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WEAK DIMENSIONS AND GORENSTEIN WEAK DIMENSIONS OF GROUP RINGS

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Abstract. Let K be a field, and let G be a group. In the present paper, we investigate when the group ring $K[G]$ has finite weak dimension and finite Gorenstein weak dimension. We give some analogous versions of Serre's theorem for the weak dimension and the Gorenstein weak dimension.

Keywords: weak dimension; Gorenstein weak dimension; principal module; group ring

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1. Introduction

Let R be a ring and G a group (finite or infinite). We denote the group ring of G over R by $R[G]$ with the elements of G as a basis and with multiplication defined distributively using the group multiplication in G. This subject is a meeting place of group theory and ring theory. In recent decades, representation and homological properties of group rings have been extensively studied (cf. [1], [4], [6], [9], [14], [15], [17], [18], [19]). Among others, Connell in [6] considered necessary and sufficient conditions on R and G so that $R[G]$ have some ring theoretic properties such as being Artinian, regular, self-injective and semiprime. Let p be a prime. A group G is called a p'-*group* provided that G has no element of order p. Let K be a field of characteristic p and let H be a subgroup of G of finite index. There is the well-known Serre's theorem (see [15]), i.e., if G is a p'-group, then the global dimension of $K[H]$ is equal to that of $K[G]$. In [1], Auslander showed that if G is a commutative group and the order of any element in G is unit in K, then the weak dimension of $K[G]$ equals the

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rank of G. Futhermore, it was shown in [17] that the weak dimension of $K[G]$ is equal to the Hirsch number of G for a field K of characteristic 0 and a solvable group G. On the other hand, Govorov proved that a module is flat if and only if it is a direct limit of free modules, which is also called the *Govorov-Lazard theorem*, see [10]. Recently, Benson and Goodearl in [4] showed that flat modules and projective modules over a group ring have a close connection. For a ring R and a finite group G, a flat $R[G]$-module which is projective as an R-module is necessarily a projective $R[G]$-module.

Motivated by this, we consider, in Section 3 of this paper, the finiteness of the weak dimension of group rings. It is shown that the weak dimension of $K[G]$ is equal to the flat dimension of a principal $K[G]$-module. Moreover, we obtain some results which generalize several properties of the global dimension of group rings. More precisely, we prove the following:

Theorem 3.10. *Let K be a field, and let H be a normal subgroup of a group G. If $wD(K[H])$ and $wD(K[G/H])$ are finite, then so is $wD(K[G])$, and we have*

$$wD(K[G]) \leqslant wD(K[H]) + wD(K[G/H]).$$

Theorem 3.13. *Let K be a field of characteristic p, and let H be a subgroup of a group G of finite index. If G is a p'-group, then $wD(K[H]) = wD(K[G])$.*

Auslander and Bridger in [2] introduced G-dimensions for finitely generated modules over commutative Noetherian rings. As an extension of the G-dimension, the Gorenstein projective dimension and the Gorenstein flat dimension of modules (necessarily finitely generated) over a general ring were defined (cf. [8], [12]). Furthermore, the Gorenstein global dimension and the Gorenstein weak dimension of a ring were given, see [3]. Those dimensions are refinements of the classical homological dimensions. In Section 4, we investigate the finiteness of the Gorenstein weak dimension of group rings. The main results of this section are the following:

Theorem 4.7. *Let K be a field, and let H be a normal subgroup of a group G. If $GwD(K[H])$ and $GwD(K[G/H])$ are finite, then so is $GwD(K[G])$, and the following hold:*
(1) *$GwD(K[G]) \leqslant GwD(K[H]) + GwD(K[G/H])$.*
(2) *If G/H is locally finite, then $GwD(K[G]) = GwD(K[H])$.*

Proposition 4.9. *Let K be a field, and let H be a subgroup of a group G of finite index. If $K[G]$ is right coherent and $GwD(K[G])$ is finite, then $GwD(K[G]) = GwD(K[H])$.*

2. Preliminaries

In this section, we set notations and discuss basic facts which will be useful in the sequel. Unless otherwise stated, R denotes an associative ring with identity and all modules are left R-modules. For an R-module M, $fd_R(M)$ and $Gfd_R(M)$ denote the flat dimension and the Gorenstein flat dimension of M, respectively. We write $wD(R)$ and $GwD(R)$ for the weak dimension and the left Gorenstein weak dimension of a ring R, respectively. More concepts and notations can be found in [2], [13], and [15].

