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# IDEAL CLASS (SEMI)GROUPS AND ATOMICITY IN PRÜFER DOMAINS

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*Abstract.* We explore the connection between atomicity in Prüfer domains and their corresponding class groups. We observe that a class group of infinite order is necessary for non-Noetherian almost Dedekind and Prüfer domains of finite character to be atomic. We construct a non-Noetherian almost Dedekind domain and exhibit a generating set for the ideal class semigroup.

*Keywords*: Prüfer domain; factorization

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## 1. Motivation

The theory of factorization dates back to the time of Euclid. As with all branches of mathematics, exploration continues in more generalized settings. Discovering fundamental relationships between two seemingly unrelated objects is also a hallmark of mathematics. In this paper, we pursue a relationship between factorization in non-Noetherian Prüfer domains and their respective ideal class semigroups.

Perhaps one of the most powerful and elegant theorems in factorization theory is the connection between the factorization properties of Dedekind domains and their respective class groups. Recall that a domain $D$ is called *Dedekind* if $D$ is Noetherian and for all maximal ideals $M$ of $D$, the localization $D_M$ is a Noetherian valuation domain. Let $D$ be a Dedekind domain with fraction field $K$. We denote the set of fraction ideals of $D$ by $\mathcal{F}(D)$ and the set of principal ideals of $D$ by $\mathcal{P}(D)$. The class group of $D$ is the quotient group $\mathcal{F}(D)/\mathcal{P}(D)$. The class group is a measure of how "far away" a Dedekind domain is from being a unique factorization domain (UFD). A Dedekind domain is a UFD if and only if $\mathcal{C}(D)$ is trivial. A Dedekind domain $D$ is a half-factorial domain (HFD) if for a nonzero non-unit $b$ all factorization of $b$ into

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irreducibles are of the same length. A Dedekind domain is an HFD if and only if $\mathcal{C}(D) \cong \mathbb{Z}_2$. The class number of a Dedekind domain is the order of the class group. If the class number of a Dedekind domain is larger than 2, it still yields information with respect to factorization.

A domain $D$ is called *almost Dedekind* if $D_M$ is a Noetherian valuation domain for all $M \in \mathrm{Max}(D)$. A domain is said to be Prüfer if $D_M$ is a valuation domain for all $M \in \mathrm{Max}(D)$. A Prüfer domain is said to be of finite character if every nonzero non-unit element of $D$ is contained in only finitely many maximal ideals. Since these domain are natural extensions of Dedekind domains, it makes sense to ask what is the relationship between their class groups and factorization properties. It should be noted that if $D$ contains non-invertible ideals, then the quotient $\mathcal{F}(D)/\mathcal{P}(D)$ is only a semigroup. We will denote the ideal class semigroup of $D$ by $\mathcal{S}(D)$. However, one can restrict this definition to only invertible ideals to still obtain a group. We will call this the *ideal class group of* $D$ and denote it $\mathcal{C}(D)$. That is $\mathcal{C}(D) = \mathcal{I}(D)/\mathcal{P}(D)$, where $\mathcal{I}(D)$ is the set of invertible fractional ideals.

We will show that if $D$ is an atomic almost Dedekind (not Dedekind) domain or an atomic Prüfer (not Dedekind) domain of finite character, then the class number of $D$ must be infinite. In some ways this seems slightly odd given that in a Dedekind domain, a small class number yields more restrictive factorization properties. However, recall that a Dedekind domain is a UFD if and only if it is a PID. Thus, we see that a trivial class group corresponds to Bézout domains and it is well known that non-Noetherian Bézout domains always fail to be atomic. Therefore, if an infinite class group is needed for a domain to be atomic, we could ask if studying the ideal class semigroups would lead to a better understating of factorization. The problem with this question is it that is almost impossible to calculate the ideal class semigroup for a Prüfer (not Dedekind) domain, especially if the domain is not of finite character. Calculating an ideal-class group of a Dedekind domain can be a monumental task if the Minkowski bound is large. When calculating the ideal class group of a Dedekind domain, one only need to look at the maximal ideals whose residue field is smaller than the Minkowski bound. When we move to calculating the ideal class group or ideal class semigroup of a Prüfer domain, we will no longer have this luxury. Thus we will need to fully understand all the maximal ideals in the domain. It is to this end that we calculate a generating set for the class semigroup of a sequence domain. Calculations in more complicated domains might be nearly impossible.

