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UNBALANCED UNICYCLIC AND BICYCLIC GRAPHS WITH EXTREMAL SPECTRAL RADIUS

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Abstract. A signed graph Γ is a graph whose edges are labeled by signs. If Γ has n vertices, its spectral radius is the number $\varrho(\Gamma) := \max\{|\lambda_i(\Gamma)|\colon 1 \leqslant i \leqslant n\}$, where $\lambda_1(\Gamma) \geqslant \ldots \geqslant \lambda_n(\Gamma)$ are the eigenvalues of the signed adjacency matrix $A(\Gamma)$. Here we determine the signed graphs achieving the minimal or the maximal spectral radius in the classes $\mathfrak{U}_n$ and $\mathfrak{B}_n$ of unbalanced unicyclic graphs and unbalanced bicyclic graphs, respectively.

Keywords: signed graph; spectral radius; bicyclic graph

MSC 2020: 05C50, 05C22

1. Introduction

A *signed graph* Γ is a pair (Γ, σ), where $G = (V(G), E(G))$ is a simple graph and $\sigma\colon E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. The (unsigned) graph G of $\Gamma = (G, \sigma)$ is called the *underlying graph*. The *sign* of a cycle C in Γ is given by $\operatorname{sign}(C) = \prod\limits_{e \in C} \sigma(e)$. A cycle is called *positive* or *negative* if $\operatorname{sign}(C)$ is 1 or -1. A signed graph is *balanced* if all cycles, if any, are positive; otherwise it is *unbalanced*. If all edges in Γ are positive or negative, then Γ is denoted by $(G, +)$ or $(G, -)$; in this case, we refer to such signature as the all-positive or all-negative one, respectively.

Most of the concepts defined for graphs are directly extended to signed graphs. For example, the girth of a signed graph is the length of a shortest cycle in its underlying graph. Moreover, a signed graph is said to be k-*cyclic* if the underlying graph is

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k-cyclic. This means that G is connected and $|E(G)| = |V(G)| + k - 1$. We use the adjectives *unicyclic* and *bicyclic* as synonyms of 1-cyclic and 2-cyclic, respectively.

Whenever a subgraph H of the underlying graph G of Γ is considered, we assume that on H the restriction of the original sign function is defined. In particular, we say that $\Lambda = (H, \tau)$ is an induced signed subgraph of $\Gamma = (G, \sigma)$ if H is an induced subgraph of G and $\tau = \sigma\big|_H$.

The concept of *signature switching* plays a pivotal role in the realm of signed graphs and cannot be eluded. Given a signed graph $\Gamma = (G, \sigma)$ and a sign function $\theta\colon V \to \{-1, +1\}$, we can build a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$, where $\sigma^\theta(e) = \theta(v_i)\sigma(e)\theta(v_j)$ for each edge $e = v_iv_j \in E(G)$. Two signed graphs Γ and Λ are *switching equivalent* and we write $\Gamma \sim \Lambda$ if there exists a switching function $\theta\colon V \to \{-1, +1\}$ such that $\Lambda = \Gamma^\theta$. Obviously, $\sim$ is an equivalence relation on the set of signed graphs with the same underlying graph. Signature switching does not affect the sign of the cycles; therefore, Γ and Γ^θ share the set of positive cycles. By [21], Lemma 5.3 it follows that a signed graph $\Gamma = (G, \sigma)$ is balanced if and only if $\Gamma \sim (G, +)$. Furthermore, two signed graphs Γ and Λ are said to be *switching isomorphic* if Γ is isomorphic to a switching of Λ. If G is bicyclic, there are at least three (and up to four) switching non-isomorphic signatures (see [5], Section 3 or [9], Section 2).

Graph matrices and their invariants can be naturally extended from graphs to signed graphs. Once we fix a labelling $v_1, \ldots, v_n$ of the vertices of G, the adjacency matrix $A(\Gamma)$ of a signed graph $\Gamma = (G, \sigma)$ is obtained from the adjacency matrix $A(G)$ by replacing 1 by -1 whenever the corresponding edge is negative. Similarly, the Laplacian matrix $L(\Gamma)$ is given by $D(G) - A(\Gamma)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. The reader is referred to [22] for basic results on the spectra of signed graphs, to [23] for a possibly complete bibliography on signed graphs, and to [24] for a glossary of terms related to signed graphs.

Since $A(\Gamma)$ is symmetric, all its eigenvalues $\lambda_1(\Gamma) \geqslant \ldots \geqslant \lambda_n(\Gamma)$ are real. Moreover, since the trace of $A(\Gamma)$ is 0, we have $\lambda_1(\Gamma)\lambda_n(\Gamma) \leqslant 0$, where the equality holds if and only the underlying graph G is empty.

The *spectral radius* of Γ is defined as the number

$$\varrho(\Gamma) := \max\{|\lambda_i(\Gamma)|\colon\ 1 \leqslant i \leqslant n\} = \max\{|\lambda_1(\Gamma)|, |\lambda_n(\Gamma)|\}.$$

Note that $A(\Gamma)$ is not in general similar to a non-negative matrix. Hence it can happen that $|\lambda_n(\Gamma)| > |\lambda_1(\Gamma)|$. As a consequence of Theorems 1.2 and 1.5 in [19], this case surely occurs when G is not bipartite and $\Gamma = (G, -)$. On the contrary, if Γ is balanced, then $\varrho(\Gamma) = \lambda_1(\Gamma)$ (see [19], Theorem 1.5).

Laplacian spectral properties of signed bicyclic graphs have been investigated in [4], [5], [9]. Here we focus on the adjacency matrix, and we address the following problem: for $k \in \{1, 2\}$, which unbalanced k-cyclic graphs have minimal or maximal spectral radius in the set of signed graphs of given order? Observe that the 0-cyclic case is trivial since all signed trees are balanced.

The identification within a fixed class of graphs of those elements which are extremal with respect to a fixed topological index is a very classical problem, and the last decades have seen growing interest for the 'spectral' sub-branch of extremal graph theory.