Module structure over group rings

(1) Let V and W be $K[G]$-modules. Then $V \otimes_K W$ becomes a $K[G]$-module under the diagonal action $g(v \otimes w) = (gv) \otimes (gw)$ for all $v \in V$, $w \in W$ and $g \in G$. It is trivial that $V \otimes_K W \cong W \otimes_K V$.

(2) The principal $K[G]$-module V_0 is a one-dimensional K-vector space in which $gv = v$ for all $v \in V_0$ and $g \in G$. For example, K with trivial G-action is a principal $K[G]$-module.

(3) Let H be a subgroup of G. Following [14], for a $K[H]$-module M, we define the induced module $M \uparrow_H^G := K[G] \otimes_{K[H]} M$ with $K[G]$ acting on the left side and the coinduced module $\mathrm{Hom}_{K[H]}(K[G], M)$. Moreover, every $K[G]$-module N can be viewed as a $K[H]$-module. We denote this restricted module by $N \downarrow_H^G$ (sometime we omit the symbol $\downarrow_H^G$ if these is not risk of confusion). Since $K[G]$ is a left and right free $K[H]$-module, the induced functor and the restricted functor are exact, and preserve projective modules. The coinduced functor preserves injective modules.

Gorenstein dimension

A complete flat resolution is an exact sequence of flat R-modules

$$\ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots,$$

which remains exact after applying the functor $I \otimes_R -$ for any injective right R-module I. An R-module M is called *Gorenstein flat* (see [12]) if it is a syzygy of a complete flat resolution, i.e., $M = \mathrm{Ker}(F^0 \to F^1)$. The Gorenstein flat dimension $Gfd_R(M)$ is at most n if there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \ldots \to G_1 \to G_0 \to M \to 0$$

with every G_i Gorenstein flat. A ring R is right coherent if every finitely generated right ideal of R is finitely presented. The following result is due to [12], Theorem 3.14.

Proposition 2.1. *Let R be a right coherent ring, and let M be a left R-module with finite Gorenstein flat dimension. Then the following are equivalent:*

(1) $Gfd_R(M) \leqslant n$;
(2) $\mathrm{Tor}_i^R(L, M) = 0$ *for all right* R*-modules* L *with finite injective dimension, and all* $i > n$;
(3) $\mathrm{Tor}_i^R(I, M) = 0$ *for all injective right* R*-modules* I*, and all* $i > n$;
(4) *for every exact sequence*

$$0 \to K_n \to G_{n-1} \to \ldots \to G_1 \to G_0 \to M \to 0,$$

where $G_0, \ldots, G_{n-1}$ *are Gorenstein flat, then also* K_n *is Gorenstein flat.*

Following [3], the left Gorenstein weak dimension of a ring R is defined as

$$GwD(R) = \sup\{Gfd_R(M) \colon M \text{ is an } R\text{-module}\}.$$

Recall that the weak dimension of R is the supremum of flat dimensions of all R-modules. It is clear that $GwD(R) \leqslant wD(R)$ and $GwD(R) = wD(R)$ provided that $wD(R)$ is finite. Recall that a ring is called a *left IF-ring* (see [5]) if every left injective module is flat. Dually, there is the definition of a right IF-ring. The ring is called an IF-ring provided that it is a right and left IF-ring. It was shown that $GwD(R) = 0$ if and only if R is an IF-ring.

3. Weak dimension

We start with the following lemmas.

Lemma 3.1. *Let* K *be a field, and let* H *be a normal subgroup of* G*. If* F *is a flat* $K[G/H]$*-module and a* $K[G]$*-module* M *is flat as a* $K[H]$*-module, then* $F \otimes_K M$ *is flat as a* $K[G]$*-module.*

Proof. By the Govorov-Lazard theorem, a flat module is a direct limit of free modules. Noting that the direct limit commutes with the tensor functor, and a direct limit (direct sum) of flat modules is also flat, we assume that $F = K[G/H]$. It is easy to verify that

$$K[G/H] \otimes_K M \cong M \downarrow_H^G \uparrow_H^G$$

by defining

$$\sigma \colon\ gH \otimes x \mapsto g \otimes g^{-1}x, \quad gH \in G/H,\ x \in M,$$

and

$$\tau \colon\ g \otimes x \mapsto gH \otimes gx, \quad g \in G,\ x \in M.$$

Thus, the result follows from [19], Proposition 2.2. □

Lemma 3.2. *Let M be a $K[G]$-module. If F is a flat $K[G]$-module, then so is $F \otimes_K M$.*

P r o o f. Let $H = \{1\}$ in Lemma 3.1. $\square$

By Lemma 3.2, we have immediately the following:

Lemma 3.3. *If N is a $K[G]$-module, then $fd_{K[G]}(N \otimes_K M) \leqslant fd_{K[G]}(N)$ for any $K[G]$-module M.*

The following result will be used in the sequel.

