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GORENSTEIN STAR MODULES AND GORENSTEIN TILTING MODULES

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Abstract. We introduce the notion of Gorenstein star modules and obtain some properties and a characterization of them. We mainly give the relationship between n-Gorenstein star modules and n-Gorenstein tilting modules, see L. Yan, W. Li, B. Ouyang (2016), and a new characterization of n-Gorenstein tilting modules.

Keywords: Gorenstein quasi-projective module; Gorenstein star module; Gorenstein tilting module

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1. Introduction and preliminaries

The tilting theory is well known and plays an important role in the representation theory of Artin algebra. The classical notion of tilting modules was first considered in the case of finite-dimensional algebras by Brenner and Butler, see [4], and by Happel and Ringel, see [7]. Colpi and Trlifaj in [5] investigated (not necessarily finitely generated) tilting modules of projective dimension not great than 1 (simply, 1-tilting modules), and then Angeleri-Hügel and Coelho in [1], and Bazzoni in [3] considered (not necessarily finitely generated) tilting modules of projective dimension not great than n (simply, n-tilting modules). Yan, Li, and Ouyang in [11] generalized Auslander-Solberg relative notions by giving the definitions of infinitely generated Gorenstein cotilting and tilting modules by means of Gorenstein exact sequences. Wei in [9] introduced n-star modules and proved that (not necessarily finitely generated) n-tilting modules are precisely n-star modules n-presenting all the injectives.

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Throughout this article, R denotes an n-Gorenstein ring (meaning that R is both left and right Noetherian, with self-injective dimension not great than n from both sides). All R-modules are, if not specified otherwise, left R-modules. We denote by R-Mod the category of all left R-modules.

In this article, we introduce the notion of Gorenstein star modules, i.e., we consider star modules in the Gorenstein homological algebra. Using the relative homological theory developed by Enochs, we obtain a characterization of n-Gorenstein star modules and some properties similar to n-star modules. Finally, the relationship between n-Gorenstein star modules and n-Gorenstein tilting modules is given. We mainly prove the following two results.

Theorem 1.1. *Let R be an n-Gorenstein ring. Then the following statements are equivalent:*
(1) *T is an n-Gorenstein tilting module.*
(2) $\mathrm{Pres}_G^n T = T^{G\perp_{1\leqslant i\leqslant n}}$.
(3) *T is an n-Gorenstein star module and $\mathcal{GI} \subseteq \mathrm{Pres}_G^n T$.*
(4) $\mathcal{GI} \subseteq \mathrm{Pres}_G^n T = \mathrm{Pres}_G^{n+1} T \subseteq T^{G\perp_1}$.

Theorem 1.2. *Let T be an R-module. Then T is an n-Gorenstein tilting module if and only if $\mathrm{Hom}(T,-)$ preserves exactness in $\mathcal{C}_n^{GT}$, $\mathcal{GI} \subseteq \mathcal{C}_n^{GT}$, and $\mathcal{C}_n^{GT}$ is closed under n-G-images.*

2. n-Gorenstein star modules

Let M and N be R-modules. Then the couple (M,N) (or $\mathrm{Ext}^n(M,N)$) means $\mathrm{Hom}_R(M,N)$ (or $\mathrm{Ext}_R^n(M,N)$, respectively).

Gorenstein homological algebra is well known. For the basic definitions and results, we refer the reader to [6], [8]. Let $\mathcal{GP}$ be the full subcategory of R-Mod of Gorenstein projective modules, the class of all Gorenstein injective modules is denoted by $\mathcal{GI}$.

