu * f)(u) = \int_{SO(n)} Tf(u)\,\mathrm{d}\mu(T).$$

The canonical pairing of $f \in C(S^{n-1})$ and $\mu \in M(S^{n-1})$ is defined by (see [25])

$$\langle \mu, f \rangle = \langle f, \mu \rangle = \int_{SO(n)} f(u)\, \mathrm{d}\mu(u).$$

If $\nu \in M(S^{n-1})$, then

$$\langle \mu * \nu, f \rangle = \langle \mu, \nu * f \rangle. \tag{4.1}$$

Let $\mathcal{H}_k^n$ be the finite dimensional vector space of spherical harmonics of dimension n and order k, and $N(n,k)$ denote the dimension of $\mathcal{H}_k^n$. The space of all finite sums of spherical harmonics of dimension n is denoted by $\mathcal{H}^n$. Moreover, the $\mathcal{H}_k^n$ are pairwise orthogonal with respect to the inner product on $C(S^{n-1})$. Clearly, $\mathcal{H}_k^n$ is invariant with respect to rotations. If $H_1, \dots, H_{N(n,k)}$ is an orthonormal basis of $\mathcal{H}_k^n$, then there is a unique polynomial $P_k^n \in C[-1,1]$ of degree k such that

$$\sum_{k=1}^{N(n,k)} H_i(u) H_i(v) = N(n,k) P_k^n(u \cdot v).$$

The polynomial P_k^n is called the *Legendre polynomial of dimension* n and *order* k.

A function $f \in C(S^{n-1})$ is called *zonal* if $Tf = f$, where $T \in SO(n-1)$ is the subgroup of rotations leaving the pole $\hat{e}$ of S^{n-1}. Zonal functions depend only on the scalar product, $\langle u, \hat{e} \rangle$. The set of continuous zonal functions and their dual on S^{n-1} will be denoted by $C(S^{n-1}, \hat{e})$ and $M(S^{n-1}, \hat{e})$, respectively. The map $\Lambda\colon C[-1,1] \to C(S^{n-1}, \hat{e})$ is defined by $\Lambda f(u) = f(\langle u, \hat{e} \rangle)$, $u \in S^{n-1}$. Clearly, Λ is an isomorphism between functions on $[-1,1]$ and zonal functions on S^{n-1} (see, e.g. [25]). If $f \in C(S^{n-1})$, $\mu \in M(S^{n-1}, \hat{e})$ and $\vartheta \in SO(n)$, then

$$(f * \mu)(\widehat{\vartheta}) = \int_{S^{n-1}} f(\vartheta u)\, \mathrm{d}\mu(u)$$

and thus, for every $\eta \in SO(n)$, $(\eta f * \mu) = \eta(f * \mu)$. Therefore, the zonal function ΛP_k^n is the unique zonal spherical harmonic in $\mathcal{H}_k^n$, up to a multiplicative constant. For $\mathcal{H}_k^n$ we can choose an orthonormal basis $\mathcal{H}_{k1}, \dots, \mathcal{H}_{kN(n,k)}$. The family $\{\mathcal{H}_{k1}, \dots, \mathcal{H}_{kN(n,k)}\}$ is a complete orthogonal system in $L^2(S^{n-1})$. In particular, for every $f \in L^2(S^{n-1})$, the series

$$f \sim \sum_{k=1}^{\infty} \pi_k f \tag{4.2}$$

converges to f, where $\pi_k f \in \mathcal{H}_k^n$ is the orthogonal projection of f on the space $\mathcal{H}_k^n$. By the properties of the Legendre polynomials, it is easy to show that $\pi_k f =$

$N(n,k)(f * \Lambda P_k^n)$. For a measure $\mu \in M(S^{n-1})$,

$$\mu \sim \sum_{k=1}^{\infty} \pi_k \mu,$$

where $\pi_k \mu \in \mathcal{H}_k^n$. Since $P_0^n(t) = 1$, $N(n,0) = 1$ and $P_1^n(t) = t$, $N(n,1) = n$, we obtain for $\mu \in M(S^{n-1})$,

$$\pi_0 \mu = \mu(S^{n-1}), \quad (\pi_1 \mu)(u) = n \int_{S^{n-1}} u \cdot v \, \mathrm{d}\mu(v).$$

For every star body $K \in \mathcal{S}^n$ it follows that

$$\omega_n \pi_0 \varrho(K, \cdot) = \widetilde{V}(B_n, K) = \frac{1}{n} \int_{S^{n-1}} \varrho(K, u) \, \mathrm{d}u.$$

In order to prove our result, we need to quote the following important lemmas.