Proposition 3.4. *Let K be a field, and let G be a group. If V_0 is a principal $K[G]$-module, then*

$$wD(K[G]) = fd_{K[G]}(V_0).$$

P r o o f. It is only to prove that $wD(K[G]) \leqslant fd_{K[G]}(V_0)$. Now suppose that $fd_{K[G]}(V_0) = n < \infty$. Then there is an exact sequence of $K[G]$-modules

$$0 \to F_n \to \ldots \to F_0 \to V_0 \to 0,$$

where each F_i is flat. For any $K[G]$-module M, it is flat as a K-module. So, it yields an exact sequence of $K[G]$-modules

$$0 \to F_n \otimes_K M \to \ldots \to F_0 \otimes_K M \to V_0 \otimes_K M \to 0.$$

By Lemma 3.2, $F_i \otimes_K M$ is flat for all i. Thus, $fd_{K[G]}(V_0 \otimes_K M) \leqslant n$. It is easy to verify that $V_0 \otimes_K M \cong M$ as $K[G]$-modules. Then $fd_{K[G]}(M) \leqslant n$, and hence $wD(K[G]) \leqslant n$. $\square$

A group is locally finite if every finite subset generates a finite subgroup. Now we list some characteristics of the weak dimension of group rings.

Proposition 3.5. *Let K be a field, and let G be a group.*
(1) *If H is a subgroup of G, then $wD(K[H]) \leqslant wD(K[G])$.*
(2) *$wD(K[G]) = 0$ if and only if G is locally finite and the order of any finite subgroup of G is unit in K.*
(3) *If G is an infinite cyclic group, then $wD(K[G]) = 1$.*

P r o o f. (1) It follows from [19], Theorem 2.7.
(2) The result can be found in [6].
(3) It follows from [1], Lemma 8. $\square$

It was shown that if the global dimension of $K[G]$, where K is a field of characteristic p, is finite, then G is a p'-group (see [15], Corollary 10.3.7). Now we can extend this result to the weak dimension.

Proposition 3.6. *Let K be a field of characteristic p, and let G be a group. If $wD(K[G])$ is finite, then G is a p'-group.*

P r o o f. Suppose that $H = (x)$ is a cyclic subgroup of order p. Let $R := K[H]$ and let

$$a = 1 - x, \quad b = 1 + x + \ldots + x^{p-1} \in R.$$

By [13], Lemma 6.2, $l_R(a) = Rb$. Thus, we have the exact sequence of R-modules

$$0 \to Rb \to R \to Ra \to 0.$$

By Proposition 3.5 (1), $wD(R)$ is finite, and hence let $fd_R(Ra) = n < \infty$. By [4], Lemma 3.2 (b) and Theorem 3.4, Ra is projective, and so $R \simeq Ra \oplus Rb$. But $b \neq 0$ annihilates both Ra and Rb, a contradiction. Therefore, G is a p'-group. $\square$

Example 3.7. Let K be a field of characteristic 3, and let G be the symmetric group of degree 3. By the proposition above, $wD(K[G])$ is infinite.

It is natural to ask when the weak dimension of $K[G]$ is finite. By Proposition 3.5 and [1], Proposition 6, we have the following:

Corollary 3.8. *Let K be a field, and let G be a locally finite group. Then the following are equivalent:*
(1) *$wD(K[G]) = 0$;*
(2) *$wD(K[G])$ is finite;*
(3) *the order of any element of G is unit in K.*

A group G is called *polycyclic-by-finite* if there is a subnormal series for G,

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G,$$

where G_i/G_{i-1} is either cyclic or finite. By Proposition 3.6 and [15], Theorem 10.3.13, we have the following result:

Corollary 3.9. *Let K be a field of characteristic p, and let G be a polycyclic-by-finite group. Then the following are equivalent:*
(1) *$wD(K[G])$ is finite;*
(2) *G is a p'-group.*

Let H be a normal subgroup of a group G. The following theorem establishes the estimate for the weak dimension of $K[G]$ by the corresponding values of $K[H]$ and $K[G/H]$.

Theorem 3.10. *Let K be a field, and let H be a normal subgroup of a group G. If $wD(K[H])$ and $wD(K[G/H])$ are finite, then so is $wD(K[G])$, furthermore,*

$$wD(K[G]) \leqslant wD(K[H]) + wD(K[G/H]).$$

P r o o f. Suppose that $wD(K[H]) = n$ and $wD(K[G/H]) = m$ are finite.

