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MONOMIAL IDEALS WITH TINY SQUARES AND FREIMAN IDEALS

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Abstract. We provide a construction of monomial ideals in $R = K[x,y]$ such that $\mu(I^2) < \mu(I)$, where μ denotes the least number of generators. This construction generalizes the main result of S. Eliahou, J. Herzog, M. Mohammadi Saem (2018). Working in the ring R, we generalize the definition of a Freiman ideal which was introduced in J. Herzog, G. Zhu (2019) and then we give a complete characterization of such ideals. A particular case of this characterization leads to some further investigations on $\mu(I^k)$ that generalize some results of S. Eliahou, J. Herzog, M. Mohammadi Saem (2018), J. Herzog, M. Mohammadi Saem, N. Zamani (2019), and J. Herzog, A. Asloob Qureshi, M. Mohammadi Saem (2019).

Keywords: Freiman ideal; number of generator; power of ideal; Ratliff-Rush closure

MSC 2020: 13E15, 13F20, 05E40

1. Introduction

Let I be a monomial ideal in the polynomial ring $K[x_1,\dots,x_n]$ over a field K. Let $\mu(I)$ denote the least number of generators of the ideal I. For a sufficiently large k, it is well known that $\mu(I^k)$ is a polynomial in k of degree $l(I)-1$, where $l(I)$ is the analytical spread of I. This implies that $\mu(I^k) < \mu(I^{k+1})$ for $k \gg 0$. But what is the behavior of $\mu(I^k)$ for small integers k? It has been proved in Herzog et al. (see [9]) that $\mu(I^k) < \mu(I^{k+1})$ for all $k \geqslant 1$ if depth $F(I) > 0$ and height $I \geqslant 2$, where $F(I)$ is the fiber ring of I. On the other hand, whenever $F(I) = 0$, Eliahou et al. in [5] constructed ideals of height 2 in $K[x,y]$ such that $\mu(I^2) = 9$ and $\mu(I) = m$, where $m \geqslant 6$ can be any integer. One goal of this paper is to provide a simpler construction of families of ideals in $K[x,y]$ with $\mu(I)-$ $\mu(I^2)$ arbitrary large. In particular, for any fixed odd integer $t \geqslant 9$ we produce an ideal I in $K[x,y]$ with $\mu(I^2) = t$ and $\mu(I)$ arbitrarily large.

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As a consequence of the well known theorem due to Freiman (see [6]) from additive number theory, Herzog et al. in [9], Theorem 1.9 showed that if I is an equigenerated monomial ideal then $\mu(I^2) \geqslant l(I)\mu(I) - \binom{l(I)}{2}$. If the equality holds, then Herzog and Zhu in [10] call the equigenerated monomial ideal I a Freiman ideal. Our investigation of $\mu(I^2)$ leads us to generalize (by dropping the equigenerated assumption) the definition of Freiman ideals in the ring $K[x,y]$. In particular, we say that an ideal I in $K[x,y]$ is a Freiman ideal whenever $\mu(I^2) = 2\mu(I) - 1$. According to this definition, we give in Subsection 4.1 a complete characterization of Freiman ideals in $K[x,y]$ and we show that they exist profoundly. In Subsection 4.2, we discuss an interesting particular case of our characterization of Freiman ideals, see Theorem 4.2. This case leads us to generalize some results from [5], [8], [9] concerning the behavior of powers of equigenerated and concave ideals. If I is an equigenerated monomial ideal in $K[x,y]$ or, more generally, the exponents of the generators of I lie in one line, then we show that $\mu(I^k) \geqslant k(\mu(I)-1)+1$ for all $k \geqslant 2$. In such a case, we give a complete characterization of the case when $\mu(I^k) = k(\mu(I)-1)+1$ for all $k \geqslant 2$, see Lemma 4.1.

In the ring $K[x,y]$, there is no loss of generality if we work with $\langle x,y\rangle$-primary ideals due to removing the common factor. Throughout this paper, I is an ideal of the form $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle \subseteq \overline{\langle y^{b_0}, x^{a_n}\rangle}$, where $\overline{\langle y^{b_0}, x^{a_n}\rangle}$ is the integral closure of $\langle y^{b_0}, x^{a_n}\rangle$. Denote this class of ideals by $\mathcal{C}$. The organization of this paper is as follows:

In Section 2, we introduce some semigroups S, S_i and T, T_j in $\mathbb{Z}^2$ and their associated ideals I_S, I_{S_i} and I_T, I_{T_j}. We explore some properties of these ideals that are needed later in this paper. We conclude this section with a decomposition of the powers of I in terms of I_{S_i} and I_{T_j} as given in Lemma 2.1. This lemma leads to the starting point of Section 3 in which we produce a formula for obtaining $\mu(I^2)$. This allows us to introduce a simple construction of ideals I with $\mu(I^2) = 9$ and $\mu(I)$ arbitrarily large in Theorem 3.1. Wee see that this construction produces monomial ideals with tiny squares. As an application of this construction, we obtain in Propositions 3.1 and 3.2 generalizations of the main result of Eliahou, Herzog and Saem, see [5], Theorem 3.1.

In Section 4 we give a complete characterization of Freiman ideals of the class $\mathcal{C}$, see Theorem 4.1. In Theorem 4.2 we show that $\mu(I^k) = k(\mu(I)-1)+1$ whenever $I_{S_2} = I = I_{T_2}$. Such a situation holds for the concave ideals (see Proposition 4.3) giving us direct proofs of Proposition 4.2 in [9] and Proposition 2.1 in [8]. In Lemma 4.1 we study $\mu(I^k)$ whenever the exponents of the generators of I lie on one line. This leads us to generalize Corollaries 3.4 and 3.5 of [9], Proposition 4.3 of [5], and Theorem 1.9 of [9] if restricted to the ring $K[x,y]$.

In Section 5 we completed this paper with an example that demonstrates how easy it is to graphically obtain S and T.

2. Semigroups and their associated ideals

Let I be an ideal of a ring R and let $\overline{I}$ be the *integral closure* of I in R. If I is a monomial ideal in a polynomial ring $K[x_1,\dots,x_n]$, where K is a field, then it is well known that, see [11], Proposition 1.4.6 or [7], Corollary 1.4.3, finding the integral closure of the ideal I is the same as finding all the integer lattice points in the convex hull in $\mathbb{R}^n$ of $\Gamma(I)$, where $\Gamma(I)$ denotes the set of exponents of all monomials in I. Therefore, if a and b are positive integers, then $x^{a_k}y^{b_k}\in H:=\overline{\langle y^b,x^a\rangle}$ if and only if $(b-b_k)/a_k\leqslant b/a$. Moreover, if $x^{a_1}y^{b_1},x^{a_2}y^{b_2}\in H$ then either $a_1+a_2\geqslant a$ (then $x^{a_1+a_2-a}y^{b_1+b_2}\in H$) or $b_1+b_2\geqslant b$ (then $x^{a_1+a_2}y^{b_1+b_2-b}\in H$); this is true since it is well known that H is a normal ideal, in particular, $H^2=\overline{\langle y^{2b},x^{2a}\rangle}$.

