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FINITE GROUPS WITH SOME SS-SUPPLEMENTED SUBGROUPS

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Abstract. A subgroup H of a finite group G is said to be SS-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is S-quasinormal in K. We analyze how certain properties of SS-supplemented subgroups influence the structure of finite groups. Our results improve and generalize several recent results.

Keywords: SS-supplemented subgroup; maximal subgroup; solvable group; minimal subgroup

MSC 2020: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite and G always denotes a finite group. Our notation and terminology are standard and the reader is referred to [4], [8]. Recall that a subgroup H of a group G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel and Deskins in 1962, see [10]. In 2012, Guo and Lu gave the definition of SS-supplemented subgroups.

Definition 1.1 ([6], Definition 2.1)**.** A subgroup H of a group G is called *SS-supplemented* in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K$ is S-quasinormal in K. In this case, we say that K is an SS-supplement of H in G.

Theorem 1.2 ([6], Theorem 3.3)**.** *A group G is solvable if and only if every maximal subgroup M of G has a subnormal SS-supplement in G.*

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The research on the SS-supplemented subgroups of a given group still continues and many related results have been recently obtained, see [11], [12]. It has been proved that the SS-supplemented subgroups are suitable for describing the structure of groups. The aim of this paper is to give a generalization of the above mentioned theorems. We investigate the solvability of some normal subgroup by using certain maximal subgroups, which is a generalization of the results known. We also study the structure of groups based on the assumption that every subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ of order p or 4 (if $p = 2$) is SS-supplemented in G, where $x \in G \setminus N_G(P)$ and $G^{\mathfrak{N}_p}$ is the p-nilpotent residual of G. Some results for a group to be p-nilpotent and supersolvable are obtained and many known results are generalized.

Recall that a formation $\mathfrak{F}$ is a class of groups which is closed under taking epimorphic images and such that every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of G and denoted by $G^{\mathfrak{F}}$. Throughout this paper, $\mathfrak{N}_{\mathfrak{p}}$ and $\mathfrak{N}$ denote the classes of p-nilpotent groups and nilpotent groups, respectively.

2. Preliminaries

In this section we present some lemmas, which are required in the proofs of our main results.

Lemma 2.1 ([6], Lemma 2.4). *Let H be an SS-supplemented subgroup of a group G. Then, the following statements hold:*

(1) *If M is a subgroup of G and $H \leqslant M$, then H is SS-supplemented in M.*
(2) *If N is a normal subgroup of G and $N \leqslant H$, then H/N is SS-supplemented in G/N.*
(3) *Let π be a set of primes. If H is a π-subgroup of G and N is a normal π'-subgroup of G, then HN/N is SS-supplemented in G/N.*

The following two lemmas are known results for S-quasinormal subgroups of a given group G.

Lemma 2.2 ([10]). *Let H be a subgroup of a group G. If H is S-quasinormal in G, then H is subnormal in G.*

Lemma 2.3 ([16], Lemma A). *If H is a p-subgroup of a group G for some prime p, then H is S-quasinormal in G if and only if $O^p(G) \leqslant N_G(H)$.*

Lemma 2.4 ([4], Lemma 14.3). *If A is a subnormal subgroup of a group G and B is a minimal normal subgroup of G, then $B \leqslant N_G(A)$.*

Lemma 2.5. *Let P be a Sylow p-subgroup of a group G and H a normal subgroup of G. If N is a normal p'-subgroup of G, then $HN \cap PN \cap P^x N = (H \cap P \cap P^{xn})N$ for some $n \in N$, where $x \in G \setminus N_G(P)$.*

P r o o f. From Sylow's theorem, we have $HN \cap PN = (HN \cap P)N = (H \cap P)N$. So $HN \cap PN \cap P^x N = (H \cap P \cap P^x N)N$. Take $P_0 = H \cap P \cap P^x N$. Then P_0 is contained in a Sylow p-subgroup of $P^x N$. Thus by Sylow's theorem again there exists an element n in N such that $P_0 \leqslant P^{xn}$. It follows that $P_0 = H \cap P \cap P^x N \geqslant H \cap P \cap P^{xn} \geqslant P_0$ and hence $P_0 = H \cap P \cap P^{xn}$. This implies that $HN \cap PN \cap P^x N = (H \cap P \cap P^{xn})N$. $\square$

A 2-group is called *quaternion-free* if it has no section isomorphic to the quaternion group of order 8.