If V_0 is a principal $K[H]$-module, then there is an exact sequence of $K[H]$-modules

$$0 \to F_n \to \ldots \to F_0 \to V_0 \to 0,$$

where each F_i is flat. By [19], Proposition 2.2, there is an exact sequence of $K[G]$-modules

$$0 \to F_n \uparrow_H^G \to \ldots \to F_0 \uparrow_H^G \to V_0 \uparrow_H^G \to 0,$$

and each $F_i \uparrow_H^G$ is flat. On the other hand, $V_0 \uparrow_H^G \cong K[G/H]$ as $K[G]$-modules, and hence

$$fd_{K[G]}(K[G/H]) = fd_{K[G]}(V_0 \uparrow_H^G) \leqslant n.$$

Thus, for every free $K[G/H]$-module F, $fd_{K[G]}(F) \leqslant n$. By the Govorov-Lazard theorem and Theorem 8.11 in [16], $fd_{K[G]}(Q) \leqslant n$ for any flat $K[G/H]$-module Q.

If W_0 is a principal $K[G/H]$-module, in view of Proposition 3.4,

$$fd_{K[G/H]}(W_0) = wD(K[G/H]) = m.$$

Then we have the exact sequences of $K[G/H]$-modules

$$0 \to W_{i+1} \to Q_i \to W_i \to 0, \quad i = 0, 1, \ldots, m-1,$$

where Q_i, $i = 0, 1, \ldots, m-1$ and W_m are flat $K[G/H]$-modules. The exact sequences above are also exact sequences of $K[G]$-modules, and W_0 is also a principal $K[G]$-module. To prove $fd_{K[G]}(W_i) \leqslant n + m - i$, we carry out the inverse induction on i.

(1) $fd_{K[G]}(W_m) \leqslant n + m - m$ because W_m is a flat $K[G/H]$-module.

(2) Suppose that $fd_{K[G]}(W_t) \leqslant n + m - t$ for $1 < t < m$. Then $fd_{K[G]}(W_t)$ and $fd_{K[G]}(Q_{t-1})$ are finite and $fd_{K[G]}(Q_{t-1}) \leqslant n$.

(3) If W_{t-1} is not flat, then

$$fd_{K[G]}(W_{t-1}) \leqslant 1+\sup\{fd_{K[G]}(W_t), fd_{K[G]}(Q_{t-1})\} \leqslant 1+(n+m-t) = n+m-(t-1).$$

In particular, when $i = 0$ we have $fd_{K[G]}(W_0) \leqslant n + m$. Therefore, in view of Proposition 3.4, $wD(K[G]) = fd_{K[G]}(W_0) \leqslant n + m$. □

By Proposition 3.5 (2) and Theorem 3.10, we have:

Corollary 3.11. *Let K be a field, and let H be a normal subgroup of G of finite index. If $[G:H]$ is unit in K, then $wD(K[H]) = wD(K[G])$.*

Proposition 3.12. *Let K be a field, and let H be a subgroup of G of finite index. If $wD(K[G])$ is finite, then $wD(K[H]) = wD(K[G])$.*

Proof. By Proposition 3.5 statement (1), it is enough to prove that $wD(K[H]) \geqslant wD(K[G])$. So suppose that $wD(K[G]) = n < \infty$. Let V_0 be a principal $K[G]$-module. By Proposition 3.4, $fd_{K[G]}(V_0) = n$. Then, for any right $K[G]$-module L, $\operatorname{Tor}_{n+1}^{K[G]}(L, V_0) = 0$, and there is at least one right $K[G]$-module N, $\operatorname{Tor}_n^{K[G]}(N, V_0) \neq 0$. Consider the following exact sequence of right $K[G]$-modules:

$$0 \to N \xrightarrow{f} \operatorname{Hom}_{K[H]}(K[G], N) \to \operatorname{Coker} f \to 0,$$

where $f(n)(\sum\limits_{g\in G} r_g g) = n \sum\limits_{g\in G} r_g g$ for $n \in N$ and $\sum\limits_{g\in G} r_g g \in K[G]$. Applying the functor $- \otimes_{K[G]} V_0$ to it, we get a long exact sequence

$$0 = \operatorname{Tor}_{n+1}^{K[G]}(\operatorname{Coker} f, V_0) \to \operatorname{Tor}_n^{K[G]}(N, V_0) \to \operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_K(K[G], N), V_0) \to \dots$$