Let U_n and B_n be the classes of unicyclic and bicyclic graphs, respectively, with n vertices. Graphs in U_n with extremal spectral radius have been considered in [7], [11], [16]. The detection of extremal graphs in B_n with respect to the spectral radius goes back to 1980s. In [8], Brualdi and Solheid found the graph in B_n with maximal spectral radius; while in [19], graphs with minimal spectral radius in B_n were identified by Simić, see [17]. Later on, extremal graphs in suitable proper subsets of B_n have been detected in [10], [13], [14], [20].

Let now $\mathfrak{U}_n$ and $\mathfrak{B}_n$ be the classes of unbalanced unicyclic and bicyclic graphs, respectively, with n vertices. Some extremal problems have been already solved in the context of signed graphs as well. For instance, the first author and others focused their attention on the index of signed graphs (i.e. the largest eigenvalue of the adjacency matrix), finding out in [2] the set of graphs with extremal index in $\mathfrak{U}_n$ (which is not empty for $n \geqslant 3$). In order to describe the extremal objects, we denote by $G(3; a_1, a_2, a_3)$ or $U(3; a_1, a_2, a_3)$ the graph obtained by attaching a_i pendant vertices or a hanging path of length a_i, respectively, to the vertex v_i of the triangle C_3, with $a_1 \geqslant a_2 \geqslant a_3$ and $a_1 + a_2 + a_3 = n - 3$.

It turns out that, among all signed graphs in $\mathfrak{U}_n$, the ones attaining the maximal index are switching isomorphic to $(G(3; n-3, 0, 0), -)$. For $n \geqslant 6$ a graph achieving the minimal index in $\mathfrak{U}_n$ is switching isomorphic to $(U(3; a_1, a_2, a_3), -)$, where $a_1 - a_3 \leqslant 1$, see [2], Theorems 3.2 and 3.11.

The remainder of the paper is structured as follows. After some preliminaries in Section 2, we determine in Section 3 the signed graphs with maximum spectral radius in $\mathfrak{U}_n$ and in the set $\mathfrak{B}_n$. Finally, Section 4 is devoted to seeking out the signed graphs in $\mathfrak{U}_n$ and $\mathfrak{B}_n$ with minimum spectral radius. It is worth noting that, for all $n \geqslant 4$, the signed graph $(G(3; n-3, 0, 0), -)$ turns out to maximize in $\mathfrak{U}_n$ both the index and the spectral radius. On the contrary, for all $n \geqslant 6$, the minimizer with respect to the index and the one with respect to the spectral radius are not isomorphic. In fact, they have girth 3 and 4, respectively.

2. Preliminaries

Let $\Gamma = (G, \sigma)$ be a signed graph of order n. Recall that throughout this paper we denote by $\lambda_1(\Gamma) \geqslant \ldots \geqslant \lambda_n(\Gamma)$ the eigenvalues of its adjacency matrix. The first known result we mention concerns the relationship between the spectrum of a signed graph $\Gamma = (G, \sigma)$ and the spectrum of the graph $-\Gamma = (G, -\sigma)$ obtained by reversing the signature on all edges. Clearly $(G, -) = -(G, +)$. The statement is an immediate consequence of the equality $A(-\Gamma) = -A(\Gamma)$.

Proposition 2.1. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n. The following formula holds:*

$$\lambda_i(-\Gamma) = -\lambda_{n-i+1}(\Gamma) \quad \textit{for } 1 \leqslant i \leqslant n.$$

When G is bipartite, it can be proved that (G, σ) and $(G, -\sigma)$ are switching equivalent. Therefore, the next corollary, which is also known, comes easily from Proposition 2.1.

Corollary 2.2. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n. If G is bipartite, then the spectrum of the adjacency matrix* $\mathrm{Spec}(A(\Gamma))$ *is symmetric with respect to* 0, *i.e.*

$$\lambda_i(\Gamma) = -\lambda_{n-i+1}(\Gamma) \quad \textit{for } 1 \leqslant i \leqslant n. \tag{2.1}$$

For the sake of brevity we simply say that $\mathrm{Spec}(A(\Gamma))$ is *symmetric* if (2.1) holds. It is known that $\mathrm{Spec}(A(G, +))$ is symmetric if and only if G is bipartite (see [19], Theorem 1.2); yet there are examples of unbalanced signed graphs Γ such that $\mathrm{Spec}(A(\Gamma))$ is symmetric, but Γ and $-\Gamma$ are not switching isomorphic. This topic is discussed in [6], Section 3.1.

The next theorem is known as the *Interlacing Theorem* for signed graphs.

Theorem 2.3. *Let Γ be a signed graph of order n, and let Λ be an induced subgraph of Γ of order m. Then*

$$\lambda_{n-m+i}(\Gamma) \leqslant \lambda_i(\Lambda) \leqslant \lambda_i(\Gamma) \quad \textit{for } 1 \leqslant i \leqslant m. \tag{2.2}$$

Corollary 2.4. *Let Γ be a signed graph, and let Λ be an induced subgraph of Γ with m vertices. The following inclusion of real intervals holds:*

$$[\lambda_m(\Lambda), \lambda_1(\Lambda)] \subseteq [\lambda_n(\Gamma), \lambda_1(\Gamma)].$$

Corollary 2.4 says that the property of a signed graph to have its spectrum included in some real interval is hereditary with respect to its vertex-induced signed subgraphs. As already noted in [1], Section 3.1, this implies that we can characterize the signed graphs whose spectrum lies in the interval $\mathcal{I}$ by providing the maximal signed graphs with such property.

The following result reproduces [2], Theorem 2.5. An alternative proof can be obtained by considering both Proposition 2.1 and [18], Theorem 3.1.

Theorem 2.5. *Let (G,σ) be a signed graph of order n with at least one edge. Then*

$$\lambda_n(G,-) \leqslant \lambda_n(G,\sigma) < \lambda_1(G,\sigma) \leqslant \lambda_1(G,+). \tag{2.3}$$

The requirement of non-emptiness for the graph G in the statement of Theorem 2.5 clearly guarantees the non-emptiness of the function σ. Conventionally, the empty signature on an empty graph (like every signature on forests) is considered to be balanced.

This section of preliminaries ends by recalling two theorems proved in [15]. Recently, they were used by the first author and others to get the spectral characterization of signed cycles, see [1].