Lemma 2.6 ([5], Theorem 2.8)**.** *If a solvable group G has a Sylow 2-subgroup P which is quaternion-free, then $P \cap Z(G) \cap G^{\mathfrak{N}} = 1$.*

Lemma 2.7. *Let H be a subgroup of a group G, then $H^{\mathfrak{N}_p} \leqslant G^{\mathfrak{N}_p}$.*

P r o o f. Since $HG^{\mathfrak{N}_p}/G^{\mathfrak{N}_p} \leqslant G/G^{\mathfrak{N}_p}$ and $G/G^{\mathfrak{N}_p}$ is p-nilpotent, we have that $H/(H \cap G^{\mathfrak{N}_p})$ is p-nilpotent and so $H^{\mathfrak{N}_p} \leqslant H \cap G^{\mathfrak{N}_p}$, as desired. $\square$

Lemma 2.8 ([1], Lemma 2)**.** *Let $\mathfrak{F}$ be a saturated formation. Assume that G is a non-$\mathfrak{F}$-group and there exists a maximal subgroup M of G such that $M \in \mathfrak{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then*
(1) *$G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$ is a chief factor of G;*
(2) *$G^{\mathfrak{F}}$ is a p-group for some prime p;*
(3) *$G^{\mathfrak{F}}$ has exponent p if $p > 2$ and exponent is at most 4 if $p = 2$;*
(4) *$G^{\mathfrak{F}}$ is either an elementary abelian group or $(G^{\mathfrak{F}})' = Z(G^{\mathfrak{F}}) = \Phi(G^{\mathfrak{F}})$ is an elementary abelian group.*

Lemma 2.9 ([17], Lemma 2.16)**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Let H be a normal subgroup of a group G. We define the following families of subgroups:

$$\mathfrak{M}(G) = \{M | M \lessdot G\},$$
$$\mathfrak{M}_{pc}(G) = \{M | M \in \mathfrak{M}(G),\ |G:M|_p = 1 \text{ and } |G:M| \text{ is composite}\},$$
$$\mathfrak{M}^{pcn}(G) = \{M | M \in \mathfrak{M}(G),\ N_G(P) \leqslant M \text{ for a Sylow } p\text{-subgroup } P \text{ of } G, M \text{ is nonnilpotent and } |G:M| \text{ is composite}\},$$
$$\mathfrak{M}_H(G) = \{M | M \in \mathfrak{M}(G) \text{ and } H \nleq M\}.$$

3. Main results

In this section, we firstly study the solvability of a normal subgroup H of a group G when some subgroups are assumed to be SS-supplemented subgroups of G.

Theorem 3.1. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$ has a subnormal SS-supplement in G, then H is solvable.*

P r o o f. If $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) = \emptyset$, then we claim that H is solvable. In fact, if $\mathfrak{M}_{pc}(G) = \emptyset$, by [13], Theorem 8, G is solvable and so is H. If $\mathfrak{M}_{pc}(G) \neq \emptyset$, then H is contained in every maximal subgroup M of G in $\mathfrak{M}_{pc}(G)$. Applying [13], Theorem 8 again, H is solvable. This proves our claim.

Now we may assume that $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) \neq \emptyset$. Let N be a minimal normal subgroup of G, and let M/N be a maximal subgroup of $\overline{G} = G/N$ with $M/N \in \mathfrak{M}_{pc}(\overline{G}) \cap \mathfrak{M}_{\overline{H}}(\overline{G})$. Then $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$. Furthermore, M/N has a subnormal SS-supplement in G/N by Lemma 2.1. It is clear that $(\overline{G}, \overline{H})$ satisfies the hypotheses of the theorem and so $\overline{H}$ is solvable by induction. If $N \nleqslant H$, then $H \cong \overline{H}$ is solvable, as desired. Hence, we may assume that $N \leqslant H$, and it follows that H/N is solvable. If G has two different minimal normal subgroups N_1 and N_2, then both H/N_1 and H/N_2 are solvable and so is $H/(N_1 \cap N_2)$. This implies that the group H is solvable. Hence we may assume that G has a unique minimal normal subgroup N.