Since $\operatorname{Tor}_n^{K[G]}(N, V_0) \neq 0$, $\operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_{K[H]}(K[G], N), V_0) \neq 0$. In addition, by [18], Lemma 9.2 and [16], Corollary 11.63, we get

$$\operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_{K[H]}(K[G], N), V_0) \cong \operatorname{Tor}_n^{K[G]}(N \otimes_{K[H]} K[G], V_0) \cong \operatorname{Tor}_n^{K[H]}(N, V_0).$$

Thus, $\operatorname{Tor}_n^{K[H]}(N, V_0) \neq 0$, and so $fd_{K[H]}(V_0) \geqslant n$. Noting that V_0 is also a principal $K[H]$-module, $wD(K[H]) = fd_{K[H]}(V_0) \geqslant n$. □

The following result provides an analogous version of Serre's theorem for the weak dimension. The idea of the proof is similar to the proof of [15], Theorem 3.12.

Theorem 3.13. *Let K be a field of characteristic p, and let H be a subgroup of G of finite index. If G is a p'-group, then $wD(K[H]) = wD(K[G])$.*

Proof. By Proposition 3.12, it suffices to show that $wD(K[G])$ is finite while $wD(K[H])$ is finite. Let V_0 be a principal $K[H]$-module and let

$$0 \to F_n \to \dots \to F_0 \to V_0 \to 0$$

be a finite flat resolution of V_0. Suppose that $[G:H] = m$, and let

$$\dots \to Q_n \to \dots \to Q_0 \to B \to 0, \tag{3.1}$$

where

$$B = \otimes^m V_0 = V_0 \otimes_K V_0 \otimes \ldots \otimes_K V_0$$

and

$$Q_t = \sum_{i_1+\ldots+i_m=t} F_{i_1} \otimes_K F_{i_2} \otimes \ldots \otimes_K F_{i_m}.$$

Choose a coset representative x_i so that $G = \cup x_i H$. If $g \in G$, let $g^{-1}x_i = x_{v_i} h_{v_i}^{-1}$ with $h_{v_i} \in H$ and define

$$g(f_1 \otimes f_2 \otimes \ldots \otimes f_m) = h_{v_1} f_{v_1} \otimes h_{v_2} f_{v_2} \otimes \ldots \otimes h_{v_m} f_{v_m}.$$

Thus, we define an action of G on Q_i. On the other hand, $B = V_0 \otimes_K V_0 \otimes \ldots \otimes_K V_0 \cong K$, and hence B is isomorphic to the principal $K[G]$-module. By the proof of [15], Theorem 3.12, (3.1) is an exact sequence of $K[G]$-modules. It will now suffice to prove that each Q_i is a flat $K[G]$-module. Noting that a flat module is a direct limit of free modules, the direct limit commutes with the tensor functor, and the direct limit (direct sum) of flat modules is also flat, it is only to show that $\otimes^m P_i$ is flat for free $K[H]$-modules P_i. It is true by the proof of [15], Theorem 3.12 again. Thus, (3.1) is a finite flat resolution of B, and hence $fd_{K[G]}(B)$ is finite. By Proposition 3.4, $wD(K[G])$ is finite, as desired. □

4. Gorenstein weak dimension

In this section, we will consider the finiteness of the Gorenstein weak dimension.

Lemma 4.1. *Let M be a $K[G]$-module. If F is a Gorenstein flat $K[G]$-module, then so is $F \otimes_K M$.*

Proof. If F is Gorenstein flat, then there is a complete flat resolution

$$F^\circ := \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots,$$

and $F = \mathrm{Ker}(F^0 \to F^1)$. Since each K-module is flat, we have the following exact sequence of $K[G]$-modules:

$$F^\circ \otimes_K M := \ldots \to F_1 \otimes_K M \to F_0 \otimes_K M \to F^0 \otimes_K M \to F^1 \otimes_K M \to \ldots,$$

and $F \otimes_K M = \mathrm{Ker}(F^0 \otimes_K M \to F^1 \otimes_K M)$. By Lemma 3.2, all $F_i \otimes_K M$ and all $F^i \otimes_K M$ are flat. Now it is enough to show that $I \otimes_{K[G]} (F^\circ \otimes_K M)$ is exact for any injective right $K[G]$-module I. In fact, we have

$$I \otimes_{K[G]} (F^\circ \otimes_K M) \cong (I \otimes_{K[G]} F^\circ) \otimes_K M.$$

Noting that F° is a complete flat resolution, the right complex is exact, as desired. □

By Lemma 4.1, the following result similar to Proposition 3.4 can be proven.