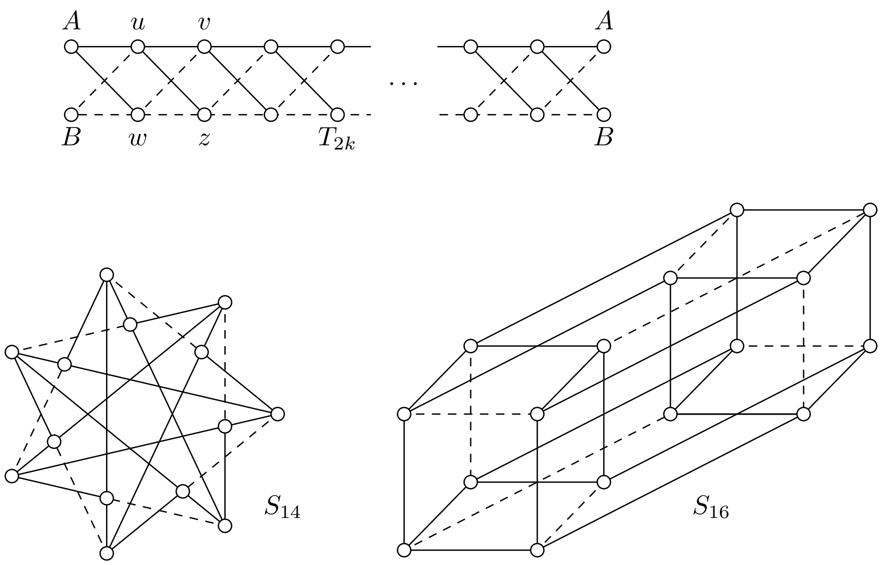


Figure 1. Maximal signed graphs whose spectrum is in $[-2,2]$. Negative edges are depicted by dashed lines.

Theorem 2.6 ([15], Theorem 1)**.** *Signed graphs with eigenvalues in* $[-2, 2]$ *are the induced subgraphs of*

(i) *the* $2k$*-vertex toral tessellation* T_{2k}*, whose spectrum is* $\{-2^{(k)}, 2^{(k)}\}$*;*
(ii) *the* 14*-vertex signed graph* S_{14}*, whose spectrum is* $\{-2^{(7)}, 2^{(7)}\}$*;*
(iii) *the* 16*-vertex signed hypercube* S_{16}*, whose spectrum is* $\{-2^{(7)}, 0^{(2)}, 2^{(7)}\}$*.*

The above mentioned signed graphs are depicted in Figure 1*.*

Theorem 2.7 ([15], Theorem 4)**.** *Up to switching isomorphism, the connected signed graphs having all their eigenvalues in* $(-2, 2)$ *are the induced subgraphs of*

(i) *the unbalanced cycle* C_{2k}^{-}*;*
(ii) *the signed graph* $Q_{h,k}$*, depicted in Figure 2;*
(iii) *the eleven sporadic examples* $U_1, U_2, \dots, U_{11}$ *with* 8 *vertices depicted in Figure* 3*.*

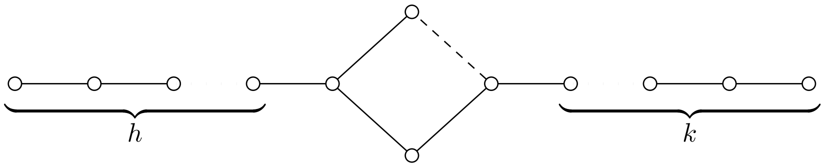


Figure 2. The signed graph $Q_{h,k}$. The only negative edge is depicted by a dashed line.

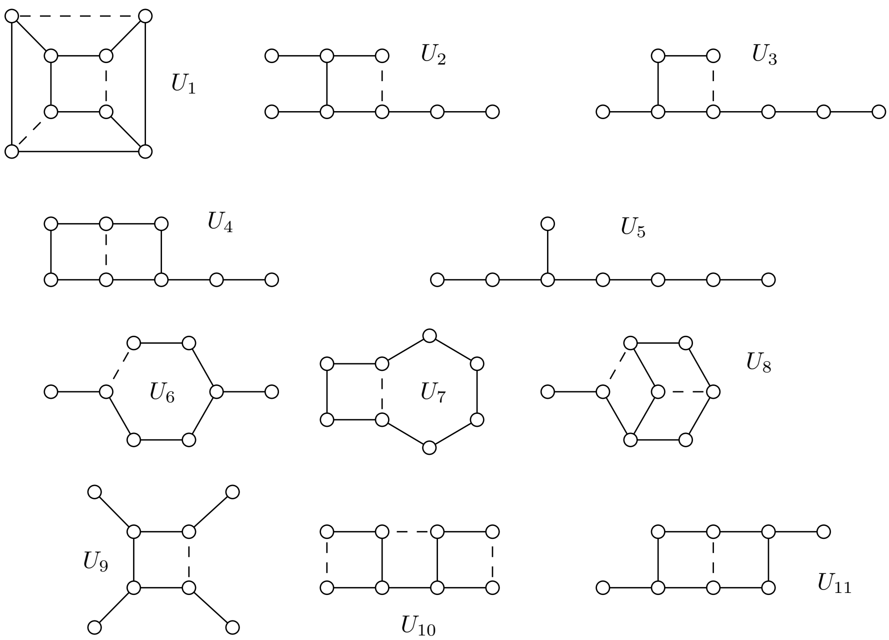


Figure 3. The eleven sporadic examples $U_1, U_2, \dots, U_{11}$. All of them have 8 vertices. Negative edges are depicted by dashed lines.

3. Unbalanced unicyclic and bicyclic graphs with maximum spectral radius

The following two theorems embody in the context of signed graphs parts of [8], Theorem 3.2, which was originally formulated for unsigned graphs. In their statements, $\mathfrak{U}_n^+$ or $\mathfrak{B}_n^+$ denotes, respectively, the class of balanced unicyclic or bicyclic signed graphs with n vertices.