Let $I=\langle y^{b_0},x^{a_1}y^{b_1},\dots,x^{a_{n-1}}y^{b_{n-1}},x^{a_n}\rangle\subseteq\overline{\langle y^{b_0},x^{a_n}\rangle}\subset K[x,y]$. Al-Ayyoub in [1] produced an algorithm for computing the Ratliff-Rush closure $\tilde{I}$ of I. We summarize this algorithm. Let Ω be the numerical semigroup in $\mathbb{Z}^2$ generated by the set $\{(a_k,b_k)\mid k=0,\dots,n\}$ with $a_0=0$ and $b_n=0$, that is,

$$\Omega=\Big\{(\alpha,\beta)=\sum_{k=0}^{n}\lambda_k(a_k,b_k)\mid\lambda_k\in\mathbb{N}\Big\}.$$

Construct two semigroups S and T,

$$\begin{aligned}S&=\{(\alpha,\beta \bmod b_0)\mid(\alpha,\beta)\in\Omega\text{ with }\alpha<a_n\},\\ T&=\{(\alpha \bmod a_n,\beta)\mid(\alpha,\beta)\in\Omega\text{ with }\beta<b_0\},\end{aligned}\tag{2.1}$$

where the modulo operation is taken to be the least positive residue. Define the ideals

$$I_S=\langle x^{\alpha}y^{\beta'}\mid(\alpha,\beta')\in S\rangle\quad\text{and}\quad I_T=\langle x^{\alpha'}y^{\beta}\mid(\alpha',\beta)\in T\rangle.\tag{2.2}$$

In fact, these two ideals were first introduced in [1], where it is proved that $\tilde{I}=I_S\cap I_T$ and $\tilde{I}=\bigcup_{n\geqslant1}(I^{n+1}:I^n)$ is the Ratliff-Rush closure of the ideal I.

In order to proceed to the results of this paper, we need to refine the semigroups given in (2.1) into new semigroups S_i and T_j that are given below. In fact, these new semigroups were first introduced in [3] and later used in [2].

$$S_i=\Big\{(\alpha,\beta \bmod b_0)\mid(\alpha,\beta):=\sum_{k=0}^{n}\lambda_k(a_k,b_k)\in\Omega\text{ with }\alpha<a_n\text{ and}\sum_{k=0}^{n}\lambda_k\leqslant i\Big\},\tag{2.3}$$

$$T_j=\Big\{(\alpha \bmod a_n,\beta)\mid(\alpha,\beta):=\sum_{k=0}^{n}\lambda_k(a_k,b_k)\in\Omega\text{ with }\beta<b_0\text{ and}\sum_{k=0}^{n}\lambda_k\leqslant j\Big\}.\tag{2.4}$$

Simply, the semigroups S, T, S_i and T_j can be obtained graphically, which is demonstrated in Example 5.1. Define the ideals

$$I_{S_i} = \langle x^{\alpha} y^{\beta'} \mid (\alpha, \beta') \in S_i \rangle \quad \text{and} \quad I_{T_j} = \langle x^{\alpha'} y^{\beta} \mid (\alpha', \beta) \in T_j \rangle.$$

Note that $I_{S_1} = I_{T_1} = I$. Before we proceed, we explore some properties that the semigroups S and T, and their corresponding ideals I_S and I_T enjoy.

Remark 2.1. Given the semigroups S an T as before. Then:
(1) If $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in S$ with $\alpha_1 + \alpha_2 < a_n$, then $(\alpha_1 + \alpha_2, (\beta_1 + \beta_2) \bmod b_0) \in S$.
(2) If $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in T$ with $\beta_1 + \beta_2 < b_0$, then $((\alpha_1 + \alpha_2) \bmod a_n, \beta_1 + \beta_2) \in T$.

Remark 2.2. It is obvious that $I_{S_2} = I \Leftrightarrow I_S = I$. Also, $I_{T_2} = I \Leftrightarrow I_T = I$.

Notation 2.1. *By $\mathcal{G}(L)$ we mean the set of the minimal generators of a monomial ideal L.*

Remark 2.3. Let $I = \langle y^{b_0}, x^{a_1} y^{b_1}, \dots, x^{a_{n-1}} y^{b_{n-1}}, x^{a_n} \rangle \subseteq \overline{\langle y^{b_0}, x^{a_n} \rangle}$. Let J be the monomial ideal given by $J = \langle x^d y^c \mid x^c y^d \in \mathcal{G}(I) \rangle$. Then $\mathcal{G}(J_S) = \{x^{\beta} y^{\alpha} \mid x^{\alpha} y^{\beta} \in \mathcal{G}(I_T)\}$ and also $\mathcal{G}(J_T) = \{x^{\beta} y^{\alpha} \mid x^{\alpha} y^{\beta} \in \mathcal{G}(I_S)\}$. Thus, $I_S = I \Leftrightarrow J_T = J$. Also, $I_T = I \Leftrightarrow J_S = J$.

Remark 2.4. Let $I = \langle y^{b_0}, x^{a_1} y^{b_1}, \dots, x^{a_{n-1}} y^{b_{n-1}}, x^{a_n} \rangle$ be such that (a_k, b_k) lies on the line connecting $(0, b_0)$ and $(a_n, 0)$ for all k. Then any minimal element of S or T lies on the same line. This implies that $\mathcal{G}(I) \subseteq \mathcal{G}(I_S)$ and $\mathcal{G}(I) \subseteq \mathcal{G}(I_T)$. In such a case, any generator of I^l is of the form $(y^{b_0})^{l-1-r} x^{\alpha} y^{\beta} (x^{a_n})^r$ with (α, β) lying on that same line and $0 \leqslant r \leqslant l-1$. In other words, if $\omega = x^{\delta_1} y^{\delta_2}$ is a generator of I^l, then (δ_1, δ_2) lies on the line connecting the points $(0, l b_0)$ and $(l a_n, 0)$.

Proof. We prove the remark for the elements of S, while the same proof works for the elements of T. Let (a_i, b_i) and (a_j, b_j) be with $a_i + a_j < a_n$, hence $b_i + b_j \geqslant b_0$. Let $d \equiv b_i + b_j \bmod b_0$, that is, $d = b_i + b_j - b_0$. The proof is over if we show that the absolute value of the slope of the line connecting the points (a_j, b_j) and $(a_i + a_j, d)$ is b_0/a_n, but this is true since $(b_j - d)/a_i = (b_0 - b_i)/a_i = b_0/a_n$. □

Recall that $I = \langle y^{b_0}, x^{a_1} y^{b_1}, \dots, x^{a_{n-1}} y^{b_{n-1}}, x^{a_n} \rangle$ with $u_k = x^{a_k} y^{b_k} \in \overline{\langle y^{b_0}, x^{a_n} \rangle}$, i.e., $(b_0 - b_k)/a_k \leqslant b_0/a_n$ for $k = 1, \dots, n-1$. Let $(\alpha, \beta) = \sum_{k=0}^{n} \lambda_k (a_k, b_k)$ with $\sum_{k=0}^{n} \lambda_k = i$, that is, $x^{\alpha} y^{\beta} \in I^i$. Notice that if $\sum_{k=0}^{n} \lambda_k a_k < a_n$ then $\sum_{k=0}^{n} \lambda_k a_k / a_n < 1$, and thus $\sum_{k=0}^{n} \lambda_k (b_0 - b_k) \leqslant \sum_{k=0}^{n} \lambda_k (a_k/a_n) b_0 < b_0$; hence $(i-1) b_0 < \sum_{k=0}^{n} \lambda_k b_k \leqslant i b_0$. This allows us to conclude the following remark and lemma.

Remark 2.5. Let I, I_{S_i} and I_{T_j} be as above and let $u_0 = y^{b_0}$ and $u_n = x^{a_n}$. Then

(1) $\xi \in I_{S_i} \Leftrightarrow u_0^{i-1}\xi \in I^i$,

(2) $\xi \in I_{T_j} \Leftrightarrow \xi u_n^{j-1} \in I^j$.