When R is an n-Gorenstein ring, Enochs and Jenda have proved that $(-,-)$ is right balanced on ${}_R\mathcal{U} \times_R \mathcal{U}$ by $\mathcal{GP} \times \mathcal{GI}$ (see [6], Theorem 12.1.4). Then we can compute right derived functors of (M,N) using a left Gorenstein projective resolution of R-module M or a right Gorenstein injective resolution of the R-module N (see [6]). We denote these derived functors by $\mathrm{Gext}^i(M,N)$. It is easy to check that:
(1) $\mathrm{Gext}^0(M,N) \cong (-,-)$.
(2) $\mathrm{Gext}^i(X,-) = 0$ for all $i \geqslant 1$ and any $X \in \mathcal{GP}$.
(3) $\mathrm{Gext}^i(-,Y) = 0$ for all $i \geqslant 1$ and any $Y \in \mathcal{GI}$.

(4) If the exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules is $\mathrm{Hom}(\mathcal{GP}, -)$ exact, then there is a long exact sequence

$$\ldots \to \mathrm{Gext}^i(Y, -) \to \mathrm{Gext}^i(X, -) \to \mathrm{Gext}^{i+1}(Z, -) \to \ldots$$

(5) If the exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules is $\mathrm{Hom}(-, \mathcal{GI})$ exact, then there is a long exact sequence

$$\ldots \to \mathrm{Gext}^i(-, Y) \to \mathrm{Gext}^i(-, X) \to \mathrm{Gext}^{i+1}(-, Z) \to \ldots$$

(6) There are natural transformations $\mathrm{Gext}^i(-, -) \to \mathrm{Ext}^i(-, -)$ which are also natural in the long exact sequence as in (4) and (5) above.

Let $G = \mathrm{Gext}^1(-, -)$. Then G is an additive subfunctor of $\mathrm{Ext}^1(-, -)$. A short exact sequence $0 \to X \to Y \to Z \to 0$ is called *G-exact* if it is in $\mathrm{Gext}^1(Z, X)$. According to [2], every pullback and pushout of a G-exact sequence $0 \to X \to Y \to Z \to 0$ given by morphisms $Z' \to Z$ and $X \to X'$ are again G-exact and the direct sum of exact sequences is G-exact if and only if each of them is G-exact. A long exact sequence

$$\ldots \to X_{n+1} \to X_n \to X_{n-1} \to \ldots$$

is G-exact if it splits into short G-exact sequences.

A homomorphism $f\colon X \to Y$ is called *G-epimorphism* if the sequence $0 \to \mathrm{Ker}\, f \to X \to Y \to 0$ is G-exact. Dually, we have the definition of G-monomorphism.

The following lemma is a characterization of G-exact sequences, which is very useful in the next section.

Lemma 2.1. *The following statements are equivalent for an exact sequence* $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$*:*

(1) *The sequence is G-exact.*
(2) $0 \to (P, X) \to (P, Y) \to (P, Z) \to 0$ *is exact for all* $P \in \mathcal{GP}$.
(3) $0 \to (Z, I) \to (Y, I) \to (X, I) \to 0$ *is exact for all* $I \in \mathcal{GI}$.
(4) π *is a G-epimorphism.*
(5) i *is a G-monomorphism.*

P r o o f. By Proposition 1.5 of [2] and Lemma 2.2 of [10]. □

For an R-module T, the short exact sequence $0 \to X \to Y \to Z \to 0$ is T-exact, if the sequence $0 \to (T, X) \to (T, Y) \to (T, Z) \to 0$ is exact. We denoted by $\mathrm{Pres}^n_G T$

the class of all R-modules X for which there exists a G-exact sequence of the form $T_n \to \ldots \to T_2 \to T_1 \to X \to 0$ with $T_i \in \operatorname{Add} T$ for any i.

Definition 2.2. An R-module T is called an *n-Gorenstein quasi-projective module* if for any G-exact sequence $0 \to X \to T_0 \to Y \to 0$ with $T_0 \in \operatorname{Add} T$ and $X \in \operatorname{Pres}_G^{n-1} T$, the G-exact sequence is T-exact.

Next, we give a characterization of n-Gorenstein quasi-projective modules.