Theorem 3.1. *Let $n \geqslant 3$. The maximal spectral radius for signed graphs in $\mathfrak{U}_n^+$ is only attained by those graphs which are switching isomorphic to $(G(3; n-3, 0, 0), +)$.*

Here and in Section 4, we denote by D_n the graph obtained by attaching $n-4$ pendant vertices to a vertex v of degree 3 in a diamond, i.e. a theta-graph of order 4 (see Figure 4).

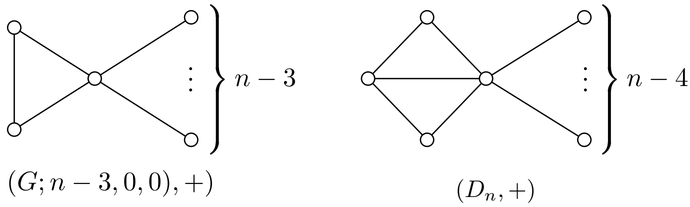


Figure 4. The graphs maximizing the spectral radius in $\mathfrak{U}_n^+$ and in $\mathfrak{B}_n^+$.

Theorem 3.2. *Let $n \geqslant 4$. The maximal spectral radius for signed graphs in $\mathfrak{B}_n^+$ is only attained by those graphs which are switching isomorphic to $(D_n, +)$.*

The characteristic polynomials of $A(G(3; n-3, 0, 0), +)$ and $A(D_n)$ were shown in [8] and can be derived by the graph divisor technique, see [12], Chapter 2.4. We exhibit them in the following corollary, included here for the sake of completeness.

Corollary 3.3. *The spectral radii $\varrho(G(3; n-3, 0, 0), +)$ and $\varrho(D_n, +)$ correspond, respectively, to the largest root of the following two polynomials:*

$$p_1(\lambda) = \lambda^3 - \lambda^2 - (n-1)\lambda + n - 3 \quad \textit{and} \quad p_2(\lambda) = \lambda^4 - (n+1)\lambda^2 - 4\lambda + 2(n-4).$$

The detection of graphs with maximum index in $\mathfrak{U}_n^+$ turns out to be a crucial step to determine the signed graphs with maximum spectral radius in $\mathfrak{U}_n$, as the proof of the following theorem shows.

Theorem 3.4. *Let $n \geqslant 3$. In the class $\mathfrak{U}_n$, a signed graph attains the maximal spectral radius if and only if it is switching isomorphic to $(G(3; n-3, 0, 0), -)$.*

Proof. Let (G,σ) be an element in $\mathfrak{U}_n$. The following inequalities hold.

$$\begin{aligned}
\lambda_1(G,\sigma) &\leqslant \lambda_1(G,+) && \text{(by Theorem 2.5)}\\
&\leqslant \lambda_1(G(3;n-3,0,0),+) && \text{(by Theorem 3.1)}\\
&= -\lambda_n(G(3;n-3,0,0),-) && \text{(by Proposition 2.1).}
\end{aligned} \tag{3.1}$$

Note that the second inequality is strict unless G is isomorphic to $G(3;n-3,0,0)$. Similarly

$$\begin{aligned}
-\lambda_n(G,\sigma) &= \lambda_1(G,-\sigma) && \text{(by Proposition 2.1)}\\
&\leqslant \lambda_1(G,+) && \text{(by Theorem 2.5)}\\
&\leqslant \lambda_1(G(3;n-3,0,0),+) && \text{(by Theorem 3.1)}\\
&= -\lambda_n(G(3;n-3,0,0),-) && \text{(by Proposition 2.1).}
\end{aligned} \tag{3.2}$$

Observe that $(G(3;n-3,0,0),-)$ is unbalanced; in fact, it contains a negative triangle. Moreover,

$$\begin{aligned}
|\lambda_n(G(3;n-3,0,0),-)| = \lambda_1(G(3;n-3,0,0),+) &> |\lambda_n(G(3;n-3,0,0),+)|\\
&= \lambda_1(G(3;n-3,0,0),-)
\end{aligned}$$

by Proposition 2.1 and [19], Theorem 1.5.

Using both (3.1) and (3.2) we get

$$\begin{aligned}
\varrho(G,\sigma) &= \max\{\lambda_1(G,\sigma),-\lambda_n(G,\sigma)\} \leqslant -\lambda_n(G(3;n-3,0,0),-)\\
&= \varrho(G(3;n-3,0,0),-),
\end{aligned}$$

where the equality holds only if (G,σ) is switching isomorphic to $(G(3;n-3,0,0),-)$ (recall that all unbalanced signatures on $G(3;n-3,0,0)$ are switching equivalent). $\square$

By comparing our Theorem 3.4 and Theorem 3.1 from paper [2], we discover that $(G(3;n-3,0,0),-)$ maximizes in $\mathfrak{U}_n$ both the index and the spectral radius.

In the remainder of this section, we identify the signed graphs with maximum spectral radius in $\mathfrak{B}_n$, the class of unbalanced bicyclic signed graphs with n vertices. Clearly $\mathfrak{B}_n$ is not empty for $n \geqslant 4$. We recall that D_n is the bicyclic graph depicted in Figure 4 (once you forget the signature). The reader immediately realizes that there exist just two switching non-isomorphic unbalanced signatures on D_n, depending on the non-negative number of unbalanced triangles: The key-point is that the signed graph $\widetilde{\Gamma}_n \colon = (D_n, \widetilde{\sigma})$ depicted in Figure 5 and $-\widetilde{\Gamma}_n$, though switching non-equivalent, are switching isomorphic.

Lemma 3.5. *Let $n \geqslant 4$. The two switching non-isomorphic unbalanced signatures on D_n give rise to signed graphs with different spectral radii.*

Proof. As representatives of the two switching non-isomorphic unbalanced signatures on D_n we choose $\widetilde{\Gamma}_n = (D_n, \widetilde{\sigma})$ and $(D_n, -)$ depicted in Figure 5.