## 2. Introduction

We provide the reader with some results from [3] that we will use throughout the paper. Let $D$ be an almost Dedekind domain. For nonzero $b \in D$, we define the norm of $b$ as

$$N(b) = (\nu_M(b))_{M \in \operatorname{Max}(D)} \in \prod_{M \in \operatorname{Max}(D)} \mathbb{N}_0.$$

Here $\nu_M$ is the valuation from $D_M$. A partial ordering on $\operatorname{Norm}(D) = \{N(b)\colon\ b \in D\}$ was introduced. More precisely, it is said that $N(a) \leqslant N(b)$ if for all $M \in \operatorname{Max}(D)$ we have $\nu_M(a) \leqslant \nu_M(b)$. We say $N(a) < N(b)$ if $N(a) \leqslant N(b)$ and there exists an $M \in \operatorname{Max}(D)$ with $\nu_M(a) < \nu_M(b)$. This partial ordering gives the following theorem, which we will use.

**Theorem 2.1.** *Let $D$ be an almost Dedekind domain and let $a, b \in D$, $N(a) \leqslant N(b)$ if and only if $a$ divides $b$. Further, if $N(a) < N(b)$, then $a$ is a proper divisor of $b$.*

It was also proved that factoring in $\operatorname{Norm}(D)$ is in one-to-one correspondence with factoring in $D$. Additionally, the elements of the normset were completely classed.

**Theorem 2.2.** *Let $D$ be an almost Dedekind domain. Then $D$ is atomic if and only if $\operatorname{Norm}(D)$ is an (additively) atomic monoid.*

**Theorem 2.3.** *Let $D$ be an almost Dedekind domain.*

$$(e_\lambda)_{\lambda \in \Lambda} \in \operatorname{Norm}(D) \textit{ if and only if } \bigcap_{\lambda \in \Lambda} M^{e_\lambda} \textit{ is principal.}$$

*Here $M^0$ is taken to be $D$ and $\Lambda = \operatorname{Max}(D)$.*

## 3. Norm($D$) when $D$ is Dedekind

We start by classifying $\operatorname{Norm}(D)$ when $D$ is Dedekind. We do this to demonstrate the importance of the class group in determining atoms. We also wish to contrast how it is "easy" to use $N(a)$ to determine if $a$ is an atom in a Dedekind domain, but that it becomes a huge challenge to use $N(a)$ to determine whether $a$ is an atom in a non-Noetherian almost Dedekind domain.

We start by giving a complete classification of $\operatorname{Norm}(D)$ when $D$ is Dedekind. Let us consider a Dedekind domain $D$ with abelian class group $G$. Let us denote the elements of $G$ by $\{g_\lambda\}_{\lambda \in \Lambda}$, where $\Lambda$ is some indexing set. We will use addition as the operation on $G$ and 0 will be the identity. Further, the set of maximal ideals of $D$ will be denoted by $\operatorname{Max}(D)$. Let $\varphi\colon\ \operatorname{Max}(D) \to G$ be the natural mapping from the set of maximal ideals to their corresponding classes in $G$.

**Definition 3.1.** For $c_i \in \mathbb{N}_0$, we say the sum $c_1g_1 + c_2g_2 + \ldots + c_ng_n = 0$ is primitive if whenever $c'_1g_1 + c'_2g_2 + \ldots + c'_ng_n = 0$ we have either $c'_i \geqslant c_i$ for all $i = 1, 2, \ldots, n$, or there is at least one $i$ with $c'_i > c_i$. We do not allow the trivial sum (i.e., at least one $c_i > 0$).

Now we can start our classification of $\mathrm{Norm}(D)$ when $D$ is Dedekind. Since all Dedekind domains are atomic, we know from Theorem 2.2 that $\mathrm{Norm}(D)$ is atomic. Thus, if we can succeed in classifying the atoms in $\mathrm{Norm}(D)$, we will have a generating set for $\mathrm{Norm}(D)$. Let $\mathcal{A}(D)$ denote the set of atoms in $\mathrm{Norm}(D)$.