Proposition 4.2. *Let K be a field and let G be a group. If V_0 is a principal $K[G]$-module, then $GwD(K[G]) = Gfd_{K[G]}(V_0)$.*

The next results give some characteristics of the Gorenstein weak dimension of group rings.

Proposition 4.3. *Let K be a field, and let G be a group.*
(1) *If H is a subgroup of G and $K[H]$ is right coherent, then $GwD(K[H]) \leqslant GwD(K[G])$.*
(2) *If H is a subgroup of G of finite index, then $GwD(K[H]) \leqslant GwD(K[G])$.*
(3) *$GwD(K[G]) = 0$ if and only if G is locally finite.*
(4) *If G is an infinite cyclic group, then $GwD(K[G]) = 1$.*

Proof. (1) It is the result of [19], Theorem 2.8.

(2) We first show that if M is a Gorenstein flat $K[G]$-module, then $M \downarrow_H^G$ is Gorenstein flat as a $K[H]$-module. Let

$$F^\circ := \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$$

be a complete flat resolution such that $M = \operatorname{Ker}(F^0 \to F^1)$. It gives rise to an exact sequence of flat $K[H]$-modules

$$F^\circ \downarrow_H^G := \ldots \to F_1 \downarrow_H^G \to F_0 \downarrow_H^G \to F^0 \downarrow_H^G \to F^1 \downarrow_H^G \to \ldots,$$

and $M \downarrow_H^G = \operatorname{Ker}(F^0 \downarrow_H^G \to F^1 \downarrow_H^G)$. Let I be any injective right $K[H]$-module. Then the coinduced module $\operatorname{Hom}_{K[H]}(K[G], I)$ is injective as a $K[G]$-module. Since H is of finite index, in view of [18], Lemma 9.2,

$$I \otimes_{K[H]} F^\circ \downarrow_H^G \cong (I \otimes_{K[H]} K[G]) \otimes_{K[G]} F^\circ \cong \operatorname{Hom}_{K[H]}(K[G], I) \otimes_{K[G]} F^\circ.$$

Noting that the right complex is exact, then $M \downarrow_H^G$ is Gorenstein flat.

If $GwD(K[G]) = \infty$, there is nothing to show. Assume $GwD(K[G]) = n$. For any $K[H]$-module V, there is an exact sequence of $K[G]$- modules

$$0 \to Q_n \to \ldots \to Q_1 \to Q_0 \to V \uparrow_H^G \to 0$$

with each Q_i Gorenstein flat. Then there is an exact sequence of $K[H]$-modules

$$0 \to Q_n \downarrow_H^G \to \ldots \to Q_1 \downarrow_H^G \to Q_0 \downarrow_H^G \to V \uparrow_H^G \downarrow_H^G \to 0,$$

where each $Q_i \downarrow_H^G$ is Gorenstein flat. Thus, $Gfd_{K[H]}(V \uparrow_H^G \downarrow_H^G) \leqslant n$. Then $Gfd_{K[H]}(V) \leqslant n$ because V is isomorphic to a direct summand of $V \uparrow_H^G \downarrow_H^G$. Therefore, $GwD(K[H]) \leqslant n$.

(3) It follows from [5], Theorem 3.

(4) By Proposition 3.5 (3), $GwD(K[G]) = wD(K[G]) = 1$. □

Remark 4.4. Let K be a field of characteristic 3, and let G be the symmetric group of order 3 (see Example 3.7). Then, $GwD(K[G]) = 0$ but $wD(K[G])$ is infinite.

Following [7], $\operatorname{sfli}(R)$ denotes the supremum of the flat lengths of all injective R-modules. We study the invariant because it is deeply related to Gorenstein weak dimension. If $GwD(R)$ is finite, in view of [7], Lemma 5.1, so is $\operatorname{sfli}(R^{\mathrm{op}})$. However, there is a right IF-ring which is not left IF (see [5], Example 2). Thus, there exists a ring with a finite $\operatorname{sfli}(R^{\mathrm{op}})$ which has infinite Gorenstein weak dimension. But, since the group ring $K[G]$ is isomorphic to its opposite ring, the following result follows from [7], Theorem 5.3.

Proposition 4.5. *Let K be a field, and let G be a group. Then the following are equivalent:*

(1) *$GwD(K[G])$ is finite;*

(2) *$\operatorname{sfli}(K[G])$ is finite.*

In this case, $GwD(K[G]) = \operatorname{sfli}(K[G])$.