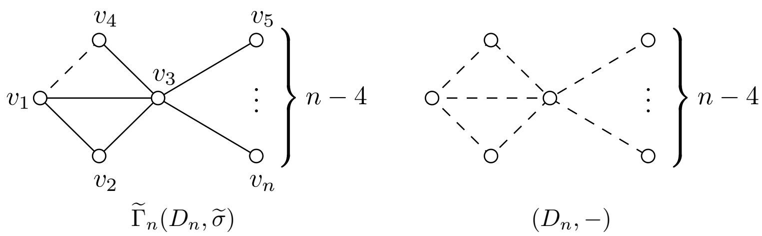


Figure 5. Representatives of the two switching non-isomorphic unbalanced graphs having D_n as the underlying graph.

First note that $\widetilde{\Gamma}_n$ and $-\widetilde{\Gamma}_n$ are switching isomorphic. By Proposition 2.1 it follows that $\mathrm{Spec}(A(\widetilde{\Gamma}_n))$ is symmetric; hence, $\varrho(\widetilde{\Gamma}_n) = \lambda_1(\widetilde{\Gamma}_n) = -\lambda_n(\widetilde{\Gamma}_n)$.

For $n = 4$ our claim comes from direct computation:

$$2 = \varrho(\widetilde{\Gamma}_4) < \frac{\sqrt{17}+1}{2} = \varrho(D_4, -).$$

For the rest of the proof we assume $n > 4$. Since $\lambda_1(\widetilde{\Gamma}_5) = \sqrt{3+\sqrt{3}} > 2$ and $\widetilde{\Gamma}_5$ is an induced subgraph of $\widetilde{\Gamma}_n$ for all $n > 4$, we deduce by Theorem 2.3 that $\varrho(\widetilde{\Gamma}_n) = \lambda_1(\widetilde{\Gamma}_n) \geqslant \lambda_1(\widetilde{\Gamma}_5) > 2$. Once we label the vertices of $\widetilde{\Gamma}_n$ according to Figure 5, and fix a $\lambda_1(\widetilde{\Gamma}_n)$-unit eigenvector $X = (x_1, \ldots, x_n)^\top$, we prove that $x_1 x_4 > 0$.

In order to see this, we solve the linear system consisting of the eigenvalue equations of X at the vertices v_1, v_2, and v_4. It turns out that

$$x_1 = \frac{\tilde{\lambda}^2}{(\tilde{\lambda}+1)(\tilde{\lambda}-2)} x_4,$$

where $\tilde{\lambda}$ stands for $\lambda_1(\widetilde{\Gamma}_n)$. Now x_1 and x_4 cannot be both 0, since in this case all the other components of X should be null as well. Hence, $x_1 x_4 > 0$ as claimed.

By using the properties of the Rayleigh quotient we now see that

$$\begin{aligned}(3.3)\quad \varrho(\widetilde{\Gamma}_n) = \lambda_1(\widetilde{\Gamma}_n) &= 2\Bigl(\sum_{i \overset{+}{\sim} j} x_i x_j - x_1 x_4\Bigr) < 2\Bigl(\sum_{i \overset{+}{\sim} j} x_i x_j + x_1 x_4\Bigr) \leqslant \lambda_1(D_n, +) \\ &= -\lambda_n(D_n, -) = \varrho(D_n, -),\end{aligned}$$

where '$i \overset{+}{\sim} j$' means that there exists a positive edge connecting v_i and v_j. $\square$

Theorem 3.6. *Let $n \geqslant 4$. In the class $\mathfrak{B}_n$, a signed graph attains the maximal spectral radius if and only if it is switching isomorphic to $(D_n, -)$.*

P r o o f. The starting argument closely resembles the one used along the proof of Theorem 3.4. Let (G, σ) be an element in $\mathfrak{B}_n$. By Theorems 2.5 and 3.2, together with Proposition 2.1, we have

$$(3.4) \qquad \lambda_1(G, \sigma) \leqslant \lambda_1(G, +) \leqslant \lambda_1(D_n, +) = -\lambda_n(D_n, -)$$

and

$$(3.5) \qquad -\lambda_n(G, \sigma) = \lambda_1(G, -\sigma) \leqslant \lambda_1(G, +) \leqslant \lambda_1(D_n, +) = -\lambda_n(D_n, -),$$

where the last inequalities in both (3.4) and (3.5) possibly hold only if $G = D_n$. Hence,

$$\varrho(G, \sigma) \leqslant \varrho(D_n, -) \quad \forall\, (G, \sigma) \in \mathfrak{B}_n,$$

where the inequality is surely strict if $G \neq D_n$.

The proof is now over since, by (3.3), $\varrho(D_n, \sigma) < \varrho(D_n, -)$ for every unbalanced graph (D_n, σ) switching non-isomorphic to $(D_n, -)$. $\square$

We explicitly point out that $(D_n, -)$ is not the right candidate to attain the maximum index in $\mathfrak{B}_n$. In fact, if we consider the graph G obtained by attaching a pendant vertex to the central vertex of the friendship graph F_2, and choose a sign function σ on $V(G)$ giving rise just to an unbalanced cycle, the largest eigenvalue of (G, σ) is bigger than $\lambda_1(D_6, -)$.

4. Unbalanced unicyclic and bicyclic graphs with minimum spectral radius

In order to detect the signed graphs minimizing the spectral radius in the set $\mathfrak{U}_n$, we consider the graphs $Q_{h,k}$ $(h \geqslant k)$ consisting of an unbalanced quadrangle having two pendant paths of length h and k at two opposite vertices of the cycle. Such graphs are depicted in Figure 2. By [3], Theorem 3.6 we easily get $\varrho(Q_{h,k}) = 2\cos(\pi/(2h+4))$.

Lemma 4.1 ([2], Lemma 3.10). *Let $\mathfrak{U}_n^{(4)}$ be the class of unbalanced unicyclic graphs of order n and girth 4. Every graph minimizing the index in $\mathfrak{U}_n^{(4)}$ is switching isomorphic to $Q_{\lceil\frac12(n-4)\rceil, \lfloor\frac12(n-4)\rfloor}$.*

We now show that the graphs minimizing the index in $\mathfrak{U}_n^{(4)}$ are also those that minimize the spectral radius in $\mathfrak{U}_n$.