Since all Dedekind domains are Noetherian, $N(b)$ has all but finitely many nonzero entries. We will write $N(b) = (b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n}$ for $\nu_{M_i}(b) = b_i$ and $\nu_{M_j}(b) = 0$ for all $j \neq 1, 2, \ldots, n$.

To classify the atoms of $\mathrm{Norm}(D)$ we will make use of Theorem 2.3.

**Theorem 3.2.** *It holds that* $(b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n}$ *is an atom if and only if* $b_1\varphi(M_1) + b_2\varphi(M_2) + \ldots + b_n\varphi(M_n) = 0$ *is primitive.*

Proof. Suppose $(b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n}$ is in $\mathcal{A}(D)$. Then

$$b_1\varphi(M_1) + b_2\varphi(M_2) + \ldots + b_n\varphi(M_n) = 0$$

by Theorem 2.3. Now if the sum is not primitive, there would be an element of smaller norm, hence $(b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n}$ would not be an atom.

Suppose $b_1\varphi(M_1) + b_2\varphi(M_2) + \ldots + b_n\varphi(M_n) = 0$ is primitive. Then

$$(b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n} \in \mathrm{Norm}(D).$$

Further there are no elements of smaller norm. Thus $(b_1, b_2, \ldots, b_n)_{M_1,M_2,\ldots,M_n}$ is an atom. $\square$

**Corollary 3.3.** *Let $D$ be a Dedekind domain. Then* $\mathrm{Norm}(D) = \langle \mathcal{A}(D) \cup \{0\}, + \rangle$, *where* $\langle \mathcal{A}(D) \cup \{0\}, + \rangle$ *is the additive monoid generated by* $\mathcal{A}(D) \cup \{0\}$.

The class group completely determines the normset for a Dedekind domain. This is yet more evidence that $\mathrm{Norm}(D)$ and $D$ share a strong relationship with respect to factorization.

**Example 3.4.** Suppose $R$ is Dedekind and a UFD. Then $G$ is trivial and every maximal ideal is principal. Thus all atoms $a \in R$ have norms of the form $N(a) = (1)_{M_i}$, where $M_i$ is any maximal ideal of $R$.

**Example 3.5.** Suppose $R$ is an HFD. Then $G \cong \mathbb{Z}_2$. Then $a$ is an atom in $R$ if and only if its norm is of the form $N(a) = (1)_{M_i}$, where $\varphi(M_i) = 0$, $N(a) = (2)_{M_i}$, where $\varphi(M_i) = \bar{1}$, or $N(a) = (1, 1)_{M_i,M_j}$, where $\varphi(M_i) = \varphi(M_j) = \bar{1}$.

Suppose $D$ and $D'$ have class groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively. The normset can differentiate these two domains even though their class numbers are the same. Let $M \in \operatorname{Max}(D)$ with $\varphi(M) = \bar{1}$, then $(4)_M$ is an atom. But every element of the $(4)_{M'}$ is reducible in $D'$ since all the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are of order 2.

When $D$ is a Dedekind domain, we have seen exactly what forms atoms can take based on the class group of $D$. Now almost Dedekind domains that are not Dedekind contain ideals that are not finitely generated, hence they are not invertible. Thus we will lose the use of the class group when trying to classify atoms of an almost Dedekind domain that are contained in a non-invertible maximal ideal. The question, we wish to explore, is, can we get a similar classification of atoms in an almost Dedekind domain by using the class semigroup.

We continue by defining a norm on the set of nonzero ideals of a one-dimensional Prüfer domain. It should be noted that an almost Dedekind domain is a one-dimensional Prüfer domain.

Let $D$ be a one-dimensional Prüfer domain with quotient field $K$. We define for nonzero $b \in K$ the norm of $b$ by $N(b) = (\nu_M(b))_{M \in \operatorname{Max}(D)}$. Now for $M \in \operatorname{Max}(D)$ and fractional ideal $I$, we define

$$T_M(I) = \{\nu_M(b)\}_{b \in I \setminus \{0\}}$$

and we set

$$s_M(I) = \inf T_M(I).$$

For a nonzero fraction ideal $I$ of $D$ we define

$$\nu_M(I) = \begin{cases} s_M(I), & s_M(I) \in T_M(I), \\ s_M(I) + \varepsilon, & s_M(I) \notin T_M(I), \end{cases}$$

where $\varepsilon$ is the fixed surreal number $1/\omega$ with $\omega$ being the cardinality of the natural numbers.

