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ON A KLEINECKE-SHIROKOV THEOREM

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Abstract. We prove that for normal operators $N_1, N_2 \in \mathcal{L}(\mathcal{H})$, the generalized commutator $[N_1, N_2; X]$ approaches zero when $[N_1, N_2; [N_1, N_2; X]]$ tends to zero in the norm of the Schatten-von Neumann class $\mathcal{C}_p$ with $p > 1$ and X varies in a bounded set of such a class.

Keywords: Kleinecke-Shirokov theorem; generalized commutator

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{C}_p(\mathcal{H})$ (or $\mathcal{C}_p$) the Schatten-von Neumann p-classes with $|\cdot|_p$, $p \geqslant 1$, being their respective norm. Furthermore, let $\mathbb{K}(\mathcal{H})$ (or $\mathbb{K}$) denote the ideal of compact operators. For arbitrary operators $S, T, X \in \mathcal{L}(\mathcal{H})$, $[S, T; X]$ denotes the generalized commutator, that is $SX - XT$, and for $S = T$ this becomes the usual commutator of S and X which is denoted by $[S; X]$.

Kleinecke in [3] and Shirokov in [4] proved that for arbitrary $S, T, X \in \mathcal{L}(\mathcal{H})$ such that $[S, T; [S, T; X]] = 0$, $[S, T; X]$ is quasi-nilpotent, that is its spectral radius $r([S, T; X])$ is zero. Ackermans-van Eijndhoven-Martens (see [2], Theorem 0.5) obtained a stronger conclusion under the additional hypothesis of normality.

Theorem 1.1 ([2], Theorem 0.5). *Let $N_1, N_2 \in \mathcal{L}(\mathcal{H})$ be normal operators and $X \in \mathcal{L}(\mathcal{H})$ be such that $[N_1, N_2; [N_1, N_2; X]] = 0$. Then $[N_1, N_2; X] = 0$.*

Furthermore, Ackermans-van Eijndhoven-Martens provided a result concerning the asymptotic dependence of $[N_1, N_2; X]$ in terms of $[N_1, N_2; [N_1, N_2; X]]$ in the context of an *algebra topology* on $\mathcal{L}(\mathcal{H})$ as follows.

Definition 1.1. A topology τ on $\mathcal{L}(\mathcal{H})$ is an *algebra topology* if
(a) τ is not finer than the uniform (norm) topology,
(b) $(\mathcal{L}(\mathcal{H}), \tau)$ is a locally convex vector space, and
(c) the mapping $X \mapsto SXT$ is τ-τ continuous for any $S, T \in \mathcal{L}(\mathcal{H})$.

Theorem 1.2 ([2], Theorem 2.5). *Let τ be an algebra topology on $\mathcal{L}(\mathcal{H})$ and let W be a τ-open neighborhood of $0_{\mathcal{H}}$. Let N_1 and N_2 be normal operators of $\mathcal{L}(\mathcal{H})$ and $K > 0$. Then there exists a τ-open neighborhood V of $0_{\mathcal{H}}$ so that $[N_1, N_2; X] \in W$ for all $X \in \mathcal{L}(\mathcal{H})$ with both $\|X\| \leqslant K$ and $[N_1, N_2; [N_1, N_2; X]] \in V$.*

2. Results

It is the purpose of this section to extend such a result to normed ideals.

Definition 2.1. A proper two-sided ideal $\mathcal{J}$ of $\mathcal{L}(\mathcal{H})$ is called a *normed ideal* if it is endowed with a norm $|\cdot|_{\mathcal{J}}$ so that
(a) $(\mathcal{J}, |\cdot|_{\mathcal{J}})$ is a Banach space,
(b) $|SXT|_{\mathcal{J}} \leqslant \|S\|\|T\||X|_{\mathcal{J}}$ for $S, T \in \mathcal{L}(\mathcal{H})$ and $X \in \mathcal{J}$,
(c) $|UXV|_{\mathcal{J}} = |X|_{\mathcal{J}}$ for any unitary operators $U, V \in \mathcal{L}(\mathcal{H})$ and $X \in \mathcal{J}$, and
(d) $|X^*|_{\mathcal{J}} = |X|_{\mathcal{J}}$ for any $X \in \mathcal{J}$.

The above definition differs from what traditionally is called a normed ideal. In what follows, $\mathcal{J}$ denotes a normed ideal according to the definition above.

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator and $X \in \mathcal{J}$. Then the function $f\colon \mathbb{R} \to \mathcal{J}$ defined by $f(t) = \mathrm{e}^{\mathrm{i}tA} X \mathrm{e}^{-\mathrm{i}tA}$ is $\mathcal{J}$-differentiable, that is*

$$D_{\mathcal{J}}(f)(t_0) := \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} = \mathrm{i}\mathrm{e}^{\mathrm{i}t_0 A}[A, X]\mathrm{e}^{-\mathrm{i}t_0 A}.$$

In particular, f is continuous.