Suppose that N is nonsolvable. Let q be the largest prime dividing the order of N and Q a Sylow q-subgroup of N. Then $G = N_G(Q)N$ by the Frattini argument. So there exists a maximal subgroup M of G which contains $N_G(Q)$, but $N \nleqslant M$. By hypothesis, $p \geqslant q$. If $p > q$, it is clear that $|G : M|_p = |N : M \cap N|_p$=1. If $p = q$, then $N_G(Q)$ contains a Sylow p-subgroup of G. Thus, we conclude that $|G : M|_p = 1$ in these two cases. If $|G : M| = r$ for some prime r, then, since $M_G = 1$, we have that G is isomorphic to a subgroup of the symmetric group S_r of degree r. This implies that $|G| \mid r!$, which is a contradiction as p is not a divisor of $r!$. Hence, we conclude that $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$.

By our hypotheses, there exists a subnormal subgroup K of G such that $G = MK$ and $M \cap K$ is S-quasinormal in K. Since K is subnormal in G, Lemma 2.2 implies that $M \cap K$ is subnormal in G. We claim that $M \cap K = 1$. Otherwise, we may take a minimal subnormal subgroup L of G contained in $M \cap K$. Since $L \cap N \trianglelefteq L$, either $L \cap N = 1$ or $L \leqslant N$. If $L \cap N = 1$, then from Lemma 2.4 $NL = N \times L$ and $L \leqslant C_G(N)$=1, a contradiction. Suppose $L \leqslant N$. We have $L^G = L^{NM} = L^M \leqslant M_G = 1$, which implies $L = 1$, a contradiction. Therefore $M \cap K = 1$. By using the same arguments, we can similarly prove that all minimal subnormal subgroups of G are contained in N. Let $N = N_1 \times \ldots \times N_r$, where each N_i is isomorphic to a fixed nonabelian simple group. It follows that $N_1, \ldots, N_r$ coincide with all minimal subnormal subgroups of G. Without loss of generality, we may assume that $N_1 \leqslant K$. Then a prime p exists such that p divides $|K| = |G : M|$. By [2], Lemma 3, we can see that N is solvable, this is a contradiction. The proof is completed. □

From Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let p be the largest prime dividing the order of a group G. Then G is solvable if and only if every maximal subgroup M of G in $\mathfrak{M}_{pc}(G)$ has a subnormal SS-supplement in G.*

Proof. From Theorem 1.2, only the sufficiency requires a proof. In fact, let $G = H$ in Theorem 3.1. Then we have the corollary. □

Remark 3.3. In Theorem 3.1, the group G is not necessary solvable. For example: Let L, H be the alternating groups of degree 5 and 4, respectively, and let $G = L \times H$. Suppose that $M = L \times C_3$, where C_3 is a cyclic group of order 3 of H. Then M is a maximal subgroup of G. It is clear that $H \nleq M$ and $|G : M| = 4$. Thus $M \in \mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G)$ and we can also see that $\mathfrak{M}_{pc}(G) \cap \mathfrak{M}_H(G) = \{M^g \colon g \in G\}$. Furthermore, it is easy to see that $G = MK_4$ and $M \cap K_4$ is S-quasinormal in K_4, where K_4 is the Klein four group contained in H. That is, M has a subnormal SS-supplement in G. However, G is not solvable.

Theorem 3.4. *Let H be a normal subgroup of a group G and p the largest prime dividing the order of G. If every maximal subgroup M of G in $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G)$ has a subnormal SS-supplement in G, then H is p-solvable.*

Proof. If $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) = \emptyset$, then we can see that H is p-solvable by [7], Lemma 2.4. Now, we may assume that $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) \neq \emptyset$. Let $P \in \mathrm{Syl}_p(G)$. If P is normal in G, then G is certainly p-solvable and so is H. So we may assume that $N_G(P) < G$.

Let N be a minimal normal subgroup of G. It is clear that G/N satisfies the hypotheses of the theorem for the normal subgroup HN/N and so HN/N is p-solvable by induction. By a routine argument, we can assume that N is contained in H and N is the unique minimal normal subgroup of G.