Now we define the norm of a fractional ideal $I$ to be $\widehat{N}(I) = (\nu_M(I))_{M \in \operatorname{Max}(D)}$. We will use the result $\widehat{N}(\overline{IJ}) = \widehat{N}(\overline{I}) + \widehat{N}(\overline{J})$. For more on this construction and a proof of the result, see [1].

We use $Db$ to denote the principal fraction ideal generated by $b \in K^*$. Let $\widehat{\operatorname{Norm}}(K) = \{\widehat{N}(Db)\colon\ b \in K^*\}$. Similarly we define

$$\widehat{\operatorname{Norm}}(D) = \{\widehat{N}(I)\colon\ I \text{ is a nonzero fractional ideal of } D\}.$$

**Theorem 3.6.** *Let $D$ be a one-dimensional Prüfer domain. Then*

$$\mathcal{S}(D) \cong \widehat{\operatorname{Norm}}(D)/\widehat{\operatorname{Norm}}(K).$$

P r o o f. Define $\varphi\colon \mathcal{S}(D) \to \widehat{\mathrm{Norm}}(D)/\widehat{\mathrm{Norm}}(K)$ by

$$\varphi(\overline{I}) \mapsto \overline{\widehat{N}(I)}.$$

First we show that $\varphi$ is well defined. Take $I \equiv J \in \mathcal{S}(D)$. Then there exists $a, b \in K$ such that $(a)I = (b)J$. Now $\varphi(I) = \widehat{N}((a)I) = \widehat{N}((a)) + \widehat{N}(I) = \widehat{N}((b)) + \widehat{N}(J)$ which yield that $\widehat{N}(\overline{I}) \equiv \widehat{N}(\overline{J}) \pmod{\widehat{\mathrm{Norm}}(K)}$. Thus $\varphi$ is well defined. Now suppose $\varphi(\overline{I}) = \varphi(\overline{J})$. Then $\widehat{N}(\overline{I}) \equiv \widehat{N}(\overline{J}) \pmod{\widehat{\mathrm{Norm}}(K)}$ if and only if $\widehat{N}(I) + \widehat{N}((a)) = \widehat{N}(J) + \widehat{N}((b))$ for some $a, b \in K$. This implies that $(a)I = (b)J$, hence $I \equiv J \pmod{\mathcal{S}(D)}$. Thus $\varphi$ is injective. By definition of $\widehat{\mathrm{Norm}}(D)$, we have that $\varphi$ is surjective. Now $\varphi(\overline{I}\overline{J}) = \widehat{N}(\overline{I}\overline{J}) = \widehat{N}(\overline{I}) + \widehat{N}(\overline{J}) = \varphi(\overline{I}) + \varphi(\overline{J})$. Thus $\varphi$ is an isomorphism and we have $\mathcal{S}(D) \cong \widehat{\mathrm{Norm}}(D)/\widehat{\mathrm{Norm}}(K)$. □

We can always discuss the class group of a domain by restricting the set of fractional ideals we consider to be the set of invertible fractional ideals. With that being said, we may ask what conditions must the class group satisfy in order for a domain to be atomic. We shall see that if the domain is a non-Dedekind almost Dedekind domain with finitely many non-invertible maximal ideals or a non-Dedekind Prüfer domain of finite character, then the class group must be infinite.

**Theorem 3.7.** *Let $D$ be a non-Dedekind almost Dedekind domain with finitely many non-invertible maximal ideals. If $D$ is atomic, then $\mathcal{C}(D)$ must be of infinite order.*