Theorem 4.2. *Let $n \geqslant 4$. In the class $\mathfrak{U}_n$, a signed graph achieves the minimal spectral radius if and only if it is switching isomorphic either to $Q_{\lceil \frac{1}{2}(n-4)\rceil, \lfloor \frac{1}{2}(n-4)\rfloor}$ or (if n is even) to the unbalanced cycle C_n^-.*

P r o o f. By Theorem 2.7 we not only infer that, for every $n \geqslant 4$, there exist graphs in $\mathfrak{U}_n$ whose spectrum is entirely contained in the interval $(-2,2)$, but also that, up to switching isomorphism, the one minimizing the spectral radius in $\mathfrak{U}_n$ has to be searched among the several $Q_{h,n-4-h}$, C_n^- (if n is even) and the induced subgraphs of the eleven sporadic examples $U_1, U_2, \ldots, U_{11}$ (if $n \leqslant 8$). Since all such graphs are bipartite their spectral radius coincides with their index. Now the statement comes from Lemma 4.1, from the equality $\operatorname{Spec}(A(C_{2l}^-)) = \operatorname{Spec}(Q_{l-2,l-2})$ (see [1], Theorem 21), and from a direct check when $5 \leqslant n \leqslant 8$. $\square$

As already observed in Section 1, for $n > 4$, the graphs $Q_{\lceil \frac{1}{2}(n-4)\rceil, \lfloor \frac{1}{2}(n-4)\rfloor}$ do not attain the minimal index in $\mathfrak{U}_n$, which, according to [2], Theorem 3.11, for $n \geqslant 6$ is achieved instead by the signed graph $(U(3;p,q,r),-)$ with

$$p = \left\lceil \frac{n-3}{3} \right\rceil \geqslant q \geqslant r = \left\lfloor \frac{n-3}{3} \right\rfloor.$$

The remainder of this section is devoted to the detection of graphs in $\mathfrak{B}_n$ minimizing the spectral radius. For any $n \geqslant 4$, we set

$$\varrho(n) = \min\{\varrho(\Gamma)\colon\ \Gamma \in \mathfrak{B}_n\}.$$

A quite important role is played by the signed theta-graph $\Theta_{2,2,c}$ with $c+3$ vertices. Its underlying graph is obtained from three disjoint paths of order 3, 3 and $c+1$, respectively, by merging their initial vertices to a single vertex, say x, and their ending vertices to a single vertex y. This graph has just one negative edge incident to y such that there is an unbalanced quadrangle as a subgraph (see Figure 7).

Lemma 4.3. *For any $c \geqslant 1$, the spectral radius of $\Theta_{2,2,c}$ is 2.*

P r o o f. The graph $\Theta_{2,2,c}$ appears among the induced subgraphs of $T_{2(c+2)}$, a set of generating vertices being the union of all the vertices of the top line in Figure 1 and of a single vertex, say w, from the bottom line. By Theorem 2.6, this implies that $\varrho(\Theta_{2,2,c}) \leqslant 2$, and such inequality cannot be strict, since, by Theorem 2.7, no graph in Figure 3 contains any $\Theta_{2,2,c}$ as an induced subgraph. $\square$

We point out that the graph $T_{2(c+2)}$ contains many other switching isomorphic copies of $\Theta_{2,2,c}$. For instance we can take as generating vertices, z, w and all vertices of the top line, vertex v excluded.

Proposition 4.4. *Let $n \geqslant 4$. The values achieved by the minimum spectral radius of signed graphs in $\mathfrak{B}_n$ can be read in Table 1.*

n	4	5	6	7	8	$\geqslant 9$
$\varrho(n)$	2	2	$\sqrt{3}$	$2\cos\dfrac{\pi}{10}$	$2\cos\dfrac{\pi}{15}$	2

Table 1. Values of the minimum spectral radius.

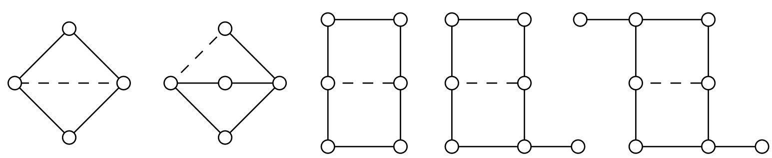
Proof. Recall that for any $n \geqslant 4$ there exists an unbalanced bicyclic graph whose spectral radius is 2; namely, $\Theta_{2,2,n-3}$. Graphs in Figure 3 do not contain induced bicyclic subgraphs of order 4 or 5 or $n \geqslant 9$. That is why, by Lemma 4.3, the values $\varrho(4)$, $\varrho(5)$, and $\varrho(n)$ for $n > 8$ are 2. To determine $\varrho(n)$ for $n = 6, 7, 8$ we compute the spectral radii of the relatively small number of unbalanced bicyclic graphs contained in the sporadic graphs $U_1, \ldots, U_{11}$. The minimizers are the graphs Γ_3, Γ_4 and Γ_5 depicted in Figure 6 and the graph U_7 in Figure 3 (Γ_5 and U_7 share the same spectrum). It turns out that

$$\varrho(6) = \varrho(\Gamma_3) = \sqrt{3}; \quad \varrho(7) = \varrho(\Gamma_4); \quad \text{and} \quad \varrho(8) = \varrho(\Gamma_5) = \varrho(U_7).$$

The values of $\varrho(7)$ and $\varrho(8)$ in Table 1 depend on the inclusions of spectra

$$\operatorname{Spec}(A(\Gamma_4)) \subset \operatorname{Spec}(A(P_{29})) \quad \text{and} \quad \operatorname{Spec}(A(\Gamma_5)) \subset \operatorname{Spec}(A(P_{14}))$$

proved in [3]. $\square$



Γ_1 Γ_2 Γ_3 Γ_4 Γ_5

Figure 6. Minimizers of the spectral radius in $\mathfrak{B}_n$ when $4 \leqslant n \leqslant 8$.

In the subsequent proofs we assign to each vertex of the toral tessellation T_{2k} coordinates in the set $(\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. Namely, A and B are identified with the pairs $(0, 1)$; $(0, 0)$, respectively, u and w with $(1, 1)$ and $(1, 0)$; and so on.