Lemma 2.1. *With the notation as before, we have*

$$I^l = u_0^{l-1} I_{S_l} + x^{a_n} y^{b_0} M + u_n^{l-1} I_{T_l} \tag{2.5}$$

for any $l \geqslant 2$, where $M = I^l : x^{a_n} y^{b_0}$.

Proof. Let $\omega = x^\alpha y^\beta \in I^l$. Write $\omega = u_{i_1} u_{i_2} \dots u_{i_l}$ and let (a_{i_j}, b_{i_j}) be the exponent of u_{i_j}, where $0 \leqslant i_j \leqslant n$. Note that $(\alpha, \beta) = \sum\limits_{j=1}^{l} (a_{i_j}, b_{i_j}) \in \Omega$. Assume $\omega \notin x^{a_n} y^{b_0} M$, then either $\alpha < a_n$ or $\beta < b_0$. If $\alpha < a_n$, then $(\alpha, \beta') \in S_l$, where $\beta' \equiv \beta \bmod b_0$ with $0 < \beta' \leqslant b_0$. According to the discussion above, we have $\beta = (l-1)b_0 + \beta'$; thus $\omega \in u_0^{l-1} I_{S_l}$. Similarly, if $\beta < b_0$ then $\omega \in u_n^{l-1} I_{T_l}$. This shows $I^l \subseteq u_0^{l-1} I_{S_l} + x^{a_n} y^{b_0} M + u_n^{l-1} I_{T_l}$. The reverse statement is true by Remark 2.5. $\square$

A natural question that may come to mind is: does the decomposition (2.5) still hold for the case when $u_k = x^{a_k} y^{b_k} \notin \overline{\langle y^{b_0}, x^{a_n} \rangle}$ for some k? The answer is positive as long as the power l is less than some integer, namely, $l < q := \min\{q_S, q_T\}$, where

$$q_S = \min\Bigl\{\sum_{k=1}^{n-1} \lambda_k \Bigm| \sum_{k=1}^{n-1} \lambda_k (b_0 - b_k) \geqslant b_0 \text{ and } \sum_{k=1}^{n-1} \lambda_k a_k < a_n\Bigr\}, \tag{2.6}$$

$$q_T = \min\Bigl\{\sum_{k=1}^{n-1} \lambda_k \Bigm| \sum_{k=1}^{n-1} \lambda_k (a_n - a_k) \geqslant a_n \text{ and } \sum_{k=1}^{n-1} \lambda_k b_k < b_0\Bigr\}. \tag{2.7}$$

Whenever $x^{a_k} y^{b_k} \notin \overline{\langle y^{b_0}, x^{a_n} \rangle}$ for some k, then S_i is still defined as in (2.3) but for $i < q_S$. Also, T_j is still defined as in (2.4) but for $j < q_T$. In such a case, the discussion before Lemma 2.1, and also its proof, guarantee that the decomposition (2.5) still apply but for $l \leqslant q - 1$.

3. Monomial ideals with tiny squares

Now we have a direct method for counting the number of generators of the square of any ideal $I \subset \overline{\langle y^{b_0}, x^{a_n}\rangle}$, or $I \not\subseteq \overline{\langle y^{b_0}, x^{a_n}\rangle}$ with $q > 2$. Applying the decomposition in (2.5) we get

$$I^2 = u_0 I_{S_2} + u_n I_{T_2}. \tag{3.1}$$

Therefore,

$$\mu(I^2) = \mu(I_{S_2}) + \mu(I_{T_2}) - 1. \tag{3.2}$$

This enables us to construct families of monomial ideals such that $\mu(I) - \mu(I^2)$ grows to infinity. Inspired by [1], Corollary 18 and Figure 5, we introduce a construction of such ideals. This construction is simple, however it generalizes a main result of Eliahou, Herzog and Saem, see [5], Theorem 3.1.

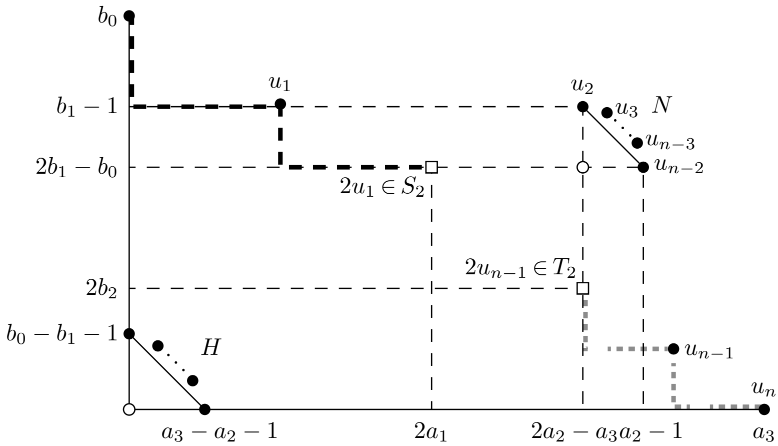

Figure 1.

Figure 1 gives a good visualization of the following construction. Given a monomial ideal $J = \langle y^{b_0}, x^{a_1}y^{b_1}, x^{a_2}y^{b_2}, x^{a_3}\rangle$ with $x^{a_3}y^{b_0}$ dividing $x^{a_1+a_2}y^{b_1+b_2}$, that is, $a_3 \leqslant a_1 + a_2$ and $b_0 \leqslant b_1 + b_2$. Write $u_0 = y^{b_0}$, $u_1 = x^{a_1}y^{b_1}$, $u_2 = x^{a_2}y^{b_2}$ and $u_3 = x^{a_3}$, and let $a_1 < a_2 < a_3$ and $b_0 > b_1 > b_2$. Since the set $\{u_0, \ldots, u_3\}$ is assumed to be the minimal set of generators of J, and since $a_3 \leqslant a_1 + a_2$ and $b_0 \leqslant b_1 + b_2$, then we must have that $a_2 > \frac{1}{2}a_3$ and $b_1 > \frac{1}{2}b_0$. Note that $J_{S_2} = J + \langle x^{2a_1}y^{2b_1-b_0}\rangle$ and $J_{T_2} = J + \langle x^{2a_2-a_3}y^{2b_2}\rangle$. Note also that the pairs (a_1, b_1) and (a_2, b_2) can be chosen so that $J \subsetneq J_{S_2} \cap J_{T_2}$. Let N be a monomial ideal with $\mathcal{G}(N) \subseteq (J_{S_2} \cap J_{T_2}) \setminus J$, that is,

$$N \subseteq \langle x^{2a_1}y^{2b_1-b_0}\rangle \cap \langle x^{2a_2-a_3}y^{2b_2}\rangle.$$

Note that $1 \leqslant \mu(N) \leqslant \min\{b_0 - b_1, a_3 - a_2\} - 1$. Now define the ideal

$$I = J + N.$$

Theorem 3.1. *For every integer $m \geqslant 5$, there exists a monomial ideal $I \subset K[x, y]$ such that $\mu(I^2) = 9$ and $\mu(I) = m$.*

Proof. Let J, N and I be the ideals as given in the construction above. The proof is over if we show that $\mu(I_{S_2}) = \mu(I_{T_2}) = 5$. In particular, we show that $I_{S_2} = J_{S_2}$ and $I_{T_2} = J_{T_2}$. We prove the first equality, while the second equality is proved similarly. We already have $J_{S_2} = J + \langle x^{2a_1} y^{2b_1 - b_0}\rangle$, hence showing $I_{S_2} = J_{S_2}$ is equivalent to showing that the set $I_{S_2} \setminus I$ contains only one minimal element whose exponent is given by $(2a_1, 2b_1 \bmod b_0) = (2a_1, 2b_1 - b_0)$. This claim is true according to the following three arguments. By $\Gamma(N)$ we mean the set of exponents of the monomials in N.