P r o o f. Let $D$ be a non-Dedekind almost Dedekind domain with finitely many non-invertible maximal ideals and suppose that $\mathcal{C}(D)$ is of finite order. Now if $D$ is atomic, it must contain an atom $\alpha \in M^*$ for some non-invertible maximal ideal $M^* \in \mathrm{Max}(D)$. Now it must be the case that $\alpha$ is contained in infinitely many maximal ideals. Moreover, $\alpha$ must be in infinitely many invertible maximal ideals contained in the same class, say $M_1, M_2, \ldots$ Now set the order of $\mathcal{C}(D)$ to be $r < \infty$. Then $M_1 M_2 \ldots M_r = (\beta)$ for some $\beta \in D$. Moreover $N(\beta) = (1, 1, \ldots, 1)_{M_1, M_2, \ldots, M_r}$, hence $N(\beta) < N(\alpha)$. But this is impossible because this implies $\beta$ divides $\alpha$. Thus, we conclude that the order of $\mathcal{C}(D)$ is infinite. □

Now while the previous result might not seem surprising, considering that an atomic non-Dedekind almost Dedekind domain must contain an atom that is contained in infinitely many maximal ideals. In a non-Dedekind atomic Prüfer domain of finite character, we have all atoms contained in only finitely many maximal ideals. We shall see in the following result that a class group of infinite order is still necessary to maintain atomicity.

**Theorem 3.8.** *Let $D$ be a one-dimensional non-Dedekind Prüfer domain of finite character. If $D$ is atomic, then $\mathcal{C}(D)$ is of infinite order.*

P r o o f. Let $D$ be an atomic non-Dedekind Prüfer domain of finite character. Then it must be the case that there exists an atom $\alpha$ that is contained in an idempotent maximal ideal, say $M$. Now $D_M$ is a non-discrete valuation domain. Let us set $\nu_M(\alpha) = s$ for some $s > 0$ in the valuation group. Let us set the order of $\mathcal{C}(D) = r < \infty$. It must be the case that there exists a $b \in D$ with $\nu_M(b) < s/r$. Now $b$ is contained in finitely many maximal ideals other that $M$, say $M_1, M_2, \ldots, M_k$. Now we find $b_i$ such that $b_i \in M$ and $b_i \notin M_i$, which we can do by prime avoidance. Consider $I = (b, b_1, \ldots, b_k)$. We have $\widehat{N}(I) = (t)_M$, where $t < s/r$. Since $I$ is finitely generated, it is invertible, hence $I^r = (\beta)$ for some $\beta \in D$ and $N(\beta) = rt < s$. Thus $\beta$ divides $\alpha$, which is a contradiction. Thus the order of $\mathcal{C}(D)$ must be infinite. $\square$

## 4. Computing class semigroups

An almost Dedekind domain is said to be a sequence domain if it has a countable number of maximal ideals which are principal with the exception of one non-invertible maximal ideal. We present an example that relies on the following Theorem 42.5 from [2].

**Theorem 4.1.** *Let $D$ be a Dedekind domain with quotient field $K$, and let $\{P_i\}_{i=1}^r$, $\{Q_i\}_{i=1}^s$, and $\{U_i\}_{i=1}^t$, where $r \geqslant 1$ be three collections of distinct maximal ideals of $D$, each with finite residue field. Then there exists a simple quadratic extension field $K(t)$ of $D$ with $t$ integral over $D$ and separable over $K$ such that if $\overline{D}$ is the integral closure of $D$ in $K(t)$, each $P_i$ is inertial with respect to $\overline{D}$, each $Q_i$ ramifies with respect to $\overline{D}$, and each $U_i$ decomposes with respect to $\overline{D}$.*

**Example 4.2.** Let $D = \mathbb{Z}_{(q)}$ for some odd prime $q$. Let $K$ denote the quotient field of $D$. We can split $(q)$ by adjoining $t_1$, which is a root of $x^2 - p$ for some prime $p \neq q$ that is a square modulo $q$. Let $K_1 = K[t_1]$. We have $(q) = (q_1)(q_1')$, where $(q_1)$ and $(q_1')$ are distinct. We set $D_1$ to be the integral closure of $D$ in $K_1$. For the remainder of the construction, $D_i$ will be the integral closure of $D$ in $K_i$ and the $t_i$ are elements of the algebraic closure of $K$. Now by Theorem 4.1 we can find $t_2$ such that $(q_1') = (q_2)(q_2')$ while $(q_1)$ remains inert. Now we set $D_2$ to be the integral closure of $D$ in $K_2 = K_1[t_2]$. We split $(q_2') = (q_3)(q_3')$ via another simple quadratic extension (add $t_3$) while keeping the three other primes inert. We set $K_3 = K_2[t_3]$ and we set $D_3$ to be the integral closure of $D$ in $K_3$. We continue by induction.