Suppose that N is not p-solvable. Then p is a divisor of the order of N. We know that $N \cap P \in \mathrm{Syl}_p(N)$ and $P \cap N$ is not a normal subgroup of N. By the Frattini argument, we have that $G = N_G(P \cap N)N$. So there exists a maximal subgroup M of G which contains $N_G(P \cap N)$ and $M \ngeqslant N$. It is clear that $N_G(P) \leqslant M$. If $|G : M| = q$ is a prime, then by Sylow's theorem, we have $q = 1 + kp$ and $q \mid |N|$. This contradicts p being the largest prime which divides the order of N. Hence $|G : M|$

must be a composite number. If M is nilpotent, then the Sylow 2-subgroup M_2 of M is not identity by [14], Theorem 10.4.2. Let $M_{2'}$ be a Hall $2'$-subgroup of M. By [15], Theorem 1, $M_{2'}$ is normal in G and therefore $P \trianglelefteq G$ since P is a characteristic subgroup of $M_{2'}$. It follows that $P \cap N \trianglelefteq G$, a contradiction. Thus, $M \in \mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G)$. By the hypotheses, M has a subnormal SS-supplement subgroup K in G. By using similar arguments as in the proof of Theorem 3.1, we can get that $|K| = |G : M| \leqslant |G : N_G(P)|$ and so $p \nmid |K|$. However, K is subnormal in G, which implies that K contains N_i for some i and hence $p \mid |K|$, a contradiction. This shows that N is p-solvable and therefore H is p-solvable. The proof of the theorem is now complete. □

From Theorem 3.4, we have the following corollary.

Corollary 3.5. *Let p be the largest prime dividing the order of a group G. Then G is p-solvable if and only if every maximal subgroup M of G in $\mathfrak{M}^{pcn}(G)$ has a subnormal SS-supplement in G.*

Proof. Only the necessity of the condition is in doubt by Theorem 3.4. Suppose that G is p-solvable and M is a maximal subgroup of G. We argue by induction on $|G|$. Assume that $M_G \neq 1$. Set $\overline{G} = G/M_G$. By induction, we can see that $\overline{M}$ has a subnormal SS-supplement $\overline{K}$ in $\overline{G}$ and so K is a subnormal SS-supplement of M in G. Hence, we may assume that $M_G = 1$ and let N be a minimal normal subgroup of G. Then $G = MN$ and $M \cap N \leqslant M_G = 1$, which implies that N is the normal SS-supplement of M in G. □

Remark 3.6. In Theorem 3.4, the group G need not be p-solvable as the following example shows. Let $H = C_2 \times C_2 \times C_2 \times C_2$ be an elementary abelian group of order 2^4. Then there is a subgroup $M = A_5$ in the automorphism group of H, where A_5 is the alternating group of degree 5. Let $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ be the corresponding semidirect product. We can deduce that $\mathfrak{M}^{pcn}(G) \cap \mathfrak{M}_H(G) = \{M^g \colon g \in G\}$. It is clear that M has a subnormal SS-supplement H in G. That is, G satisfies the hypotheses of Theorem 3.4 for normal subgroup H. However, G is not 5-solvable.

Finally we study the p-nilpotency and supersolvability of a group G by looking at certain minimal subgroups, leading to generalizations of known results.

Theorem 3.7. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G and when $p = 2$, either cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then G is not p-nilpotent. Noticing that all its Sylow p-subgroups are conjugate in G, we see that the hypotheses of our theorem are a subgroup-closure by Lemma 2.1. Consequently, G is a minimal non-p-nilpotent group (that is, every proper subgroup of a group is p-nilpotent but is not p-nilpotent itself). Now, by a result of Itô (see [14], Theorem 10.3.3), G must be a minimal nonnilpotent group. By a result of Schmidt (see [14], Theorem 9.1.9 and Exercise 9.1.11), we know that G is of order $p^a q^b$, where q is a prime which is different from p, P is normal in G and any Sylow q-subgroup Q of G is cyclic. Moreover, $P = G^{\mathfrak{N}_p}$ and P is of exponent p when p is odd and of exponent at most 4 when $p = 2$.

Let P_1 be a minimal subgroup of P. Then by hypotheses there exists a subgroup K of G such that $G = P_1K$ and $P_1 \cap K$ is S-quasinormal in K. Assume that $P_1 \cap K = 1$. Since p is the smallest prime divisor of the order of G, we get that K is normal in G. Noticing that K is a proper subgroup of G, we have that K is nilpotent. It follows that the Sylow q-subgroup of K is normal in G and therefore G is nilpotent, which is a contradiction. Hence, $P_1 \leqslant K$ and so P_1 is S-quasinormal in G. Therefore every minimal subgroup of P is S-quasinormal in G.