Lemma 4.1 ([23], [24]). *Let $\Phi\colon C(S^{n-1}) \to C(S^{n-1})$ be a monotone, linear map. Then Φ is $SO(n)$ equivariant if and only if there is a measure $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ such that $\Phi f = f * \mu$.*

In view of the above lemmas, we give the proof of Theorem 1.3.

Theorem 4.1. *$\Phi\colon \mathcal{S}^n \to \mathcal{S}^n$ is a radial Minkowski homomorphism if and only if there is a nonnegative measure $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ such that $\varrho(\Phi K, \cdot) = \varrho(K, \cdot) * \mu$.*

Proof. Assume that $\varrho(\Phi K, \cdot) = \varrho(K, \cdot) * \mu$. Let $K_i, K \in \mathcal{S}^n$. If $K_i \to K$, then we have

$$\Phi K_i \to \Phi K,$$

i.e., Φ is continous. For any $T \in SO(n)$, we obtain

$$\varrho(\Phi T K, \cdot) = \varrho(TK, \cdot) * \mu = \varrho(K, T^{-1} \cdot) * \mu = \varrho(\Phi K, T^{-1} \cdot) = \varrho(T \Phi K, \cdot).$$

In view of the notion of radial sum, we obtain

$$\varrho(\Phi K \widetilde{+} \Phi L, \cdot) = \varrho(\Phi K, \cdot) + \varrho(\Phi L, \cdot) = \varrho(K, \cdot) * \mu + \varrho(L, \cdot) * \mu = \varrho(\Phi(K \widetilde{+} L), \cdot).$$

Thus, by the definition of radial Minkowski homomorphism, we know that the map Φ is a radial Minkowski homomorphism.

Conversely, since the vector space $\{\varrho(K, \cdot) - \varrho(L, \cdot)\colon K, L \in \mathcal{S}^n\}$ coincides with $C(S^{n-1})$, we consider the operator $\widetilde{\Phi}\colon C(S^{n-1}) \to C(S^{n-1})$ defined by

$$\widetilde{\Phi} f = \varrho(\Phi K_1, \cdot) - \varrho(\Phi L_1, \cdot),$$

where $f = \varrho(K_1,\cdot) - \varrho(L_1,\cdot)$ is a linear extension of Φ to $C(S^{n-1})$ that intertwines with rotations. Since the cone of radial functions is invariant under $\widetilde{\Phi}$, it is also monotone.

Similarly, we define $\widetilde{\Phi}g = \varrho(\Phi K_2,\cdot) - \varrho(\Phi L_2,\cdot)$. This yields

$$\begin{aligned}\widetilde{\Phi}f + \widetilde{\Phi}g &= \varrho(\Phi K_1,\cdot) - \varrho(\Phi L_1,\cdot) + \varrho(\Phi K_2,\cdot) - \varrho(\Phi L_2,\cdot)\\ &= \varrho(\Phi K_1 \widetilde{+} \Phi K_2,\cdot) - \varrho(\Phi L_1 \widetilde{+} \Phi L_2,\cdot) = \varrho(\Phi(K_1 \widetilde{+} K_2),\cdot) - \varrho(\Phi(L_1 \widetilde{+} L_2),\cdot)\\ &= \widetilde{\Phi}(\varrho(K_1 \widetilde{+} K_2,\cdot) - \varrho(L_1 \widetilde{+} L_2,\cdot)) = \widetilde{\Phi}(f+g),\end{aligned}$$

and so $\widetilde{\Phi}$ is linear. In terms of Lemma 4.1, we have $\widetilde{\Phi}f = f * \mu$. The desired result is obtained. □

By the dual mixed volume, we have the relation $\widetilde{V}(K,\Phi L) = \widetilde{V}(\Psi K, L)$, where $K, L \in \mathcal{S}^n$ the map Φ is a radial Minkowski homomorphism and the map Ψ is a radial Blaschke-Minkowski homomorphism, see [23]. Indeed, in terms of the notation of radial Minkowski homomorphism and radial Blaschke-Minkowski homomorphism, we have

$$\widetilde{V}(K,\Phi L) = \frac{1}{n}\langle \varrho_{\Phi L}, \varrho_K\rangle = \frac{1}{n}\langle \varrho_L * \mu, \varrho_K\rangle = \frac{1}{n}\langle \varrho_L, \varrho_K * \mu\rangle = \widetilde{V}(\Psi K, L).$$

Since the spherical convolution operators are multiplier transformations, by Theorem 4.1 we obtain:

Lemma 4.2. *Let $K \in \mathcal{S}^n$. If Φ is a radial Minkowski homomorphism which is generated by the zonal measure μ, then*

$$\pi_k \varrho(\Phi K,\cdot) = \mu_k \pi_k \varrho(K,\cdot),$$

where μ_k are the Legendre coefficients of μ.