(i) If $(c_1, d_1), (c_2, d_2) \in \Gamma(N)$, then $c_1, c_2 \geqslant g := \max\{2a_1, 2a_2 - a_3\} \geqslant \frac{1}{2}a_3$ since $a_1 + a_2 \geqslant a_3$; thus, $c_1 + c_2 \geqslant a_3$. Therefore, no minimal element in $I_{S_2} \setminus I$ can be produced using combinations of elements of $\Gamma(N) \cup \{(a_2, b_2)\}$.

(ii) If $(c, d) \in \Gamma(N)$, then $c \geqslant 2a_2 - a_3$; thus, $a_1 + c \geqslant a_2 + (a_1 + a_2) - a_3 \geqslant a_2$ as $a_1 + a_2 \geqslant a_3$. Also, since $d \geqslant 2b_2$, then $b_1 + d - b_0 \geqslant b_2 + (b_2 + b_1) - b_0 \geqslant b_2$ as $b_1 + b_2 \geqslant b_0$. Thus, $(a_1 + c, (b_1 + d) \bmod b_0)$ is a multiple of (a_2, b_2). Therefore, no minimal element in $I_{S_2} \setminus I$ can be produced using any combination of (a_1, b_1) along with an element from $\Gamma(N)$.

(iii) Since $a_3 \leqslant a_1 + a_2$, then no minimal element in $I_{S_2} \setminus I$ can be produced using any combination that involves both (a_1, b_1) and (a_2, b_2). □

Proposition 4.1 of [5] gives a construction for ideals that are generated in two degrees for which $\mu(I) - \mu(I^2)$ is arbitrary large and such that $\mu(I^2) = 9$ and I^2 is generated in a single degree. In the following proposition, whose proof is a direct application of the above theorem, we generalize this by giving a construction for ideals generated in multiple degrees and such that I^2 is also generated in multiple degrees. Figure 1 visualizes such a construction, where $\mathcal{G}(I) = \{u_0, u_1, u_2, \ldots, u_{n-2}, u_{n-1}, u_n\}$.

Proposition 3.1. *Let $J = \langle y^{b_0}, x^{a_1} y^{b_1}, x^{a_2} y^{b_2}, x^{a_3}\rangle$ be such that $x^{a_3} y^{b_0}$ divides $x^{a_1+a_2} y^{b_1+b_2}$ and $b_0 - b_1 \leqslant \frac{1}{4} b_0$ and $a_3 - a_2 \leqslant \frac{1}{4} a_3$. Let H be the integral closure of $\langle x^{a_3 - a_2 - 1}, y^{b_0 - b_1 - 1}\rangle$ and let $N = x^{2a_2 - a_3} y^{2b_1 - b_0} H$. Define the ideal I as*

$$I = J + N.$$

Then $\mu(I) = \mu(J) + \mu(N) = 4 + \min\{b_0 - b_1, a_3 - a_2\}$ and $\mu(I^2) = 9$.

The following proposition shows that for any even integer $m \geqslant 6$ there exists an ideal I with $\mu(I^2) = 2m+1$ and $\mu(I)$ is arbitrary large. Figure 2 gives a visualization for the construction of such ideals.

Proposition 3.2. *Let $m \geqslant 6$ be an even integer. Choose $c \geqslant 2$ and let $a = mc$. Let $H = \overline{\langle y^{c-1}, x^{c-1}\rangle}$ and define*

$$N = x^{3c}y^{a-3c}H + x^{5c}y^{a-5c}H + \ldots + x^{a-3c}y^{3c}H.$$

Let $J = \langle x^c y^{a-c}\rangle + \langle x^{a-c}y^c\rangle + \langle x^{2ic}y^{a-2ic}\rangle_{i=0}^{m/2}$. Define the ideal

$$I = J + N.$$

Then $\mu(I^2) = 2m+1$ and $\mu(I) = (\frac{1}{2}m - 2)(c+1) + 5$.

Proof. From Figure 2, it is direct to see that $I_{S_2} = I_{T_2} = \langle x^{ic}y^{a-ic}\rangle_{i=0}^{a/c}$; therefore, we have $\mu(I^2) = \mu(I_{S_2}) + \mu(I_{T_2}) - 1 = 2m+1$. Note that $\mu(N) = \frac{1}{2}(m-4)\mu(H) = (\frac{1}{2}m-2)c$ as $\mu(H) = c$. Also, note that $\mu(J) = \frac{1}{2}m+3$. Now, since $\mathcal{G}(J) \cap \mathcal{G}(N) = \varphi$, then we have $\mu(I) = \mu(J) + \mu(N) = (\frac{1}{2}m - 2)(c+1) + 5$. $\square$

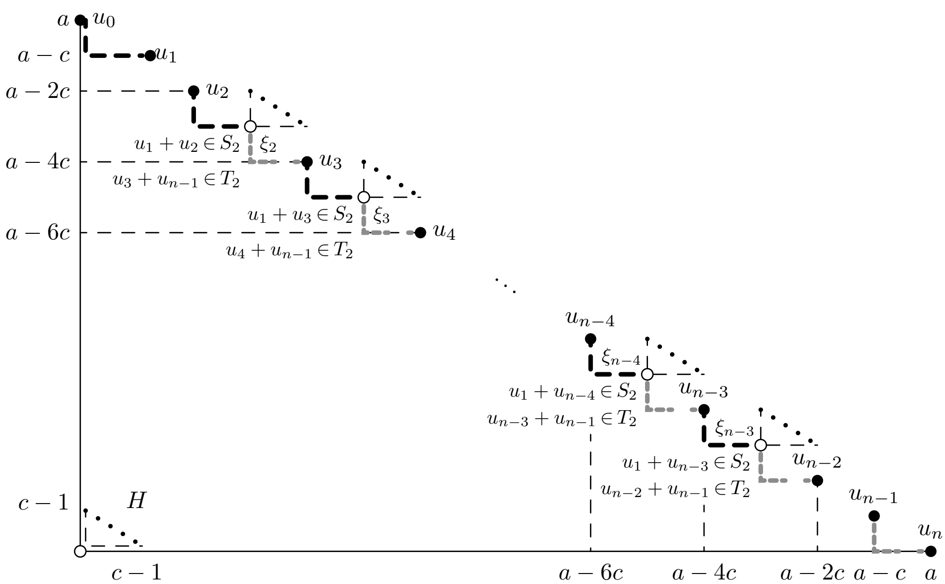

Figure 2.

4. Freiman ideals and their powers

Let $\mathcal{C}$ denote the class of ideals with $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle \subseteq \overline{\langle y^{b_0}, x^{a_n}\rangle}$. Our investigation of $\mu(I^2)$ leads us to study the Freiman ideals and their powers. In turn, this leads us to generalize some results from [5], [8] and [9]. Herzog et al. in [9], Theorem 1.9 showed that if I is an equigenerated monomial ideal then $\mu(I^2) \geqslant l(I)\mu(I) - \binom{l(I)}{2}$, where $l(I)$ is the analytical spread of I. Herzog and Zhu in [10] defined an equigenerated monomial ideal to be the *Freiman ideal* if $\mu(I^2) = l(I)\mu(I) - \binom{l(I)}{2}$. In the ring $K[x, y]$, there is no loss of generality if we work with $\langle x, y\rangle$-primary ideals due to removing the common factor; therefore, the last equality becomes

$$\mu(I^2) = 2\mu(I) - 1. \tag{4.1}$$

In $K[x, y]$, we generalize the definition of Freiman ideals by dropping the *equigenerated* assumption from the original definition of [10]. We say that a monomial ideal $I \in \mathcal{C}$ is the *Freiman ideal* if it satisfies the equality (4.1).