Proposition 2.3. *For an R-module T, if $\operatorname{Pres}_G^{n-1} T = \operatorname{Pres}_G^{n} T$, then the following statements are equivalent.*

(1) *T is an n-Gorenstein quasi-projective module.*

(2) *If $0 \to X \to T_0 \to Y \to 0$ is G-exact with $T_0 \in \operatorname{Add} T$ and $Y \in \operatorname{Pres}_G^{n} T$, then $X \in \operatorname{Pres}_G^{n-1} T$ if and only if the G-exact sequence is T-exact.*

Proof. (2) $\Rightarrow$ (1): Obvious.

(1) $\Rightarrow$ (2) The if-part is clear. On the other hand, by assumption, there is a G-exact sequence $0 \to Y' \to T_0' \to Y \to 0$ with $T_0' \in \operatorname{Add} T$ and $Y' \in \operatorname{Pres}_G^{n-1} T$. Note that the sequence is T-exact since T is an n-Gorenstein quasi-projective module. By Lemma 2.3 of [9], we obtain that $T_0' \oplus X \cong T_0 \oplus Y' \in \operatorname{Pres}_G^{n-1} T = \operatorname{Pres}_G^{n} T$. Consequently, there is a G-exact sequence $0 \to X' \to T_0'' \to T_0' \oplus X \to 0$ with $T_0'' \in \operatorname{Add} T$ and $X' \in \operatorname{Pres}_G^{n-1} T$, and it is also T-exact by (1). We consider the commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] & & \\
0 \arrow[r] & Y'' \arrow[r] \arrow[d] & T_0'' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_0' \arrow[r] \arrow[d] & T_0' \bigoplus X \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}
$$

According to [2], every pullback and pushout of a G-exact sequence are again G-exact. It is easy to verify that all exact sequences are T-exact and G-exact in above diagram. There is a G-exact sequence $0 \to X'' \to T_{X'} \to X' \to 0$ with $T_{X'} \in \operatorname{Add} T$ and $X'' \in \operatorname{Pres}_G^{n-2} T$ since $X' \in \operatorname{Pres}_G^{n-1} T$. So we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X'' \arrow[r, equal] \arrow[d] & X'' \arrow[d] & & \\
0 \arrow[r] & T_{X'} \arrow[r] \arrow[d] & T_{X'} \bigoplus T'_0 \arrow[r] \arrow[d, "\alpha"] & T'_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & T'_0 \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

The existence of α is based on that the sequence $0 \to X' \to Y'' \to T'_0 \to 0$ is T-exact. It follows from the snake lemma that the second column is G-exact and then $Y'' \in \operatorname{Pres}_G^{n-1} T$. So $X \in \operatorname{Pres}_G^n T = \operatorname{Pres}_G^{n-1} T$. $\square$

Definition 2.4. An R-module T is called an *n-Gorenstein star module* if T is $(n+1)$-Gorenstein quasi-projective and $\operatorname{Pres}_G^n T = \operatorname{Pres}_G^{n+1} T$.

For an n-Gorenstein star module T, it is easy to see that for any $X \in \operatorname{Pres}_G^n T$ there is an infinite G-exact sequence

$$\ldots \to T_n \to \ldots \to T_1 \to T_0 \to X \to 0$$

with $T_i \in \operatorname{Add} T$ for all i.

Lemma 2.5. *Assume that T is an n-Gorenstein star module. If the sequence $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ is G-exact with $X, Y \in \operatorname{Pres}_G^n T$, then Z is also in $\operatorname{Pres}_G^n T$.*