Definition 4.1. If Φ is a radial Minkowski homomorphism generated by the zonal measure μ, then the subset $\mathcal{S}^n(\Phi)$ of $\mathcal{S}^n$, defined by

$$\mathcal{S}^n(\Phi) = \{K \in \mathcal{S}^n\colon\ \pi_k \varrho(K,\cdot) = 0 \text{ if } \mu_k = 0\},$$

is called the *injectivity set* of Φ.

It is not hard to show that $\mathcal{S}^n(\Phi)$ is a nonempty rotation and dilatation invariant subset, and it is closed under radial sum. By Lemma 4.2, we know that $K \in \mathcal{S}^n(\Phi)$ is uniquely determined by its image Φ. A star body $K \in \mathcal{S}^n$ is called *polynomial* if $\varrho(K,\cdot) \in \mathcal{H}^n$.

Theorem 4.2. *If Φ is a radial Minkowski homomorphism with $\mathcal{S}^n_e \subset \mathcal{S}^n(\Phi)$, then for every polynomial $L \in \mathcal{S}^n_e$ there exists origin-symmetry star bodies $K_1, K_2 \in \mathcal{S}^n_e$ such that $L \widetilde{+} \Phi K_2 = \Phi K_1$.*

P r o o f. By (4.1), we get that

$$\varrho(L,\cdot)=\sum_{k=1}^{m}\pi_k\varrho(L,\cdot),$$

where L is a polynomial. From the properties of the orthogonal projection of f on $\mathcal{H}_k^n$, we have $\pi_k\varrho(L,\cdot)=0$ for all odd $k\in\mathbb{N}$. Since $\mathcal{S}_e^n\subset\mathcal{S}^n(\Phi)$, this yields $\mu_k\neq 0$ for all even $k\in\mathbb{N}$. Thus, we define

$$f:=\sum_{k=1}^{m}c_k\pi_k\varrho(L,\cdot),$$

where $c_k=0$ if k is odd, when k is even, $c_k=\mu_k^{-1}$. It is easy to get that f is an even continuous function on S^{n-1}, since spherical convolution operators are multiplier transformations, we have

$$f*\mu=\sum k=1^m c_k\mu_k\pi_k\varrho(L,\cdot)=\sum k=1^m\mu_k\pi_k\varrho(L,\cdot)=\varrho(L,\cdot).$$

Let f^+, f^- denote the positive and negative parts of f, respectively, and $K_1,K_2\in\mathcal{S}^n$ with $\varrho(K_1,\cdot)=f^+$, $\varrho(K_2,\cdot)=f^-$, and so

$$\varrho(K_1,\cdot)*\mu=\varrho(K_2,\cdot)+\varrho(L,\cdot).$$

It follows from Theorem 4.1 that $L\widetilde{+}\Phi K_2=\Phi K_1$. □

References

[1] *J. Abardia-Evéquoz, A. Colesanti, E. Saorín Gómez*: Minkowski additive operators under volume constraints. J. Geom. Anal. *28* (2018), 2422–2455. zbl MR doi
[2] *J. Abardia-Evéquoz, A. Colesanti, E. Saorín Gómez*: Minkowski valuations under volume constraints. Adv. Math. *333* (2018), 118–158. zbl MR doi
[3] *S. Alesker*: Continuous rotation invariant valuations on convex sets. Ann. Math. (2) *149* (1999), 977–1005. zbl MR doi
[4] *B. Chen, G.-C. Rota*: Totally invariant set functions of polynomial type. Commun. Pure Appl. Math. *47* (1994), 187–197. zbl MR doi
[5] *A. Colesanti, M. Ludwig, F. Mussnig*: Minkowski valuations on convex functions. Calc. Var. Partial Differ. Equ. *56* (2017), Article ID 162, 29 pages. zbl MR doi
[6] *F. Dorrek*: Minkowski endomorphisms. Geom. Funct. Anal. *27* (2017), 466–488. zbl MR doi
[7] *P. Dulio, R. J. Gardner, C. Peri*: Characterizing the dual mixed volume via additive functionals. Indiana Univ. Math. J. *65* (2016), 69–91. zbl MR doi
[8] *R. J. Gardner*: A positive answer to the Busemann-Petty problem in three dimensions. Ann. Math. (2) *140* (1994), 435–447. zbl MR doi
[9] *R. J. Gardner*: Intersection bodies and the Busemann-Petty problem. Trans. Am. Math. Soc. *342* (1994), 435–445. zbl MR doi