4.1. Characterization of Freiman ideals. In the following theorem we give a complete characterization of the Freiman ideals in $\mathcal{C}$.

Theorem 4.1. *An ideal $I \in \mathcal{C}$ is Freiman if and only if $\mu(I_{S_2}) + \mu(I_{T_2}) = 2\mu(I)$.*

Proof. The proof is direct since $\mu(I^2) = \mu(I_{S_2}) + \mu(I_{T_2}) - 1$ according to (3.2). □

The following corollary is a direct consequence of the above theorem, it coincides with Corollary 2.4 of [10].

Corollary 4.1. *Let I be an equigenerated monomial ideal in $K[x, y]$ of height 2. Then I is Freiman if and only if $I = \langle y^a, x^a\rangle^t$ for some integers a and t.*

Proof. The proof is a direct application of Theorem 4.1 as well as Lemmas 4.1 and 4.2. □

Freiman ideals profoundly exist in the ring $K[x, y]$. This assertion is supported by the following two propositions, and also by Corollaries 4.2, 4.3, and 4.4.

Proposition 4.1. *For any integer $d \geqslant 2$ there exists a Freiman ideal. In particular, if $I = \langle y^{3d}, x^d y^{2d}, x^{2d+1}y^d, x^{3d}\rangle$, then I is Freiman.*

Proof. The proof is direct since $I_{S_2} = \langle y^{3d}, x^d y^{2d}, x^{2d} y^d, x^{3d}\rangle$ and $I_{T_2} = I$. □

Proposition 4.2. *Let $r, q \geqslant 3$ be integers with $r + 2 < q < 2r$. Let $J = \langle y^{3q}, x^r y^{3q-1}, x^{r+q} y^q, x^{r+2q}\rangle$ and choose any two monomials w_1 and w_2 from $\langle x^{2r+1} y^{2q}\rangle \setminus J$ so that $w_i \nmid w_j$. Put*

$$I = J + \langle w_1, w_2\rangle.$$

Then I is Freiman. Therefore, there are many Freiman ideals for every pair $r, q \geqslant 3$.

Proof. Note that the bigger q is the more choices there are for the monomials w_1 and w_2. Write $u_1 = x^r y^{3q-1}$ and $u_2 = x^{r+q} y^q$. Figure 3 above explains the computation of I_{S_2} and I_{T_2}. Notice that y-deg$(u_2 u_1)$, y-deg$(u_2 w_i) > 3q$; hence, the only minimal element in $I_{T_2} \setminus I$ is obtained by $2u_2 \leftrightarrow x^r y^{2q}$. Also, notice that u_1, w_1 and w_2 are multiples of this new element; therefore,

$$I_{T_2} = \langle y^{3q}, x^r y^{2q}, u_2, x^{r+2q}\rangle.$$

Since $y - \deg(w_i) > 2q$, then $y - \deg(u_1 w_i) - 3q > y - \deg(u_2)$. Also, since we have $q < 2r$ and $x - \deg(w_i) \geqslant 2r + 1$, then $x - \deg(u_1 w_i) \geqslant x - \deg(u_2)$. This implies that the minimal elements in $I_{S_2} \setminus I$ are obtained by $2u_1 \leftrightarrow x^{2r} y^{3q-2}$ and $u_1 + u_2 \leftrightarrow x^{2r+q} y^{q-1}$; therefore,

$$I_{S_2} = \langle y^{3q}, u_1, x^{2r} y^{3q-2}, w_1, w_2, u_2, x^{2r+q} y^{q-1}, x^{r+2q}\rangle.$$

Now $2\mu(I) = \mu(I_{S_2}) + \mu(I_{T_2})$, hence the ideal I is Freiman by Theorem 4.1. $\square$

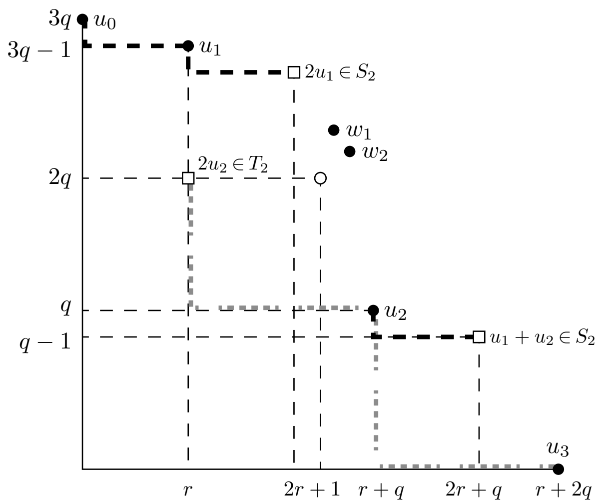

Figure 3.

4.2. Freiman ideals with reduction number one. A particular case of Theorem 4.1 occurs if $I_{S_2} = I = I_{T_2}$. Having an ideal to satisfy this case has turned out to be useful as we see in the remaining of this paper. Note that this case is equivalent to having $r(I) = 1$ for the reduction number of I. Therefore, any power I^k is a union of translations of the ideal I itself.

Theorem 4.2. *Let $I \subseteq \overline{\langle y^b, x^a \rangle}$ with $y^b, x^a \in I$. Then the following statements are equivalent.*

(1) $I_S = I = I_T$.

(2) $J := \langle y^b, x^a \rangle$ *is a reduction of I with $I^2 = JI$, hence $r(I) = 1$.*

In such a case, the ideal I is Freiman and

$$I^k = \sum_{j=0}^{k-1} (y^b)^{k-1-j} I (x^a)^j; \tag{4.2}$$

therefore, $\mu(I^k) = k(\mu(I) - 1) + 1$ for all $k \geqslant 2$.

Proof. (1) $\Rightarrow$ (2) By (3.1) we have $I^2 = y^b I_{S_2} + x^a I_{T_2} = y^b I + x^a I = JI$.

(2) $\Rightarrow$ (1) $y^b I_{S_2} + x^a I_{T_2} = I^2 = JI = y^b I + x^a I$; thus, the y-degree and the x-degree counts force $I_{S_2} = I$ and $I_{T_2} = I$, respectively. Hence, $I_S = I$ and $I_T = I$ by the virtue of Remark 2.2.

Since $I^2 = JI$ and $J = \langle y^b, x^a \rangle$, then it follows that $I^k = \sum_{j=0}^{k-1} (y^b)^{k-1-j} I (x^a)^j$ for all $k \geqslant 2$. Therefore, if $\omega \in \mathcal{G}(I^k)$ then $\omega = (y^b)^{k-1-j} u (x^a)^j$ for some $u \in \mathcal{G}(I)$ and some $j \in \{0, \ldots, k-1\}$. Thus, to prove $\mu(I^k) = k(\mu(I) - 1) + 1$ it suffices to show that if $u \in \mathcal{G}(I)$ then $(y^b)^{k-1-j} u (x^a)^j \in \mathcal{G}(I^k)$ for every $j \in \{0, \ldots, k-1\}$. Suppose not, that is, suppose $\delta := (y^b)^{k-1-j} u (x^a)^j$ is not a generator of I^k for some $u \in \mathcal{G}(I)$ and some $j \in \{0, \ldots, k-1\}$. This implies that there is a generator $\omega \in \mathcal{G}(I^k)$ such that $\omega \mid \delta$. But ω has to be of the form $(y^b)^{k-1-j'} v (x^a)^{j'}$ for some $v \in \mathcal{G}(I)$ and some $j' \in \{0, \ldots, k-1\}$. Since $(y^b)^{k-1-j'} v (x^a)^{j'} \mid (y^b)^{k-1-j} u (x^a)^j$, then the y-degree and the x-degree counts force that $j = j'$; thus $v \mid u$, a contradiction unless $v = u$. $\square$