P r o o f. By assumption, there are two G-exact sequences $0 \to X' \to T_X \xrightarrow{\alpha} X \to 0$ and $0 \to Y_1 \to T_Y \xrightarrow{\beta} Y \to 0$ with $T_X, T_Y \in \operatorname{Add} T$ and $X', Y_1 \in \operatorname{Pres}_G^n T$. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_X \arrow[r, "\binom{1}{0}"] \arrow[d, "\alpha"] & T_X \bigoplus T_Y \arrow[r, "{(0,1)}"] \arrow[d, "{(i\alpha,\beta)}"] & T_Y \arrow[r] \arrow[d, "\pi\beta"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Note that the sequence $0 \to Y_1 \to T_Y \xrightarrow{\beta} Y \to 0$ is T-exact since T is an n-Gorenstein star module, so the second column in the diagram above is also T-exact. It follows from Proposition 2.3 that $Y' \in \mathrm{Pres}^n_G T$. By the snake lemma, it is easy to verify that the second column and the first row are G-exact. Repeating the process with the G-exact sequence $0 \to X' \to Y' \to Z' \to 0$, we obtain that $Z \in \mathrm{Pres}^n_G T$. □

Proposition 2.6. *Assume that T is an n-Gorenstein star module and the G-exact sequence $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ is T-exact. If two of three terms X, Y, Z are in $\mathrm{Pres}^n_G T$, so is the third one.*

Proof. If X and Y are in $\mathrm{Pres}^n_G T$, the result holds by Lemma 2.5.

If X and Z are in $\mathrm{Pres}^n_G T$, then there are two G-exact sequences $0 \to X' \to T_X \xrightarrow{\alpha} X \to 0$ and $0 \to Z' \to T_Z \xrightarrow{\beta} Z \to 0$ with $T_X, T_Z \in \mathrm{Add}\, T$ and $X', Z' \in \mathrm{Pres}^n_G T$. Since $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ is T-exact, there exists a morphism $\gamma\colon T_Z \to Y$ such that $\beta = \pi\gamma$. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_X \arrow[r, "\binom{1}{0}"] \arrow[d] & T_X \bigoplus T_Z \arrow[r, "{(0,1)}"] \arrow[d, "{(i\alpha,\gamma)}"] & T_Z \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Similarly to the proof of Lemma 2.5, we obtain that $Y \in \mathrm{Pres}^n_G T$.

For the last case, assume that Y and Z are in $\mathrm{Pres}^n_G T$. Then there exists a G-exact sequence $0 \to Y' \to T_Y \to Y \to 0$ with $T_Y \in \mathrm{Add}\, T$ and $Y' \in \mathrm{Pres}^n_G T$. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y' \arrow[r, equal] \arrow[d] & Y' \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & T_Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Since the bottom row and the middle column are T-exact, it is not difficult to verify that the middle row is also T-exact. Note that every pullback of a G-exact sequence is again G-exact. It is easy to see that all exact sequences in the diagram above are G-exact. By Proposition 2.3, $M \in \operatorname{Pres}^n_G T$ since $Z \in \operatorname{Pres}^{n+1}_G T$. It follows from Lemma 2.5 that $X \in \operatorname{Pres}^n_G T$. $\square$

Proposition 2.7. *Let T be an n-Gorenstein star module. If the exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{Pres}^n_G T$ is G-exact, then it is also T-exact.*

Proof. By assumption, there are two G-exact sequences $0 \to X' \to T_X \xrightarrow{\alpha} X \to 0$ and $0 \to Y_1 \to T_Y \xrightarrow{\beta} Y \to 0$ with $T_X, T_Y \in \operatorname{Add} T$ and $X', Y_1 \in \operatorname{Pres}^n_G T$. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_X \arrow[r, "\binom{1}{0}"] \arrow[d] & T_X \bigoplus T_Y \arrow[r, "(0,1)"] \arrow[d, "(i\alpha,\beta)"] & T_Y \arrow[r] \arrow[d, "\pi\beta"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

As proved in Lemma 2.5, it is easy to see that $Z' \in \operatorname{Pres}^n_G T$ and then the right-hand column is T-exact since T is $(n+1)$-Gorenstein quasi-projective. For any morphism $f\colon T \to Z$, there is a morphism $g\colon T \to T_Y$ such that $f = \pi\beta g$. It follows that $\operatorname{Hom}(T, \pi)$ is surjective, i.e., the result holds. $\square$

To prove Theorem 1.1, we need to the following characterization of n-Gorenstein star modules.