For a group ring $K[G]$, the ring homomorphism $\varepsilon\colon K[G] \to K$, $\sum r_g g \to \sum r_g$, is called the *augmentation mapping* of $K[G]$ and its kernel, denoted by $\Delta(K[G])$, is

$$\Delta(K[G]) = \left\{ \sum_{g \in G} a_g(g-1)\colon\ 1 \neq g,\ a_g \in K \right\}.$$

Proposition 4.6. *Let K be a field, and let H be a normal subgroup of G. Then*

(1) $\operatorname{sfli}(K[G]) \leqslant \operatorname{sfli}(K[H]) + \operatorname{sfli}(K[G/H])$.

(2) *If G/H is locally finite, then* $\operatorname{sfli}(K[H]) = \operatorname{sfli}(K[G])$.

P r o o f. (1) For convenience, we set $G/H := \overline{G}$. Suppose that $\operatorname{sfli}(K[H]) = n$ and $\operatorname{sfli}(K[\overline{G}]) = m$ are finite. For any injective $K[G]$-module I, it is sufficient to show that $fd_{K[G]}(I) \leqslant m + n$. Note that the augmentation sequence

$$0 \to \Delta(K[\overline{G}]) \to K[\overline{G}] \to K \to 0$$

yields an exact sequence of $K[\overline{G}]$-modules

$$0 \to K \to A \to B \to 0,$$

where $A = \mathrm{Hom}_K(K[\overline{G}], K)$ and $B = \mathrm{Hom}_K(\Delta(K[\overline{G}]), K)$. Hence I is a direct summand of $A \otimes_K I$, and so it is enough to prove that $fd_{K[G]}(A \otimes_K I) \leqslant m + n$.

Since K is an injective K-module, A is an injective $K[\overline{G}]$-module, and hence $fd_{K[\overline{G}]}(A) \leqslant m$. Let

$$F^\circ := 0 \to F_m \to F_{m-1} \to \ldots \to F_0 \to A \to 0$$

be a $K[\overline{G}]$-flat resolution of A. Choose a $K[G]$-flat resolution of I and

$$Q^\circ := 0 \to Q_n \to Q_{n-1} \to \ldots \to Q_0 \to I \to 0$$

is the truncation, where Q_i is $K[G]$-flat for $i = 0, \ldots, n-1$ and Q_n is $K[H]$-flat. Then the total complex $F^\circ \otimes_K Q^\circ$ is a $K[G]$-complex over $A \otimes_K I$ of length $m + n$. Since A is flat as a K-module, $F^\circ \otimes_K Q^\circ$ is a $K[G]$-resolution of $A \otimes_K I$ by the Künneth formula. Finally, we claim that $F^\circ \otimes_K Q^\circ$ is a $K[G]$-flat resolution. To prove this, it suffices to show that $F_m \otimes_K Q_n$ is a flat $K[G]$-module. By Lemma 3.1, it is true because F_m is $K[\overline{G}]$-flat and Q_n is $K[H]$-flat.

(2) By Proposition 4.3 and the result above, it is enough to show that $\mathrm{sfli}(K[H]) \leqslant \mathrm{sfli}(K[G])$. Now assume that $\mathrm{sfli}(K[G])$ is finite and I is an injective $K[H]$-module, by [14], Corollary 2.2 and the fact that the coinduced functor preserves injective modules, $\mathrm{Hom}_{K[H]}(K[G], I) \downarrow_H^G$ is injective. Since I is a direct summand of $\mathrm{Hom}_{K[H]}(K[G], I) \downarrow_H^G$, in view of [19], Remark 2.9,

$$fd_{K[H]}(I) \leqslant fd_{K[H]}(\mathrm{Hom}_{K[H]}(K[G], I) \downarrow_H^G) \leqslant fd_{K[G]}(\mathrm{Hom}_{K[H]}(K[G], I)).$$

Thus, $\mathrm{sfli}(K[H]) \leqslant \mathrm{sfli}(K[G])$. □

By Propositions 4.5 and 4.6, we get the next result.

Theorem 4.7. *Let K be a field, and let H be a normal subgroup of a group G. If $GwD(K[H])$ and $GwD(K[G/H])$ are finite, then so is $GwD(K[G])$, and the following hold:*
1. *$GwD(K[G]) \leqslant GwD(K[H]) + GwD(K[G/H])$.*
2. *If G/H is locally finite, then $GwD(K[G]) = GwD(K[H])$.*

Lemma 4.8. *Let K be a field, and let H be a subgroup of G. If $K[G]$ is right coherent, then the following hold:*
1. *If H is of finite index, then $K[H]$ is right coherent.*
2. *If H is a finite generated normal subgroup of G, then $K[G/H]$ is right coherent.*