[10] *R. J. Gardner, A. Koldobski, T. Schlumprecht*: An analytical solution to the Busemann-Petty problem on sections of convex bodies. Ann. Math. (2) *149* (1999), 691–703. zbl MR doi
[11] *E. Grinberg, G. Zhang*: Convolutions, transforms, and convex bodies. Proc. Lond. Math. Soc., III. Ser. *78* (1999), 77–115. zbl MR doi
[12] *L. Guo, H. Jia*: The dual Blaschke addition. J. Geom. Anal. *30* (2020), 3026–3034. zbl MR doi
[13] *L. Ji, Z. Zeng*: Some inequalities for radial Blaschke-Minkowski homomorphisms. Czech. Math. J. *67* (2017), 779–793. zbl MR doi
[14] *M. Kiderlen*: Blaschke- and Minkowski-endomorphisms of convex bodies. Trans. Am. Math. Soc. *358* (2006), 5539–5564. zbl MR doi
[15] *D. A. Klain*: Star valuations and dual mixed volumes. Adv. Math. *121* (1996), 80–101. zbl MR doi
[16] *D. A. Klain*: Invariant valuations on star-shaped sets. Adv. Math. *125* (1997), 95–113. zbl MR doi
[17] *M. Ludwig*: Intersection bodies and valuations. Am. J. Math. *128* (2006), 1409–1428. zbl MR doi
[18] *E. Lutwak*: Dual mixed volumes. Pac. J. Math. *58* (1975), 531–538. zbl MR doi
[19] *E. Lutwak*: Intersection bodies and dual mixed volumes. Adv. Math. *71* (1988), 232–261. zbl MR doi
[20] *R. Schneider*: On Steiner point of convex bodies. Isr. J. Math. *9* (1971), 241–249. zbl MR doi
[21] *R. Schneider*: Equivariant endomorphisms of the space of convex bodies. Trans. Am. Math. Soc. *194* (1974), 53–78. zbl MR doi
[22] *R. Schneider*: Convex Bodies: The Brunn-Minkowski Theory. Encyclopedia of Mathematics and its Applications 151. Cambridge University Press, Cambridge, 2014. zbl MR doi
[23] *F. E. Schuster*: Volume inequalities and additive maps of convex bodies. Mathematika *53* (2006), 211–234. zbl MR doi
[24] *F. E. Schuster*: Convolutions and multiplier transformations of convex bodies. Trans. Am. Math. Soc. *359* (2007), 5567–5591. zbl MR doi
[25] *F. E. Schuster*: Valuations and Busemann-Petty type problems. Adv. Math. *219* (2008), 344–368. zbl MR doi
[26] *P. Tradacete, I. Villanueva*: Radial continuous valuations on star bodies. J. Math. Anal. Appl. *454* (2017), 995–1018. zbl MR doi
[27] *P. Tradacete, I. Villanueva*: Continuity and representation of valuations on star bodies. Adv. Math. *329* (2018), 361–391. zbl MR doi
[28] *I. Villanueva*: Radial continuous rotation invariant valuations on star bodies. Adv. Math. *291* (2016), 961–981. zbl MR doi
[29] *W. Wang, L. Liu*: Fourier transform and valuations. J. Math. Anal. Appl. *470* (2019), 1167–1184. zbl MR doi
[30] *G. Zhang*: A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$. Ann. Math. (2) *149* (1999), 535–543. zbl MR doi
[31] *C.-J. Zhao*: On Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms. J. Geom. Anal. *26* (2016), 1523–1538. zbl MR doi

Author's address: Lewen Ji, School of Science, East China University of Technology, 418 Guanglan Ave, Qingshanhu District, Nanchang, Jiangxi, 330013, P. R. China, e-mail: `jilewen2008@163.com`.