Having an ideal I to be Freiman does not necessarily imply that $\mu(I^k) = k(\mu(I) - 1) + 1$ for every $k \geqslant 3$. To see this, let

$$I = \langle y^{14}, x^5 y^{12}, x^{11} y^{11}, x^{12} y^{10}, x^{16} y^2, x^{17} \rangle,$$

then routine computations with Singular (see [4]) show that $\mu(I^2) = 11$, hence I is Freiman, and $\mu(I^3) = 18 \neq 3(\mu(I) - 1) + 1$. On the other hand, it is direct to see that $I_{S_2} = \langle y^{14}, x^5 y^{12}, x^{10} y^{10}, x^{16} y^2, x^{17} \rangle$ and $I_{T_2} = I + \langle x^{15} y^4 \rangle$. This also gives that not all Freiman ideals satisfy the above theorem. On the other hand, powers of ideals that satisfy Theorem 4.2 show a systematic behavior which makes them easy to study. In the following three corollaries we produce families of such ideals. In Subsections 4.2.1 and 4.2.2 we discuss two of such families that have been of interest for some authors.

Corollary 4.2. *Let $I = \overline{\langle y^b, x^a\rangle}$ with a and b being integers. Then I satisfies Theorem* 4.2.

P r o o f. Clearly, $I_{S_2}, I_{T_2} \subseteq \overline{\langle y^b, x^a\rangle} = I$; hence $I_{S_2} = I = I_{T_2}$. □

Corollary 4.3. *If $I = \langle y^b, x^a\rangle + J$ with $J \subseteq \langle y^{\lceil b/2\rceil} x^{\lceil a/2\rceil}\rangle$, then I satisfies Theorem* 4.2.

P r o o f. Since the sum of the x-exponents of any two generators of I is greater than or equal to a, then $I_{S_2} = I$. Similarly, since the sum of the y-exponents of any two generators of I is greater than or equal to b, then $I_{T_2} = I$. □

Corollary 4.4. *For every pair of integers c and d, there are many monomial ideals that satisfy Theorem* 4.2. *More precisely, if we choose $a \in (c, \frac{3}{2}c]$ then the monomial ideal $I = \langle y^{3d}, x^{2c-a}y^{2d}, x^c y^d, x^a\rangle$ satisfies Theorem* 4.2.

P r o o f. It is direct to see that $I_{T_2} = I$ since $(2c \bmod a, 2d) = (2c - a, 2d)$ is the exponent of $x^{2c-a}y^{2d} \in \mathcal{G}(I)$. Now, we show that $I_{S_2} = I$. Since $3c \geqslant 2a$, then $2c \geqslant a$ and $3c - a \geqslant a$, that is, no element in $I_{S_2} \setminus I$ can be produced either by using two copies of the exponent of the monomial $x^c y^d$, or by using a combination of the exponents of the two monomials $x^c y^d$ and $x^{2c-a}y^{2d}$. Also, if $4c - 2a < a$ then $(4c - 2a, 4d \bmod (3d)) = (4c - 2a, d)$ is a multiple of (c, d), hence no element in $I_{S_2} \setminus I$ can be produced using two copies of $(2c - a, 2d)$. □

In fact, if we replace the generator $x^{2c-a}y^{2d}$ of the ideal in the above corollary with any set of monomials each of which is a multiple of $x^{2c-a}y^{2d}$, then the conclusion of the corollary still holds. Namely, let $J = \langle y^{3d}, x^c y^d, x^a\rangle$, and let N be any ideal with $N \subset \langle x^{2c-a}y^{2d}\rangle$ and $\mathcal{G}(J) \cap \mathcal{G}(N) = \varphi$, then the ideal $I = J + N$ satisfies Theorem 4.2.

4.2.1. Equigenerated ideals. Assume that the ideal I is equigenerated, or more generally, the exponents of the generators of I lie on one line. The following two lemmas treat such a case. These lemmas generalize Corollaries 3.4 and 3.5 of [9] and Proposition 4.3 of [5]. Also, if restricted to the ring $K[x, y]$, they generalize Theorem 1.9 of [9].

Lemma 4.1. *Let $I = \langle y^b, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^a\rangle$, where (a_i, b_i) lies on the line connecting $(0, b)$ and $(a, 0)$ for every i. Then:*
(a) $\mu(I^k) \geqslant k(\mu(I) - 1) + 1$ *for all* $k \geqslant 2$.
(b) *The following statements are equivalent.*
(1) $I_S = I = I_T$.
(2) $I^2 = JI$, *where* $J = \langle y^b, x^a\rangle$.
(3) $\mu(I^k) = k(\mu(I) - 1) + 1$ *for all* $k \geqslant 2$.
(4) $\mu(I^k) = k(\mu(I) - 1) + 1$ *for* $k = 2$.

P r o o f. (a) By Remark 2.4, $(y^b)^{k-1-j}x^{a_i}y^{b_i}(x^a)^j$ is a minimal generator of I^k for every i, that is, $\bigcup_{j=0}^{k-1}(y^b)^{k-1-j}\mathcal{G}(I)(x^a)^j \subseteq \mathcal{G}(I^k)$. This implies that

$$k(\mu(I) - 1) + 1 \leqslant \mu(I^k).$$

(b) According to Theorem 4.2, it suffices to prove (4) $\Rightarrow$ (1). Assume $x^\alpha y^\beta \in I_{S_2} \setminus I$. Then Remark 2.4 implies that (α, β) lies on the line connecting $(0, b)$ and $(a, 0)$, hence $x^{a_i}y^{b_i} \notin \langle x^\alpha y^\beta\rangle$ for every i; thus, $\mu(I) < \mu(I_{S_2})$. This proves that if $I \subsetneq I_{S_2}$, then $\mu(I) < \mu(I_{S_2})$. Similarly, if $I \subsetneq I_{T_2}$, then $\mu(I) < \mu(I_{T_2})$. Therefore, if $I \subsetneq I_{S_2}$ or $I \subsetneq I_{T_2}$, then $2(\mu(I) - 1) + 1 < \mu(I_{S_2}) + \mu(I_{T_2}) - 1 = \mu(I^2)$. $\square$

It worth noting that the inequality $\mu(I^k) \geqslant k(\mu(I) - 1) + 1$ is not necessarily true in general. In the following we give a complete characterization of the ideals that satisfy part (b) of Lemma 4.1 above.

Lemma 4.2. *Let $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle$, where (a_i, b_i) lies on the line connecting $(0, b_0)$ and $(a_n, 0)$ for every i, and $a_i < a_{i+1}$ and $b_i > b_{i+1}$ for all $0 \leqslant i \leqslant n - 1$. Then $I_S = I = I_T$ if and only if $I = \langle y^{b_{n-1}}, x^{a_1}\rangle^q$ for some q.*

P r o o f. The sufficient part is direct from the definitions of S and T. Thus we only need to prove the necessary part. We present the proof for the case $a_1 \leqslant a_n - a_{n-1}$, where we use the hypothesis $I_S = I$. The proof for the case that $a_1 > a_n - a_{n-1}$ follows by the symmetry (as in Remark 2.3), where we use the hypothesis $I = I_T$.