In $D_i$ we have $i+1$ maximal ideals, namely $(q_1), (q_2), \ldots, (q_i)$ and $(q_i')$. Now we add $t_i$ that splits $(q_i') = (q_{i+1})(q_{i+1}')$ while keeping the other primes inert. Note that we can do this because there are only finitely many primes at each step. Now we set $K_{i+1} = K_i[t_i]$ and $D_{i+1}$ to be the integral closure of $D$ in $K_{i+1}$. We set $D_q = \bigcup_{i=1}^{\infty} D_i$. Now $D_q$ is an almost Dedekind domain. Moreover, it is a sequence domain with non-invertible ideal $M^* = (q_1', q_2', q_3', \ldots)$. For more on sequence domains see [4].

In order to calculate the ideal class semigroup, we will need to characterize the set of non-invertible ideals. Theorem 2.1 will serve to make this task easier to understand. A useful observation is that $N(q_i') = (1; 1, 1, \ldots)_{M^*, (q_{i+1}), (q_{i+2}), \ldots}$. Moreover, $D_q$ is a sequence domain. Since all sequence domains are Bézout (see [4]), we have that $\mathcal{C}(D_q)$ is trivial. Moreover, for all $b \in D_q$, $\{\nu_M(b) \colon M \in \operatorname{Max}(D_q)\}$ is bounded. Another way of saying this is there exists a $w \in \mathbb{N}$ such that $I \not\subset (q_i)^w$ for all $i$, because otherwise $I = (0)$. We can see this is true by realizing that all $b \in D_q$ appear at some point in the construction. Domains with this property are called *SP-domains*. For a characterization of $SP$-domains, see [5].

Let $\{a_i\} = \{a_i\}_{i=1}^{\infty}$ be a sequence of zeros and ones. That is, $a_i \in \{0,1\}$ for all $i \in \mathbb{N}$. Now the collection of all sequences of this form is uncountable. For each $\{a_i\}$, let $S = \{q_i \colon a_i = 1\}$ and $S^c = \{q_i \colon a_i = 0\}$. We make the following definition. We rewrite $S = \{s_1, s_2, \ldots\}$, noting that $S$ may or may not be finite. We denote the cardinality of $S$ by $|S|$.

$$I_S = \begin{cases} \left(\dfrac{q}{q_{s_1} q_{s_2} \ldots q_{s_k}}\right), & |S| = k, \\ (q_{s_1} q_{s_2} \ldots q_{s_k}), & |S^c| = k, \\ \left(q, \dfrac{q}{q_{s_1}}, \dfrac{q}{q_{s_1} q_{s_2}}, \dfrac{q}{q_{s_1} q_{s_2} q_{s_3}}, \ldots\right), & |S| = |S^c| = \infty, \end{cases}$$

where $I_S$ is defined to be a principal ideal if the sequence $\{a_i\}$ converges and $I_S$ is defined to be a non-finitely generated (non-invertible) ideal if the $\{a_i\}$ does not converge. We should also point out that $I_S = (q)$ when $S = \emptyset$.

We claim that $H = \{I_{S_\lambda} \colon \lambda \in \Lambda\} \cup M^*$ is a generating set for the non-identity elements of the ideal class semigroup, where $\Lambda$ is some uncountable index set corresponding to the non-convergent sequences of zeros and ones. First note that if $S_\lambda \triangle S_\gamma = (S_\lambda \setminus S_\gamma) \cup (S_\gamma \setminus S_\lambda)$ is finite, then $I_{S_\lambda} \equiv I_{S_\gamma} \pmod{\mathcal{P}(D)}$. Thus we are not claiming that $H$ is a minimal generating set.