Theorem 2.8. *The following statements are equivalent for an R-module T:*

(1) *T is an n-Gorenstein star module.*

(2) *If the exact sequence $0 \to X \to T_0 \to Y \to 0$ is G-exact with $T_0 \in \operatorname{Add} T$ and $Y \in \operatorname{Pres}^n_G T$, then $X \in \operatorname{Pres}^n_G T$ if and only if the G-exact sequence is T-exact.*

(3) *If the exact sequence $0 \to X \to Y \to Z \to 0$ is G-exact with $Y, Z \in \operatorname{Pres}^n_G T$, then $X \in \operatorname{Pres}^n_G T$ if and only if the G-exact sequence is T-exact.*

Proof. (1) $\Rightarrow$ (2) The only-if-part is clear by the definition of n-Gorenstein quasi-projective modules. The only-if-part is proved easily by Proposition 2.3.

(2) $\Rightarrow$ (1) It is enough to prove that $\mathrm{Pres}_G^n T = \mathrm{Pres}_G^{n+1} T$. For any $M \in \mathrm{Pres}_G^n T$, there are G-exact sequences $0 \to M' \overset{\alpha}{\to} T_0 \to M \to 0$ with $T_0 \in \mathrm{Add}\, T$ and $M' \in \mathrm{Pres}_G^{n-1} T$. Then we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\alpha"] \arrow[d, "\gamma"] & T_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M'' \arrow[r] & T_0' \arrow[r, "\beta"] & M \arrow[r] & 0,
\end{tikzcd}$$

where β is the evaluation map. Indeed, the diagram above is a pushout of the morphism α and the morphism γ. It follows that the second row is G-exact. By the assumption, we have that $M'' \in \mathrm{Pres}_G^n T$, i.e., $M \in \mathrm{Pres}_G^{n+1} T$.

(1) $\Rightarrow$ (3) By Propositions 2.7 and 2.6.

(3) $\Rightarrow$ (2) Obvious. $\square$

In order to prove Theorem 1.2, we need the following two propositions. Recall that a class $\mathcal{C}$ of R-modules is said to be closed under G-extension, if for any G-exact sequence $0 \to X \to Y \to Z \to 0$ with X, $Z \in \mathcal{C}$ it is $Y \in \mathcal{C}$.

Proposition 2.9. *The following statements are equivalent for an R-module T:*

(1) *T is an n-Gorenstein star module and $\mathrm{Pres}_G^n T$ is closed under G-extension.*

(2) $\mathrm{Pres}_G^n T = \mathrm{Pres}_G^{n+1} T \subseteq T^{G\perp_1}$.

Proof. (1) $\Rightarrow$ (2) It remains to show that $\mathrm{Pres}_G^n T \subseteq T^{G\perp_1}$. For any $M \in \mathrm{Pres}_G^n T$, we only need to verify that the G-exact sequence $0 \to M \to N \to T \to 0$ is split. By the assumption, $N \in \mathrm{Pres}_G^n T$. It follows from Proposition 2.7 that the G-exact sequence is T-exact, and then it is split.

(2) $\Rightarrow$ (1) Consider any G-exact $0 \to X \to T_0 \to Y \to 0$ with $T_0 \in \mathrm{Add}\, T$ and $X \in \mathrm{Pres}_G^n T$. Clearly, it is T-exact since $\mathrm{Pres}_G^n T \subseteq T^{G\perp_1}$, i.e., T is an n-Gorenstein star module. Now consider any G-exact $0 \to L \to M \to N \to 0$ with $L, N \in \mathrm{Pres}_G^n T$. Obviously, it is T-exact since $\mathrm{Pres}_G^n T \subseteq T^{G\perp_1}$. By Proposition 2.6, $M \in \mathrm{Pres}_G^n T$, i.e., $\mathrm{Pres}_G^n T$ is closed under G-extension. $\square$