Write $a_n = pa_1 + r$ with $0 < r \leqslant a_1$. Then

$$(pa_1, pb_1 - (p - 1)b_0) = (pa_1, pb_1 \bmod b_0) \in S,$$

hence $x^{pa_1}y^{pb_1-(p-1)b_0}$ is a generator of I_S; thus, $x^{a_n-r}y^{pb_1-(p-1)b_0}$ is a generator of I as $I_S = I$. Note that $a_{n-1} \leqslant a_n - a_1 \leqslant a_n - r$. Now, since the ideal I has no generator δ with $a_{n-1} < x-\deg(\delta) < a_n$, then we must conclude that $a_{n-1} = a_n - r$, hence $r = a_1$; thus, a_1 divides a_n and $a_n = qa_1$ with $q = p + 1$. Since (a_{n-1}, b_{n-1}) lies on the line connecting $(0, b_0)$ and $(a_n, 0)$, then $b_0/a_n = b_{n-1}/(a_n - a_{n-1})$. But

$a_n - a_{n-1} = a_1$ and $a_n = qa_1$, thus $b_0 = qb_{n-1}$. Also, since (a_1, b_1) lies on the line connecting $(0, b_0)$ and $(a_n, 0)$, then $b_0/a_n = (b_0 - b_1)/a_1$; thus, $b_{n-1} = b_0 - b_1$.

We show $\langle y^{b_{n-1}}, x^{a_1}\rangle^q \subseteq I$. If $M = \langle y^{b_0}, x^{a_1}y^{b_1}, x^{a_n}\rangle$, then

$$M_S = \langle y^{b_0}, x^{ia_1}y^{ib_1 \bmod b_0}, x^{a_n}\rangle_{i=1}^{q-1}.$$

Note $ib_1 \bmod b_0 \equiv ib_1 - (i-1)b_0 = i(b_0 - b_{n-1}) - (i-1)b_0 = b_0 - ib_{n-1}$, hence we have $M_S = \langle y^{b_0}, x^{ia_1}y^{b-ib_{n-1}}, x^{a_n}\rangle_{i=1}^{q-1} = \langle y^{b_{n-1}}, x^{a_1}\rangle^q$. Since $M \subseteq I$, then $M_S \subseteq I_S$. But $I_S = I$, thus we have $\langle y^{b_{n-1}}, x^{a_1}\rangle^q \subseteq I$ as required.

The proof is over if we show that $I \subseteq \langle y^{b_{n-1}}, x^{a_1}\rangle^q$. For this purpose, suppose $x^{a_i}y^{b_i} \notin \langle y^{b_{n-1}}, x^{a_1}\rangle^q$ for some i. Write $a_n - a_i = sa_1 + r$ with $0 \leqslant r < a_1$. Note that $(a_i + sa_1, (b_i + sb_1) \bmod b_0)$ is a minimal element of S, hence it corresponds to a generator of $I_S = I$; thus, $x^{a_i+sa_1}y^{b_i+s(b_1-b_0)}$ is a generator of I. This gives a contradiction since $a_i + sa_1 = a_n - r > a_n - a_1 = a_{n-1}$ and since I has no generator δ with $a_{n-1} < x - \deg(\delta) < a_n$. □

4.2.2. Concave ideals. Let

$$0 = a_0 < a_1 < \ldots < a_n \quad \text{and} \quad b_0 > b_1 > \ldots > b_{n-1} > b_n = 0.$$

According to [8], the ideal $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle$ is called *concave* if $2a_i \geqslant a_{i-1} + a_{i+1}$ and $2b_i \geqslant b_{i-1} + b_{i+1}$ for $1 \leqslant i \leqslant n-1$. This implies that

$$a_{i+1} - a_i \leqslant a_i - a_{i-1} \quad \text{and} \quad b_i - b_{i+1} \geqslant b_{i-1} - b_i. \tag{4.3}$$

Therefore,

$$a_{h+1} - a_h \leqslant a_{k+1} - a_k \quad \text{and} \quad b_h - b_{h+1} \geqslant b_k - b_{k+1} \quad \text{for } h > k \geqslant 0. \tag{4.4}$$

Remark 4.1. Let I be a concave ideal as given above. If $a_i + a_j < a_n$, then $i + j < n$.

P r o o f. Assume $i + j \geqslant n$ and let $i + j - s = n$ with $s \geqslant 0$. Then

$$\begin{aligned} a_n &= a_1 + (a_2 - a_1) + \ldots + (a_i - a_{i-1}) + (a_{i+1} - a_i) \\ &\quad + \ldots + (a_{i+j-s} - a_{i+j-s-1}) \\ \text{(by (4.4))} &\leqslant a_1 + (a_2 - a_1) + \ldots + (a_i - a_{i-1}) + a_1 + (a_2 - a_1) \\ &\quad + \ldots + (a_{j-s} - a_{j-s-1}) \\ &\leqslant \underbrace{a_1 + (a_2 - a_1) + \ldots + (a_i - a_{i-1})}_{a_i} + \underbrace{a_1 + (a_2 - a_1) + \ldots + (a_j - a_{j-1})}_{a_j} \end{aligned}$$

and the proof is over. □

In [8], Proposition 2.1, it is proved that concave ideals satisfy part (2) of Theorem 4.2 of this paper. Also, in [9], Proposition 4.2, it is shown that if I is concave then $\mu(I^2) = 2\mu(I) - 1$. In the following proposition we show that concave ideals satisfy part (1) of Theorem 4.2, hence the results of [8] and [9] follow directly.

Proposition 4.3. *Let I be a concave ideal as given above. Then $I_S = I = I_T$. Therefore, $I^k = JI^{k-1}$ and $\mu(I^k) = k(\mu(I)-1)+1$ for all $k \geqslant 2$, where $J = \langle y^{b_0}, x^{a_n}\rangle$.*

P r o o f. We prove $I_S = I$, while the proof for $I_T = I$ follows by symmetry as in Remark 2.3. By the virtue of Remark 2.2, it suffices to show that $I_{S_2} = I$. We show that if $a_i + a_j < a_n$, then $(a_i + a_j, b_i + b_j - b_0)$ is a multiple of (a_{i+j}, b_{i+j}) with $i + j \leqslant n - 1$ by Remark 4.1. Assume $i \leqslant j$. Consider

$$\begin{aligned} a_i + a_j &= a_1 + (a_2 - a_1) + \ldots + (a_i - a_{i-1}) + a_j \\ \text{(by (4.4))} \qquad &\geqslant (a_{i+j} - a_{i+j-1}) + \ldots + (a_{j+1} - a_j) + a_j \\ &= a_{i+j}, \\ b_i + b_j &= b_0 - (b_0 - b_1) - \ldots - (b_{i-1} - b_i) + b_j \\ \text{(by (4.4))} \qquad &\geqslant b_0 - (b_j - b_{j+1}) - \ldots - (b_{i+j-1} - b_{i+j}) + b_j \\ &= b_0 + b_{i+j}, \end{aligned}$$

hence, the proof is over. □

In [8], a concave ideal $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle$ is said to have a *corner* at the point (a_i, b_i) if $2a_i > a_{i-1} + a_{i+1}$ and $2b_i > b_{i-1} + b_{i+1}$. Combining Proposition 4.3 and Lemma 4.2 we obtain the following result which replaces Proposition 2.3 of [8].