Proof. Let A and X be as in the hypothesis and we prove that

$$\lim_{t \to t_0} \left| \frac{f(t) - f(t_0)}{t - t_0} - \mathrm{i}\mathrm{e}^{\mathrm{i}t_0 A}[A, X]\mathrm{e}^{-\mathrm{i}t_0 A} \right|_{\mathcal{J}} = 0.$$

Indeed,

$$\begin{aligned} \frac{f(t) - f(t_0)}{t - t_0} - \mathrm{i}\mathrm{e}^{\mathrm{i}t_0 A}[A, X]\mathrm{e}^{-\mathrm{i}t_0 A} &= \frac{\mathrm{e}^{\mathrm{i}tA} X \mathrm{e}^{-\mathrm{i}tA} - \mathrm{e}^{\mathrm{i}t_0 A} X \mathrm{e}^{-\mathrm{i}t_0 A}}{t - t_0} - \mathrm{i}\mathrm{e}^{\mathrm{i}t_0 A}[A, X]\mathrm{e}^{-\mathrm{i}t_0 A} \\ &= \mathrm{e}^{\mathrm{i}t_0 A}\Big(\frac{\mathrm{e}^{\mathrm{i}(t-t_0)A} X \mathrm{e}^{-\mathrm{i}(t-t_0)A} - X}{t - t_0} - \mathrm{i}[A; X]\Big)\mathrm{e}^{-\mathrm{i}t_0 A} \\ &= \mathrm{e}^{\mathrm{i}t_0 A}\Big(\frac{\mathrm{e}^{\mathrm{i}(t-t_0)A} X - X \mathrm{e}^{\mathrm{i}(t-t_0)A}}{t - t_0} - \mathrm{i}[A; X]\mathrm{e}^{\mathrm{i}(t-t_0)A}\Big)\mathrm{e}^{-\mathrm{i}tA}. \end{aligned}$$

Since the operator e^{isA} is unitary for any $s \in \mathbb{R}$, it is enough to show

$$\lim_{u\to 0}\left|\frac{e^{iuA}X - Xe^{iuA}}{u} - i[A;X]e^{iuA}\right|_{\mathcal{J}} = 0.$$

Indeed,

$$e^{iuA}X - Xe^{iuA} = \sum_{k=0}^{\infty}\frac{(iuA)^k}{k!}X - X\frac{(iuA)^k}{k!} = i[A;X] + \sum_{k=2}^{\infty}\frac{(iu)^k}{k!}[A^kX - XA^k]$$

and by an induction argument one can prove that

$$|A^kX - XA^k|_{\mathcal{J}} \leqslant k\|A\|^{k-1}|[A;X]|_{\mathcal{J}},$$

and thus

$$\left|\frac{e^{iuA}X - Xe^{iuA}}{u} - i[A;X]e^{iuA}\right|_{\mathcal{J}} \leqslant \left(\|I - e^{iuA}\| + \sum_{k=2}^{\infty}\frac{|u|^{k-1}}{(k-1)!}\|A\|^{k-1}\right)|[A;X]|_{\mathcal{J}}.$$

Furthermore, $\|I - e^{iuA}\| \leqslant e^{|u|\|A\|} - 1$ and consequently

$$\left|\frac{e^{iuA}X - Xe^{iuA}}{u} - i[A;X]e^{iuA}\right|_{\mathcal{J}} \leqslant 2(e^{|u|\|A\|} - 1)|[A;X]|_{\mathcal{J}},$$

which ends the proof. $\square$

Theorem 2.1. *Let A be a self-adjoint operator in $\mathcal{L}(\mathcal{H})$ and $K > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ so that for any $X \in \mathcal{J}$ with $|X|_{\mathcal{J}} \leqslant K$, the inequality $|[A;[A;X]]|_{\mathcal{J}} < \delta$ implies $|[A;X]|_{\mathcal{J}} < \varepsilon$.*

Proof. For A and X as in the hypothesis, let f be the function defined above. According to Lemma 2.1, f is twice $\mathcal{J}$-differentiable and

$$i[A;X] = D_{\mathcal{J}}(f)(0) = \frac{f(u) - f(0)}{u} - \frac{1}{u}\int_0^u\left(\int_0^t D_{\mathcal{J}}(D_{\mathcal{J}}(f))(s)\,ds\right)dt.$$