Recall that a class $\mathcal{C}$ of R-modules is said to be closed under n-G-images if for any G-exact sequence $C_n \to \ldots \to C_2 \to C_1 \to X \to 0$ with $C_i \in \mathcal{C}$ for all i it is $X \in \mathcal{C}$. Obviously, the class $\mathrm{Pres}_G^1 T$ is closed under 1-G-images. We do not know whether or not $\mathrm{Pres}_G^n T$ is closed under n-G-images in general. But if T is an n-Gorenstein star module such that $\mathrm{Pres}_G^n T$ is closed under G-extensions, we have the following result.

Proposition 2.10. *Let T be an n-Gorenstein star module such that $\mathrm{Pres}_G^n T$ is closed under G-extensions. Then $\mathrm{Pres}_G^k(\mathrm{Pres}_G^n T) = \mathrm{Pres}_G^k T$ for all $k \geqslant 1$. In particular, $\mathrm{Pres}_G^n T$ is closed under n-G-images.*

P r o o f. It is sufficient to show that $\mathrm{Pres}_G^k(\mathrm{Pres}_G^n T) \subseteq \mathrm{Pres}_G^k T$. We proceed by induction on k. In case $k = 1$, for any $X \in \mathrm{Pres}_G^1(\mathrm{Pres}_G^n T)$, there is a G-exact sequence $0 \to X' \to Y \to X \to 0$ with $Y \in \mathrm{Pres}_G^n T$ and then we have also a G-exact sequence $0 \to Y' \to T_Y \to Y \to 0$ with $T_Y \in \mathrm{Add}\, T$ and $Y' \in \mathrm{Pres}_G^n T$. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y' \arrow[r, equal] \arrow[d] & Y' \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & T_Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Note that the second row is G-exact. It follows that $X \in \mathrm{Pres}_G^1(T)$.

Assume that $\mathrm{Pres}_G^j(\mathrm{Pres}_G^n T) \subseteq \mathrm{Pres}_G^j T$ is correct for all $1 \leqslant j \leqslant k$. Next we show that the result holds for $j = k+1$. Let $M \in \mathrm{Pres}_G^{k+1}(\mathrm{Pres}_G^n T)$, i.e., there is a G-exact $C_{k+1} \to \ldots \to C_2 \to C_1 \xrightarrow{f} M \to 0$ with $C_i \in \mathrm{Pres}_G^n T$ for all i. Set $\mathrm{Ker}\, f = M_1$, then $M_1 \in \mathrm{Pres}_G^k(\mathrm{Pres}_G^n T) = \mathrm{Pres}_G^k T$ by the induction assumption. Since T is an n-Gorenstein star module, we obtain a G-exact sequence $0 \to C' \to T' \to C_1 \to 0$ with $T' \in \mathrm{Add}\, T$ and $C' \in \mathrm{Pres}_G^n T$. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & C' \arrow[r, equal] \arrow[d] & C' \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

It is easy to verify that all rows and columns are G-exact. Since $M_1 \in \mathrm{Pres}_G^k T$, we obtain a G-exact sequence $0 \to M_1' \to T'' \to M_1 \to 0$ with $T'' \in \mathrm{Add}\, T$ and

$M_1' \in \mathrm{Pres}_G^{k-1}T$. We have the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & M_1' \arrow[r, equal] \arrow[d] & M_1' \arrow[d] & \\
0 \arrow[r] & C' \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & T'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & X \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

It is easy to see that all rows and columns are G-exact. Since C' and T'' are in $\mathrm{Pres}_G^n T$, then $Y \in \mathrm{Pres}_G^n T$. Since $M_1' \in \mathrm{Pres}_G^{k-1}T = \mathrm{Pres}_G^{k-1}(\mathrm{Pres}_G^n T)$, then $X \in \mathrm{Pres}_G^k(\mathrm{Pres}_G^n T) = \mathrm{Pres}_G^k T$. It follows from the middle row in the first diagram that M is in $\mathrm{Pres}_G^{k+1}T$.