**Theorem 4.3.** *Let $I \subset D_q$ be a non-invertible ideal. Then $I \equiv I_1 I_2 \ldots I_k \pmod{\mathcal{P}(D_q)}$ for some $I_1, I_2, \ldots, I_k \in H$. This representation is not unique and the ideals in the product may not be distinct.*

P r o o f. Let $I \subset D_q$ be a non-invertible ideal. Note that if $I$ is only contained in $M^*$, then $I = (M^*)^k$ for some $k \in \mathbb{N}$. If $I$ is contained in only finitely many ideals, then $I = (a)(M^*)^k$ for some $k \in \mathbb{N}$ and some $a \in D_q$. Thus $I \equiv (M^*)^k \pmod{\mathcal{P}(D_q)}$. Now suppose that $I$ is contained in infinitely many maximal ideals. Since $D_q$ is an $SP$-domain, there exists a $w \in \mathbb{N}$ such that $I \not\subset (q_i)^w$ for all maximal ideals $(q_i)$. We take $w$ to be the smallest such positive integer. For $1 \leqslant k \leqslant w-1$, consider $S_k^c = \{i\colon I \subset (q_i)^k\}$. We now have that $I = I_{S_1} I_{S_2} \dots I_{S_{w-1}} (M^*)^r$ for some $r \geqslant 0$, where it is understood that $(M^*)^0 = D_q$. If $S_k^c$ or $S_k$ is finite, it corresponds to a principal ideal. We can reduce the product using the equivalence relation to get the desired result. $\square$

To aid the understanding of the proof, we give an example for the reader.

**Example 4.4.** Consider the ideal $I$ with $\widehat{N}(I) = (5; 4, 1, 2, 3, 1, 2, 3, \dots)$. The first entry is the value at $M^*$. The pattern continues with $1, 2, 3, \dots$ The first value (the one right after the semicolon) will be used to demonstrate how principal ideals can appear in the product. We consider $S_k^c$ for $1 \leqslant k \leqslant 4$.

$$S_1^c = \mathbb{N}, \quad S_2^c = \{1, 3, 4, 6, 7, 9, 10, \dots\}, \quad S_3^c = \{1, 4, 7, 10, 13, \dots\}, \quad S_4^c = \{1\}.$$

This yields $I_1 = (q)$ and $I_4 = (q_1)$. The noninvertible ideals are

$$I_2 = \Bigl(q, \frac{q}{q_2}, \frac{q}{q_2 q_5}, \frac{q}{q_2 q_5 q_8}, \dots\Bigr), \quad I_3 = \Bigl(q, \frac{q}{q_2}, \frac{q}{q_2 q_3}, \frac{q}{q_2 q_3 q_5}, \frac{q}{q_2 q_3 q_5 q_6}, \dots\Bigr).$$

We can also think about this in terms of our norm.

$$\begin{aligned}
\widehat{N}(I_1) &= (1; 1, 1, 1, \dots),\\
\widehat{N}(I_2) &= (1; 1, 0, 1, 1, 0, 1, 1, 0, \dots),\\
\widehat{N}(I_3) &= (1; 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \dots),\\
\widehat{N}(I_4) &= (0; 1, 0, 0, 0, \dots).
\end{aligned}$$

Using the fact that $\widehat{N}(IJ) = \widehat{N}(I) + \widehat{N}(J)$, it is fairly easy to see that

$$I = I_1 I_2 I_3 I_4 (M^*)^2 \equiv I_2 I_3 (M^*)^2 \pmod{\mathcal{P}(D_q)}$$

**Remark 4.5.** Let $I$ be an ideal and let $t$ be the largest power such that $I \subset (q_i)^t$ for infinitely many $i$. Then $I \subset (M^*)^t$. To see this, let $b \in I$. It must be the case that $b \in (q_k')^t$ for all $k \geqslant n$ for some $n$. Thus $b \in (M^*)^t$. This explains why the power on $M^*$ in our representation can be taken to be non-negative.

This gives a description of a generating set for $\mathcal{S}(D_q)$. The equivalence classes can be described under the relation $S_\lambda \equiv S_\gamma$ if and only if $S_\lambda \bigtriangleup S_\gamma$ is finite and the equivalence classes corresponding to the powers of $M^*$.

Another interesting property of $\mathcal{S}(D_q)$ is that for all nontrivial $I \in \mathcal{S}(D_q)$, there exists $I' \in \mathcal{S}(D_q)$ such that $II' \cong (M^*)^k \pmod{\mathcal{P}(D_q)}$ for some $k \geqslant 1$. To see this, write $I = I_S$ and $I' = I_{S^c}$ as before. Now $II' = (q)(M^*)^k$.

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