Let $\varepsilon > 0$ and $X \in \mathcal{J}$ with $|X|_{\mathcal{J}} \leqslant K$ and $u > 0$; thus $|f(u) - f(0)|_{\mathcal{J}} \leqslant 2K$ and $|(f(u) - f(0))/u|_{\mathcal{J}} < \frac{1}{2}\varepsilon$ for $u > 4K/\varepsilon$. On the other hand, let $|[A;[A;X]]|_{\mathcal{J}} < \delta$ with δ to be described later; thus

$$\left|\frac{1}{u}\int_0^u\left(\int_0^t D_{\mathcal{J}}(D_{\mathcal{J}}(f))(s)\,ds\right)dt\right|_{\mathcal{J}} \leqslant \frac{1}{u}\int_0^u\left(\int_0^t |[A;[A;X]]|_{\mathcal{J}}\,ds\right)dt \leqslant \frac{u}{2}\delta.$$

Selecting $\delta < \varepsilon/u$, and since u has to be large enough, precisely $u > 4K/\varepsilon$, then $\delta < \varepsilon^2/(4K)$ ensures that $|(1/u)\int_0^u(\int_0^t D_{\mathcal{J}}(D_{\mathcal{J}}(f))(s)\,ds)\,dt|_{\mathcal{J}} \leqslant \frac{1}{2}\varepsilon$, and consequently $|[A;X]|_{\mathcal{J}} < \varepsilon$. $\square$

Corollary 2.1. *Let A and B be self-adjoint operators in $\mathcal{L}(\mathcal{H})$ and $K > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $X \in \mathcal{J}$ with $|X|_{\mathcal{J}} \leqslant K$, the inequality $|[A, B; [A, B; X]]|_{\mathcal{J}} < \delta$ implies $|[A, B; X]|_{\mathcal{J}} < \varepsilon$.*

P r o o f. Put $C = A \oplus B$ and $\widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, and observe that $[C; \widetilde{X}] = \begin{pmatrix} 0 & [A, B; X] \\ 0 & 0 \end{pmatrix}$. Moreover, for an arbitrary $X \in \mathcal{J}$ is equivalent to $\widetilde{X} \in \mathcal{J}$ and $|X|_{\mathcal{J}} = |\widetilde{X}|_{\mathcal{J}}$, and by applying Theorem 2.1, the proof is done. □

The result can be extended to normal operators, but relative to normed ideals for which the Fuglede-Putnam theorem is known to be valid.

Theorem 2.2 ([1], [5]). *If N_1, N_2 are normal operators and $X \in \mathcal{L}(\mathcal{H})$ so that $[N_1, N_2; X] \in \mathcal{C}_p$ with $p > 1$, then $[N_1^*, N_2^*; X] \in \mathcal{C}_p$ and*

$$|[N_1^*, N_2^*; X]|_p < c(p)|[N_1, N_2; X]|_p.$$

On other the hand, the Fuglede-Putnam theorem is not valid if $p = 1$ (cf. [6], Corollary 8.6), more precisely there exist a normal operator N and a compact operator X so that $[N; X]$ is a rank one operator (thus, a trace-class operator) and $[N^*, X]$ is not a trace-class operator.

Theorem 2.3. *Let N_1 and N_2 be normal operators in $\mathcal{L}(\mathcal{H})$, $p > 1$ and $K > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ so that for any $X \in \mathcal{C}_p$ with $|X|_p \leqslant K$, the inequality $|[N_1, N_2; [N_1, N_2; X]]|_p < \delta$ implies $|[N_1, N_2; X]|_p < \varepsilon$.*

P r o o f. Let $A_j + \mathrm{i}B_j = N_j$, $j = 1, 2$, be the Cartesian decomposition of N_j. Let N_1 and N_2 be normal operators that satisfy $|[N_1, N_2; [N_1, N_2; X]]|_p < \delta$. According to Theorem 2.2,

$$|[N_1^*, N_2^*; [N_1, N_2; X]]|_p < c(p)\,\delta.$$

Since $[N_1^*, N_2^*; [N_1, N_2; X]] = [N_1, N_2; [N_1^*, N_2^*; X]]$, it implies

$$|[N_1, N_2; [N_1^*, N_2^*; X]]|_p < c(p)\,\delta,$$

and after one more application of Theorem 2.2,

$$|[N_1^*, N_2^*; [N_1^*, N_2^*; X]]|_p < c(p)^2\,\delta.$$

Consequently,

$$|[C_1, C_2; [C_1, C_2; X]]|_p < d(p)\,\delta,$$

where $C_1 = A_1$, $C_2 = A_2$ or $C_1 = B_1$, $C_2 = B_2$ and $d(p)$ is a constant that depends only on p, which proves that $|[C_1, C_2; [C_1, C_2; X]]|_p$ becomes as small as necessary if $|[N_1, N_2; [N_1, N_2; X]]|_p$ does so. For an arbitrary $\varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$, where δ_1 and δ_2 are the two positive δ's resulting by applying Corollary 2.1 for the pairs A_1, A_2 and B_1, B_2, and thus $|[A_1, A_2; X]|_p < \varepsilon$, $|[B_1, B_2; X]|_p < \varepsilon$. Consequently $|[N_1, N_2; X]|_p < 2\varepsilon$ and the proof is finished. □

The hypothesis of normality can be relaxed as follows.