□

3. Gorenstein tilting modules

In this section, we mainly give the relationship between n-Gorenstein star modules and n-Gorenstein tilting modules, and a new characterization of n-Gorenstein tilting modules.

Definition 3.1 ([11], Definition 3.2)**.** An R-module T is called an *n-Gorenstein tilting module* if it satisfies the following three conditions:
(1) $\mathrm{pd}_G T \leqslant n$.
(2) $\mathrm{Gext}^i(T, T^I) = 0$ for each $i > 0$ and all sets I.
(3) There exists a long G-exact sequence $0 \to P \to T^0 \to T^1 \to \ldots \to T^n \to 0$ with each $T^i \in \mathrm{Add}\, T$ and any $P \in \mathcal{GP}$.

Theorem 3.2. *Let R be an n-Gorenstein ring. Then the following statements are equivalent:*
(1) *T is an n-Gorenstein tilting module.*
(2) $\mathrm{Pres}_G^n T = T^{G\perp_{1\leqslant i\leqslant n}}$.
(3) *T is an n-Gorenstein star module and $\mathcal{GI} \subseteq \mathrm{Pres}_G^n T$.*
(4) $\mathcal{GI} \subseteq \mathrm{Pres}_G^n T = \mathrm{Pres}_G^{n+1} T \subseteq T^{G\perp_1}$.

P r o o f. (1) $\Rightarrow$ (2) By Theorem 3.16 of [11].

(2) $\Rightarrow$ (3) $\mathcal{GI} \subseteq T^{G\perp_{1\leqslant i\leqslant n}} \subseteq \mathrm{Pres}^n_G T$. Now we show that T is an n-Gorenstein star module for any G-exact sequence $0 \to X \to T^0 \to Y \to 0$ with $T^0 \in \mathrm{Add}\, T$ and $Y \in \mathrm{Pres}^n_G T$. It is easy to see that $X \in T^{G\perp_{2\leqslant i\leqslant n}}$ by the dimension shifting. The induced sequence $0 \to (T, X) \to (T, T^0) \to (T, Y) \to \mathrm{Gext}^1(T, X) \to 0$ is exact since $T^0 \in \mathrm{Add}\, T \subseteq \mathrm{Pres}^n_G T = T^{G\perp_{1\leqslant i\leqslant n}}$. So the G-exact sequence $0 \to X \to T^0 \to Y \to 0$ is T-exact if and only if $\mathrm{Gext}^1(T, X) = 0$ if and only if $X \in T^{G\perp_{1\leqslant i\leqslant n}} = \mathrm{Pres}^n_G T$. It follows from Theorem 2.8 that T is an n-Gorenstein star module.

(3) $\Rightarrow$ (4) It is enough to prove that $\mathrm{Pres}^n_G T \subseteq T^{G\perp_1}$. For any $X \in \mathrm{Pres}^n_G T$, there exists a G-exact sequence $0 \to X \to I \to X' \to 0$ with $I \in \mathcal{GI} \subseteq \mathrm{Pres}^n_G T$ by the assumption and Lemma 2.1. It follows from Lemma 2.5 that $X' \in \mathrm{Pres}^n_G T$. By Proposition 2.7, the G-exact sequence is T-exact, and then $X \in T^{G\perp_1}$ since $\mathcal{GI} \subseteq T^{G\perp_1}$, i.e., $\mathrm{Pres}^n_G T \subseteq T^{G\perp_1}$.