Corollary 4.5. *Let $I = \langle y^{b_0}, x^{a_1}y^{b_1}, \ldots, x^{a_{n-1}}y^{b_{n-1}}, x^{a_n}\rangle$ be a concave ideal as given above. Then I has no corner points if and only if $I = \langle y^{b_{n-1}}, x^{a_1}\rangle^q$ for some q.*

P r o o f. Suppose that I has no corner points, that is, $2a_i = a_{i-1} + a_{i+1}$ and $2b_i = b_{i-1} + b_{i+1}$ for $1 \leqslant i \leqslant n - 1$. But this simply implies that the points (a_{i-1}, b_{i-1}), (a_i, b_i), and (a_{i+1}, b_{i+1}) lie on the same line for all i. By Proposition 4.3 we have $I_S = I = I_T$ since I is concave. Hence, we are finished by Lemma 4.2. Conversely, it is obvious that an ideal of the form $\langle y^c, x^d\rangle^q$ is concave with no corner points for all positive integers c, d, and q. □

5. Appendix

Since the semigroups S and T play a major role in this paper and the proposed future work, we see that it is worthy to demonstrate how easy obtaining these semigroups becomes if we work graphically. This is carried out in the following example. First, we must agree on the term "*path*" as follows; given that $u_0 = (0, b_0)$, $u_n = (a_n, 0)$ and $u_i = (c, d)$. A $u_0 \searrow u_i$ *path* means moving $b_0 - d$ units downward and then c units rightward. A $u_i \nwarrow u_n$ *path* means moving $a_n - c$ units leftward and then d units upward.

Example 5.1. Given the ideal $I = \langle y^{20}, x^4y^{18}, x^{10}y^{15}, x^{14}y^{12}, x^{21}y^7, x^{26}y^3, x^{30}\rangle$. Recall that $T = \left\{ \sum_{i=0}^{6} \lambda_i(a_i \bmod 30, b_i) \,\middle|\, \sum_{i=0}^{6} \lambda_i b_i \leqslant 20 \right\}$. The dashed path in Figure 4 represents the $u_5 \nwarrow u_6$ path, while the vacuumed path represents the $u_4 \nwarrow u_6$ path. The figure below represents all the possible combinations of paths that lead to the minimal generators of the ideal I_T. For instance, combining two $u_4 \nwarrow u_6$ paths followed by one $u_5 \nwarrow u_6$ path, we reach the point $\tau_7 \leftrightarrow 2u_4 + u_5$ that corresponds to x^8y^{17} as a minimal generator of I_T. Also, combining one $u_4 \nwarrow u_6$ path followed

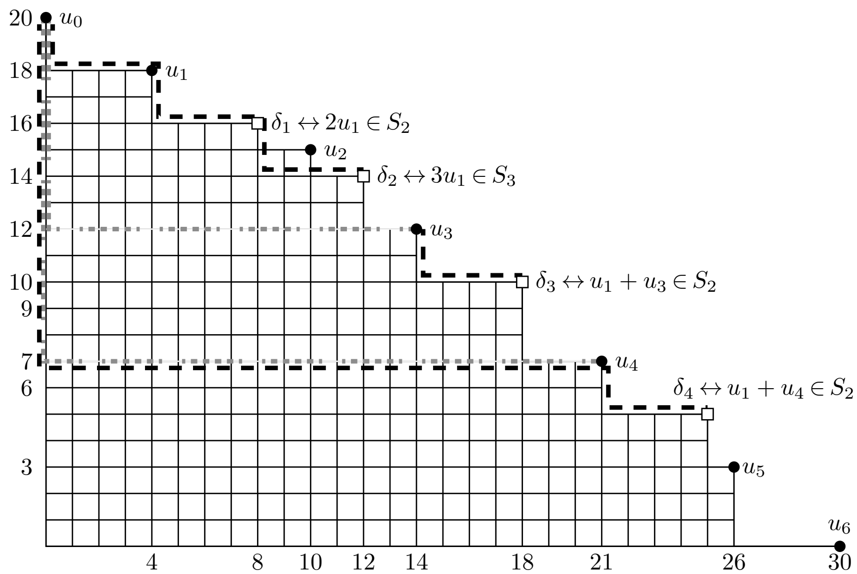

Figure 4.

by three $u_5 \nwarrow u_6$ paths, we reach the point $\tau_6 \leftrightarrow u_4 + 3u_5$ that corresponds to x^9y^{16} as a minimal generator of I_T. And also, combining three $u_5 \nwarrow u_6$ paths, we reach the point $\tau_2 \leftrightarrow 3u_5$ that corresponds to $x^{18}y^9$ as a minimal generator of I_T. All other points with τ_i are reached in a similar manner. Note that combining one $u_3 \nwarrow u_6$

path followed by one $u_5 \nwarrow u_6$ path, we reach $(10, 15)$ which already corresponds to a minimal generator of I_T, thus such a walk does not appear in the figure. Combining one $u_2 \nwarrow u_6$ path followed by one $u_5 \nwarrow u_6$ path does not correspond to a minimal generator of I_T. Same thing applies for combining four $u_5 \nwarrow u_6$ paths. Such walks do not appear in the figure.

Recall that $S = \left\{\sum_{i=0}^{n} \lambda_i(a_i, b_i \bmod 20) \,\middle|\, \sum_{i=0}^{n} \lambda_i a_i \leqslant 30\right\}$. The dashed path in Figure 5 represents the $u_0 \searrow u_1$ path, while the vacuumed path represents the $u_0 \searrow u_3$ path. The figure below represents all the possible combinations of paths that lead to the minimal generators of the ideal I_S. For instance, combining three $u_0 \searrow u_1$ paths, we reach the point $\delta_2 \leftrightarrow 3u_1$ that corresponds to $x^{12}y^{14}$ as a minimal generator of I_S. Also, combining one $u_0 \searrow u_3$ path followed by one $u_0 \searrow u_1$ path, we reach the point $\delta_3 \leftrightarrow u_1 + u_3$ that corresponds to $x^{18}y^{10}$ as a minimal generator of I_S. Note that combining two $u_0 \searrow u_1$ paths, we reach the point that corresponds to $x^{20}y^{10}$ which not a minimal generator of I_S, thus such a walk does not appear in the figure.

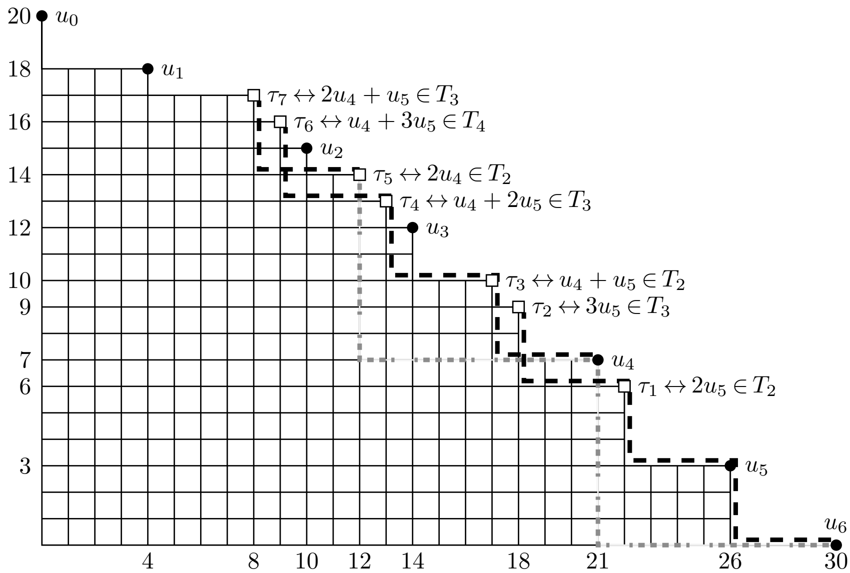

Figure 5.

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