Theorem 2.4. *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be such that T_1 and T_2^* are subnormal operators, and let $p > 1$ and $K > 0$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ so that for any $X \in \mathcal{C}_p$ with $|X|_p \leqslant K$, the inequality $|[T_1, T_2; [T_1, T_2; X]]|_p < \delta$ implies $|[T_1, T_2; X]|_p < \varepsilon$.*

P r o o f. Let T_1 and T_2^* be subnormal operators. One may assume that there are some normal operators $N_i \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, $i = 1, 2$, so that $N_1 = \begin{pmatrix} S_1 & A \\ 0 & B \end{pmatrix}$ and $N_2 = \begin{pmatrix} S_2 & 0 \\ C & D \end{pmatrix}$, after an extension by zero if necessary. Thus,

$$\begin{aligned} [N_1, N_2; X \oplus 0] &= [S_1, S_2; X] \oplus 0 \\ [N_1, N_2; [N_1, N_2; X \oplus 0]] &= [S_1, S_2; [S_1, S_2; X]] \oplus 0, \end{aligned}$$

and

$$\begin{aligned} |[N_1, N_2; X \oplus 0]|_p &= |[S_1, S_2; X] \oplus 0|_p \\ |[N_1, N_2; [N_1, N_2; X \oplus 0]]|_p &= |[S_1, S_2; [S_1, S_2; X]] \oplus 0|_p \end{aligned}$$

as well, and Theorem 2.3 can be applied. □

3. Remarks

Let $\pi\colon \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbb{K}$ denote the canonical projection onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathbb{K}$ which is a C^*-algebra. Let $N_1, N_2 \in \mathcal{L}(\mathcal{H})$ be essentially normal operators (that is, their self-commutator is a compact operator, or equivalently $\pi(N_i)$ is a normal operator, $i = 1, 2$) and let $X \in \mathcal{L}(\mathcal{H})$ be such that $[N_1, N_2; [N_1, N_2; X]] \in \mathbb{K}$, or equivalently $[\pi(N_1), \pi(N_2); [\pi(N_1), \pi(N_2); \pi(X)]] = 0$. According to Theorem 1.1, $[N_1, N_2; X] \in \mathbb{K}$.

It is natural to ask a similar question whether $[N_1, N_2; [N_1, N_2; X]] \in \mathcal{J}$ implies $[N_1, N_2; X] \in \mathcal{J}$ relative to a normed ideal $\mathcal{J}$ when $X \in \mathcal{L}(\mathcal{H})$, not necessarily in a normed ideal. The most appropriate choice of a normed ideal is the class of Hilbert-Schmidt operators $\mathcal{C}_2$.

Let $N \in \mathcal{L}(\mathcal{H})$ be a normal operator and $X \in \mathcal{L}(\mathcal{H})$ be such that $[N;[N;X]] \in \mathcal{C}_2$. Does it imply that $[N;X] \in \mathcal{C}_2$?

The following example shows that the answer to the above question is negative. In what follows, the operators act on $l^2(\mathbb{N})$, the Hilbert space of square-summable complex sequences, and $\{e_i\}_{i\geqslant 0}$ is its cononical orthonormal basis.

Example 3.1. Let D be a diagonal operator with the diagonal entries d_i, $i \geqslant 1$, described below. Let X be the unilateral shift operator. Then $[D;[D;X]] \in \mathcal{C}_2$ and $[D;X] \notin \mathcal{C}_2$.

Indeed, for $i \geqslant 1$, the entry $(i, i-1)$ of $Y = [D;X]$ is $y_{i,i-1} = (d_i - d_{i-1})$ and that of $Z = [D;[D;X]]$ is $z_{i,i-1} = (d_i - d_{i-1})^2$, and all other entries of Y and Z are equal to zero. Let $d_i - d_{i-1} = a_{i-1}$, $i \geqslant 1$ and $Z \in \mathcal{C}_2$ be equivalent to $\sum\limits_{i=1}^{\infty} |a_i|^4 < \infty$ and $Y \notin \mathcal{C}_2$ be equivalent to $\sum\limits_{i=1}^{\infty} |a_i|^2 = \infty$. Furthermore, the boundedness of D be equivalent to the boundedness of the partial sums of the series $\sum\limits_{i=1}^{\infty} a_i$. An instance of such a sequence is $a_i = (-1)^i / i^{\alpha}$ with $\alpha \in (\frac{1}{4}, \frac{1}{2}]$.

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