Lemma 4.5. *Let $k \geqslant 4$. Every bicyclic induced subgraph of the toral tessellation T_{2k} with $2k$ vertices has girth 4.*

P r o o f. Along the proof we refer to the depiction of T_{2k} in Figure 1. In our hypotheses T_{2k} is triangle-free. Let $\Gamma = (G, \sigma)$ be a bicyclic induced subgraph of T_{2k} and let C be a cycle of G. If C contains two vertices having the same ascissa then its girth is 4. In order to see this, assume that u and w in Figure 1 are in $V(C)$. Such set contains also a neighbor of u, say A. This means that C is one of the following three quadrangles: $AuBw$, $Auvw$, or $Auzw$. In particular, the girth of C is 4. If the girth of C is bigger than 4, the non-existence of pairs of vertices with the same x-coordinate implies that the girth is k and C is one of the four following cycles:

$$(0,0)(1,0)\ldots(k,0); \quad (0,1)(1,1)\ldots(k,1);$$
$$(0,0)(1,1)\ldots(j,(1-(-1)^j)/2)\ldots(k,0); \quad (0,1)(1,0)\ldots(j,(1+(-1)^j)/2)\ldots(k,1).$$

(the third and the fourth ones occur only if k is even). Let now t be a vertex in $V(G) \setminus V(C)$. We leave to the reader to check that, in all four cases, t is adjacent to two vertices in $V(C)$ having a common neighbor in C. In other words, G contains a quadrangle. $\square$

Proposition 4.6. *Let $k \geqslant 4$, and let (p, q, r) be a triple of non-negative integers. Every bicyclic induced subgraph of the toral tessellation T_{2k} with $2k$ vertices is switching isomorphic to either $\Theta_{2,2,k-2}$, Θ'_l $(0 \leqslant l \leqslant k-4)$, Θ''_l $(0 \leqslant l \leqslant k-5)$, $Q_{p,q,r}$ $(p+q+r \leqslant k-6)$, $Q'_{p,q,r}$ $(p+q+r \leqslant k-7)$, or $Q''_{p,q,r}$ $(p+q+r \leqslant k-8)$, all depicted in Figure 7.*

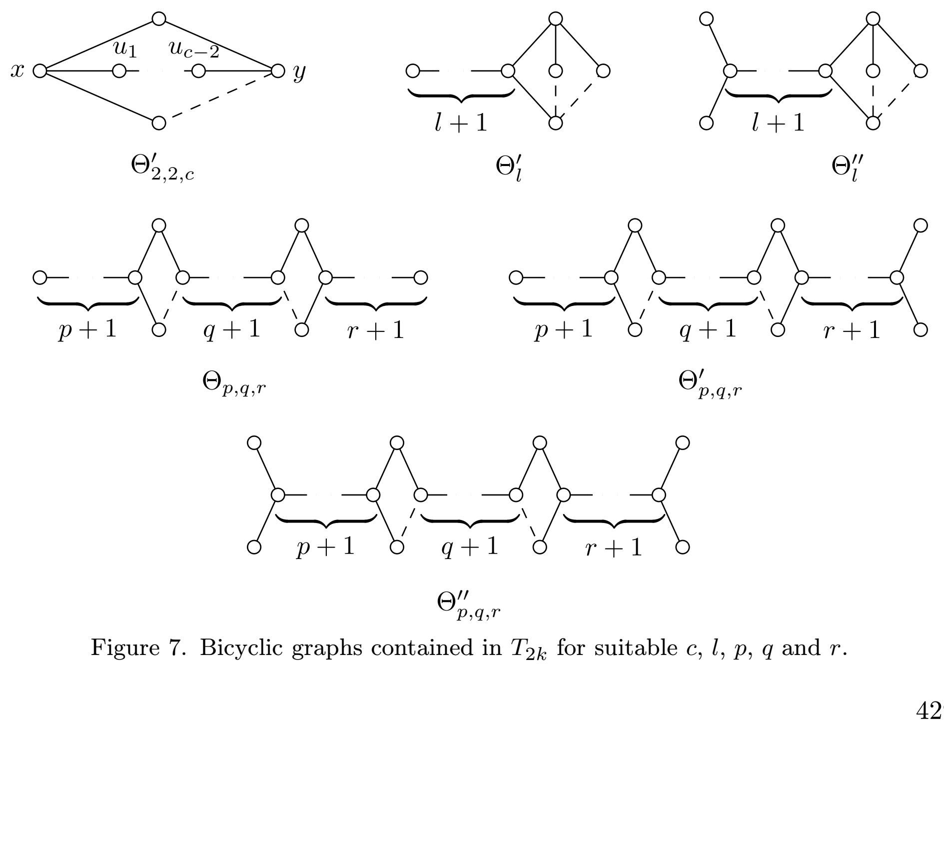


Figure 7. Bicyclic graphs contained in T_{2k} for suitable c, l, p, q and r.

Proof. Once again we consider the graph T_{2k} as depicted in Figure 1. Let $\Gamma = (G, \sigma)$ be a bicyclic induced subgraph of T_{2k}. By Lemma 4.5, G contains a quadrangle Q.

In order to examine all cases, we first suppose that three vertices in Q have the same y-coordinate. To fix ideas, we assume that $Q = Auvw$. Now, if neither B nor z are in $V(G)$, then Γ is switching isomorphic to $\Theta_{2,2,k-2}$. If instead $\{B, z\} \cap V(G) \neq \emptyset$, we assume that $B \in V(G)$ (the case $x \in V(G)$ could be treated similarly). The degree of B is two, otherwise Γ would not be bicyclic. Vertices in $V(G) \setminus \{A, u, v, w, B\}$, if existing, necessarily form a tree attached to v. Bicyclicity of Γ ensures that such signed graph is switching isomorphic to a graph of type Θ'_l or Θ''_l.