P r o o f. (1) To prove that $K[H]$ is right coherent, it is enough to show that $\prod F$ is flat for any flat $K[H]$-module F. Since H is of finite index, we have the following isomorphisms:

$$K[G] \otimes_{K[H]} \Big(\prod F\Big) \cong \operatorname{Hom}_{K[H]}\Big(K[G], \prod F\Big) \cong \prod \operatorname{Hom}_{K[H]}(K[G], F)$$
$$\cong \prod (K[G] \otimes_{K[H]} F).$$

By [19], Proposition 2.2, $K[G] \otimes_{K[H]} F$ is a flat $K[G]$-module. Then $\prod(K[G] \otimes_{K[H]} F)$ is flat because $K[G]$ is right coherent, and so $K[G] \otimes_{K[H]} (\prod F)$ is flat. By [19], Proposition 2.2 again, $(\prod F) \uparrow_H^G \downarrow_H^G$ is flat as a $K[H]$-module. Thus, $\prod F$ is flat because $\prod F$ is a direct summand of $(\prod F) \uparrow_H^G \downarrow_H^G$.

(2) By [6], Proposition 1, $K[G/H] \cong K[G]/\omega H$, where ωH is a right ideal of $K[G]$ generated by $\{h_i - 1\colon h_i \in H\}$. Since H is finite generated, ωH is finite generated. Thus, $K[G/H]$ is right coherent in terms of [9], Theorem 4.1.1. □

Similarly to Proposition 3.12, one can prove the next results:

Proposition 4.9. *Let K be a field, and let H be a subgroup of a group G of finite index. If $K[G]$ is right coherent and $GwD(K[G])$ is finite, then $GwD(K[G]) = GwD(K[H])$.*

P r o o f. By Proposition 4.3 (2), $GwD(K[H]) \leqslant GwD(K[G])$. So suppose that $GwD(K[G]) = n < \infty$. Let V_0 be a principal $K[G]$-module. By Proposition 4.2,

$$Gfd_{K[G]}(V_0) = GwD(K[G]) = n.$$

Then, in view of Proposition 2.1, $\operatorname{Tor}_{n+1}^{K[G]}(L, V_0) = 0$ for any injective right $K[G]$-module L, and there is at least one injective right $K[G]$-module N,

$$\operatorname{Tor}_n^{K[G]}(N, V_0) \neq 0.$$

Consider the split exact sequence of right $K[G]$-modules

$$0 \to N \xrightarrow{f} \operatorname{Hom}_{K[H]}(K[G], N) \to \operatorname{Coker} f \to 0.$$

Since N is also injective as a right $K[H]$-module, we see that the coinduced module $\operatorname{Hom}_{K[H]}(K[G], N)$ is injective, and hence $\operatorname{Coker} f$ is injective. Applying the functor $- \otimes_{K[G]} V_0$ to it, we get a long exact sequence

$$0 = \operatorname{Tor}_{n+1}^{K[G]}(\operatorname{Coker} f, V_0) \to \operatorname{Tor}_n^{K[G]}(N, V_0) \to \operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_{K[H]}(K[G], N), V_0) \to \dots$$

Since $\operatorname{Tor}_n^{K[G]}(N, V_0) \neq 0$, $\operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_{K[H]}(K[G], N), V_0) \neq 0$. In addition, by [18], Lemma 9.2 and [16], Corollary 11.63, we get

$$\operatorname{Tor}_n^{K[G]}(\operatorname{Hom}_{K[H]}(K[G], N), V_0) \cong \operatorname{Tor}_n^{K[G]}(N \otimes_{K[H]} K[G], V_0) \cong \operatorname{Tor}_n^{K[H]}(N, V_0).$$

Thus, $\operatorname{Tor}_n^{K[H]}(N, V_0) \neq 0$, and so $Gfd_{K[H]}(V_0) \geqslant n$ in terms of Proposition 2.1 and Lemma 4.8. Therefore, by Proposition 4.2, $GwD(K[H]) = Gfd_{K[H]}(V_0) \geqslant n$. $\square$

Corollary 4.10. *Let K be a field, and let H be a subgroup of a group G of finite index. If $K[G]$ is right coherent and* $\operatorname{sfli}(K[G])$ *is finite, then $GwD(K[G]) = GwD(K[H])$.*

Question 4.11. Holm in [11] mentioned the meta-theorem: every result in classical homological algebra has a counterpart in Gorenstein homological algebra. Thus, the question is whether the condition that $K[G]$ is right coherent in Proposition 4.9 and the corresponding corollary can be omitted or not.

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