Let Q be a Sylow q-subgroup of G. Then P_1Q is a proper group of G and P_1Q is nilpotent by the minimality of G. It follows that $Q \subseteq C_G(P_1)$ and hence $Q \subseteq C_G(\Omega_1(P))$. If $C_G(\Omega_1(P)) < G$, then $C_G(\Omega_1(P))$ is nilpotent and so $Q \trianglelefteq G$, a contradiction. This leads to $C_G(\Omega_1(P)) = G$ and $\Omega_1(P) \leqslant Z(G)$. If $p > 2$, then from Itô's Lemma (see [9]) G is nilpotent, a contradiction. Hence $p = 2$. If P is quaternion-free, then by Lemma 2.6, we get that $\Omega_1(P) \leqslant P \cap G^{\mathfrak{N}_p} \cap Z(G) \leqslant P \cap G^{\mathfrak{N}} \cap Z(G) = 1$, a contradiction. Now assume that every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G. Let $A = \langle a \rangle$ be a cyclic subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ with order 4. Then there exists a subgroup T of G such that $G = AT$ and $A \cap T$ is S-quasinormal in T. Noticing that $\langle a^2 \rangle \subseteq Z(G)$, we see that $\langle a^2 \rangle T$ is a subgroup of G. If $|G : T| = 4$, then $|G : \langle a^2 \rangle T| = 2$ and $\langle a^2 \rangle T$ is normal in G. This implies that the Sylow q-subgroup of $\langle a^2 \rangle T$ is normal in G and therefore G is nilpotent, this is a contradiction. If $|G : T| = 2$, then T itself is a normal subgroup and T is nilpotent. Since the normal p-complement of T is the normal p-complement of G, it follows that G is nilpotent, a contradiction. Consequently, $T = G$ and so A is S-quasinormal in G. If $A = P$, then G is nilpotent, a contradiction. Thus, $A \neq P$. Since G is a minimal nonnilpotent group and the exponent of P is at most 4, we have $P \leqslant C_G(Q)$ and therefore $G = P \times Q$, a contradiction. The proof is complete.

□

We say that a group G is a Sylow tower group of supersolvable type if $p_1 > p_2 > \ldots > p_r$ are the distinct prime divisors of the order of G, then there exists a series of normal subgroups of G,

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_r = G,$$

such that G_i/G_{i-1} is a Sylow p_i-subgroup of G/G_{i-1} for $i = 1, \ldots, r$. Given a group G, observing that $H^{\mathfrak{N}_p} \leqslant G^{\mathfrak{N}_p}$ for every subgroup H of G by Lemma 2.7 and using Lemma 2.1 and Theorem 3.7, we obtain at once the following result.

Corollary 3.8. *Let G be a group. Suppose that for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, either cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then G is a Sylow tower group of supersolvable type.*

Theorem 3.9. *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersolvable groups and N be a normal subgroup of G such that $G/N \in \mathfrak{F}$. Suppose that for every prime p dividing the order of N and for every Sylow p-subgroup P of N, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then $G \in \mathfrak{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemma 2.1 and Corollary 3.8, we know that N is a Sylow tower group of supersolvable type. Thus if p is the largest prime dividing the order of N and P is a Sylow p-subgroup of N, then P must be normal in G and $G/P/N/P \cong G/N \in \mathfrak{F}$. It is clear that G/P satisfies the hypotheses of our theorem for its normal subgroup N/P by Lemmas 2.5 and 2.1. Then the minimality of G implies that $G/P \in \mathfrak{F}$.

Now, when G is not in $\mathfrak{F}$, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is nontrivial. Since $G/G^{\mathfrak{N}}$ is nilpotent and therefore $G/G^{\mathfrak{N}}$ belongs to $\mathfrak{F}$, necessarily $G/(P \cap G^{\mathfrak{N}})$ belongs to $\mathfrak{F}$ as well. It follows that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}}$. Furthermore, we claim that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}_p}$. Let P^* be a Sylow p-subgroup of G. As $G/G^{\mathfrak{N}_p}$ is p-nilpotent, we can see that $P^*G^{\mathfrak{N}_p} \cap O^p(G)G^{\mathfrak{N}_p} = G^{\mathfrak{N}_p}$ and so $P^* \cap O^p(G) \leqslant G^{\mathfrak{N}_p}$, which means that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}_p}$. A similar argument shows that $P^* \cap O^p(G) = P \cap G^{\mathfrak{N}}$ and this proves our claim. By [3], Theorem 3.5, there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$ and $G/M_G \notin \mathfrak{F}$. Then $G = MG^{\mathfrak{F}}$ and so $G = MF(G)$ since $G^{\mathfrak{F}}$ is a p-group, where $F(G)$ is the Fitting subgroup of G. It is now clear that M satisfies the hypotheses of our theorem for its normal subgroup $M \cap P$. Hence, the minimality of G implies that $M \in \mathfrak{F}$.