Finally we suppose that the quadrangle Q has only two vertices with 1 as the y-coordinate. To fix ideas, let $Q = Auzw$. If $\{B, z\} \cap V(G) \neq \emptyset$ we return to the previous case. If v and B are not in $V(G)$, aside from $Auzw$ the graph Γ must contain another cycle C. By the topology of T_{2k}, the cycle C has at most one vertex in common with $Auzw$. It is now straightforward to realize that Γ is switching isomorphic to a graph of type $Q_{p,q,r}$, $Q'_{p,q,r}$ or $Q''_{p,q,r}$, where the number $p + q + r$ satisfies the inequalities of the statement. As a final remark note that the induced subgraphs of type Θ''_l and $Q''_{p,q,r}$ are maximal, in the sense that they are not properly contained in another bicyclic induced subgraph of T_{2k}. $\square$

We explicitly note that the bicyclic graphs in the statement of Proposition 4.6 are all bipartite. Their spectrum is symmetric (see Corollary 2.2). In particular the index of all of them is 2. We specify their order.

$$\Theta_{2,2,c} \in \mathfrak{B}_{c+3}, \qquad \Theta'_l \in \mathfrak{B}_{l+5}, \qquad \Theta''_l \in \mathfrak{B}_{l+7}$$
$$Q_{p,q,r} \in \mathfrak{B}_{p+q+r+7}, \quad Q'_{p,q,r} \in \mathfrak{B}_{p+q+r+9}, \quad Q''_{p,q,r} \in \mathfrak{B}_{p+q+r+11}.$$

Besides these families, up to switching isomorphism there are still twelve sporadic bicyclic graphs whose spectral radius is 2. We depict them in Figure 8. They have 9 or 10 vertices. Consistently with Theorem 2.6 and Proposition 4.6, they are induced subgraphs of either S_{14} or S_{16}. Table 2 establishes whether a certain Λ_i with $1 \leqslant i \leqslant 12$ in Figure 8 is an induced subgraph of S_j $(j \in \{14, 16\})$ or not.

	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}	Λ_{11}	Λ_{12}
S_{14}				✓	✓	✓						
S_{16}	✓	✓	✓	✓		✓	✓	✓	✓	✓	✓	✓

Table 2. Induced subgraphs of S_{14} and S_{16}.

The non-existence of other sporadic bicyclic graphs inside S_{14} and S_{16} with minimal spectral radius was proved through the following sieve-type algorithm: having

Corollary 2.4 on disposal, and bipartiteness of S_{14} and S_{16} as well, we first identified, for $4 \leqslant n \leqslant 16$, the graphs $\Gamma = (G, \sigma) \in \mathfrak{B}_n$ not containing odd cycles or pendant vertices such that $\varrho(\Gamma) \leqslant 2$, and then we added in all possible ways pendant trees to such Γ's, one vertex at time up to $16 - |V(G)|$, being careful that, at each step, the spectral radius did not exceed 2. It turned out that, for $n \geqslant 9$, the spectral radius of all signed graphs in $\mathfrak{B}_n$ is bigger than 2 whenever they are not switching isomorphic to any graph in Figures 7 or 8. Our final theorem summarizes the results so far achieved on the unbalanced bicyclic signed graphs of order n ($n \geqslant 9$).

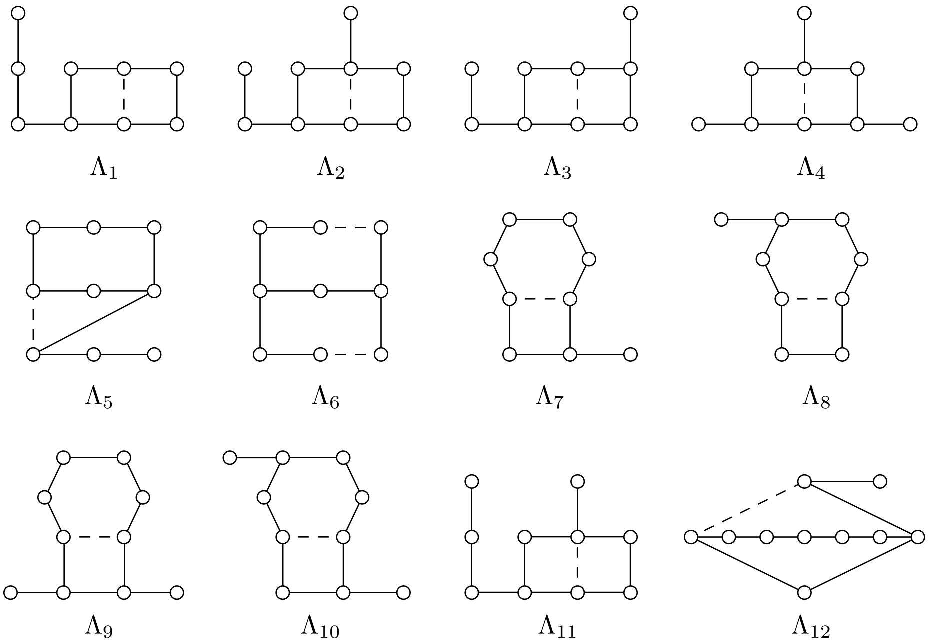


Figure 8. The twelve sporadic graphs with order $n \in \{9, 10\}$ whose spectral radius is 2.

Theorem 4.7. *Let $n \geqslant 9$. Up to switching isomorphism, the graphs in $\mathcal{B}_n$ minimizing the spectral radius are the following.*

- ▷ *For $n = 9$: Λ_i, with $1 \leqslant i \leqslant 8$; $\Theta_{2,2,6}$; Θ'_4; Θ''_2; $Q_{2,0,0}$; $Q_{1,0,1}$; $Q_{1,1,0}$; $Q'_{0,0,0}$.*
- ▷ *For $n = 10$: Λ_i, with $9 \leqslant i \leqslant 12$; $\Theta_{2,2,7}$; Θ'_5; Θ''_3; $Q_{p,q,r}$, with $p+q+r = 3$; $Q'_{p,q,r}$, with $p+q+r = 1$.*
- ▷ *For $n > 10$: $\Theta_{2,2,n-3}$; Θ'_6; Θ''_4; $Q_{p,q,r}$, with $p+q+r = n-7$; $Q'_{p,q,r}$, with $p+q+r = n-9$; $Q''_{p,q,r}$, with $p+q+r = n-11$.*

The problem of identifying the graphs in $\mathfrak{B}_n$ minimizing the index is still open. In the light of experimental results, it seems that the minimizers do not appear among the graphs listed in Theorem 4.7.

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