Now, by Lemma 2.8, we get that $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$ is a minimal normal subgroup of $G/\Phi(G^{\mathfrak{F}})$, G has exponent p when $p > 2$ and exponent at most 4 when $p = 2$. Let $\Phi = \Phi(G^{\mathfrak{F}})$ and A/Φ be any subgroup of $G^{\mathfrak{F}}/\Phi$ with order p, $a \in A \setminus \Phi$ and $X = \langle a \rangle$. Then $|X| = p$ or $|X| = 4$ and so X is SS-supplemented in G. Thus, there exists a subgroup K of G such that $G = XK$ and $X \cap K$ is S-quasinormal in K. Clearly, $(X\Phi/\Phi)(K/\Phi) = G/\Phi$. Assume that $X \nleq K$, then $X\Phi/\Phi \nleq K\Phi/\Phi$. Hence, the minimality of $G^{\mathfrak{F}}/\Phi$ implies that $(G^{\mathfrak{F}} \cap K)/\Phi = 1$, since $G^{\mathfrak{F}}/\Phi \cap K/\Phi \trianglelefteq G/\Phi$. By order comparison, $|G^{\mathfrak{F}}/\Phi| = p$. Assume that $X \leqslant K$, then $K = G$ and X is S-quasinormal in G. It follows that $A/\Phi = X\Phi/\Phi$ is S-quasinormal in G/Φ. By Lemma 2.3, $O^p(G/\Phi) \leqslant N_{G/\Phi}(A/\Phi)$ and so $|G/\Phi : N_{G/\Phi}(A/\Phi)| = p^a$ for some $a \in \mathbb{N}$. Thus if $\{A_1/\Phi, \dots, A_t/\Phi\}$ is the set of all minimal subgroups of $G^{\mathfrak{F}}/\Phi$, then it follows from [8], III, 8.5 Hilfssatz, that $|G/\Phi : N_{G/\Phi}(A_i/\Phi)| = 1$ for some $i \in \{1, \dots, t\}$. Hence, A_i/Φ is normal in G/Φ. The minimality of $G^{\mathfrak{F}}/\Phi$ also implies that $|G^{\mathfrak{F}}/\Phi| = p$.

Now $(G/\Phi)/(G^{\mathfrak{F}}/\Phi) \cong G/G^{\mathfrak{F}} \in \mathfrak{F}$ and $G^{\mathfrak{F}}/\Phi$ is a cyclic group of order p. Hence, $(G/\Phi, G^{\mathfrak{F}}/\Phi)$ satisfies the hypotheses of the theorem. If $\Phi \neq 1$, then by the minimality of G, $G/\Phi \in \mathfrak{F}$. It follows that $G \in \mathfrak{F}$, a contradiction. Thus $\Phi = 1$ and so $G^{\mathfrak{F}}$ is a cyclic group of order p. By Lemma 2.9, we can conclude that $G \in \mathfrak{F}$, a contradiction.

There remains the case, where $p = 2$ and P is quaternion-free. Let R be a Sylow r-subgroup of G with $r \neq 2$ and $G_1 = RG^{\mathfrak{F}}$. Then $G^{\mathfrak{F}}$ is a Sylow 2-subgroup of G_1. Observing that $G^{\mathfrak{F}} \leqslant P \cap G^{\mathfrak{N}_p}$, we have that G_1 is 2-nilpotent by Theorem 3.7. It follows that $G^{\mathfrak{F}} \leqslant C_G(R)$ and therefore $Z(G) \cap G^{\mathfrak{F}} \neq 1$. Since $G^{\mathfrak{F}} \leqslant G^{\mathfrak{N}}$, we have $Z(G) \cap G^{\mathfrak{N}} \cap P \neq 1$, in contradiction to Lemma 2.6. This completes the proof of the theorem. □

As an immediate consequence of Theorem 3.9, we have:

Corollary 3.10. *Let G be a group. Suppose that, for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every minimal subgroup of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is SS-supplemented in G, and when $p = 2$, every cyclic subgroup of order 4 of $P \cap P^x \cap G^{\mathfrak{N}_p}$ is also SS-supplemented in G for all $x \in G \setminus N_G(P)$ or P is quaternion-free. Then G is supersolvable.*

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