(4) $\Rightarrow$ (1) Note that T is an n-Gorenstein star module and $\mathrm{Pres}^n_G T$ is closed under G-extension by Proposition 2.9. We only need to verify that $\mathrm{Pres}^n_G T = T^{G\perp_{\geqslant 1}}$. For any $X \in \mathrm{Pres}^n_G T$, there is a long G-exact sequence $0 \longrightarrow X \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \longrightarrow \ldots \xrightarrow{f_n} I_n \longrightarrow 0$ with $I_i \in \mathcal{GI}$ for all i by the assumption and Lemma 2.1. Set $\mathrm{Ker}\, f_i = X_{i-1}$, $1 \leqslant i \leqslant n$, where $X_0 = X$. By $\mathcal{GI} \subseteq \mathrm{Pres}^n_G T$ and Lemma 2.5, we get that $X_i \in \mathrm{Pres}^n_G T \subseteq T^{G\perp_1}$ for all i. It is not difficult to see that $X \in T^{G\perp_{\geqslant 1}}$ by the dimension shifting, i.e., $\mathrm{Pres}^n_G T \subseteq T^{G\perp_{\geqslant 1}}$. On the other hand, for any $Y \in T^{G\perp_{\geqslant 1}}$, there is a long G-exact sequence $0 \longrightarrow Y \xrightarrow{g_0} J_0 \xrightarrow{g_1} J_1 \longrightarrow \ldots \xrightarrow{g_n} J_n \longrightarrow 0$ with $J_j \in \mathcal{GI}$ for all j by the assumption and Lemma 2.1. Set $\mathrm{Ker}\, g_j = Y_{j-1}$, $1 \leqslant j \leqslant n$, where $Y_0 = Y$. By the dimension shifting, $Y_j \in T^{G\perp_{\geqslant 1}}$ for all j. Since $\mathcal{GI} \subseteq \mathrm{Pres}^n_G T$, then $Y_{n-1} \in \mathrm{Pres}^n_G T$ by Proposition 2.6, and then $Y \in \mathrm{Pres}^n_G T$, i.e., $T^{G\perp_{\geqslant 1}} \subseteq \mathrm{Pres}^n_G T$. Consequently, $T^{G\perp_{\geqslant 1}} = \mathrm{Pres}^n_G T$. $\square$

Denote by $\mathcal{C}^{GT}_n$ the subcategory. There is a G-exact sequence $T_n \to T_{n-1} \ldots \to T_1 \to T_0 \to X \to 0$ with $T_i \in \mathrm{Add}\, T$ and the exact sequence is T-exact.

Lemma 3.3. *Let $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ be a G-exact sequence.*

(1) *If X and Y are in $\mathcal{C}^{GT}_n$, then $Z \in \mathrm{Pres}^{n+1}_G T$. Moveover, if the sequence is T-exact, then $Z \in \mathcal{C}^{GT}_n$.*

(2) *If Y and Z are in $\mathcal{C}^{GT}_n$ and the sequence is T-exact, then $X \in \mathcal{C}^{GT}_{n-1}$.*

P r o o f. (1) We prove the statement by induction on n. For $n = 0$, the result is clear.

Now, we assume that the conclusion holds for $n - 1$. Let $X, Y \in \mathcal{C}^{GT}_n$, then we have two G-exact sequences $0 \to X' \to T_X \xrightarrow{\alpha} X \to 0$ and $0 \to Y' \to T_Y \xrightarrow{\beta} Y \to 0$ with $T_X, T_Y \in \mathrm{Add}\, T$ and $X', Y' \in \mathcal{C}^{GT}_{n-1}$ such that both the sequences are T-exact.

We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_X \arrow[r, "\binom{1}{0}"] \arrow[d] & T_X \bigoplus T_Y \arrow[r, "(0,1)"] \arrow[d, "(i\alpha,\beta)"] & T_Y \arrow[r] \arrow[d, "\pi\beta"] & 0 \\
0 \arrow[r] & X \arrow[r, "i"] \arrow[d] & Y \arrow[r, "\pi"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

It is easy to see that the middle column and the first row are G-exact. Then we have a new G-exact sequence $0 \to X_1 \to Y_1 \to Z_1 \to 0$, where $X_1 := X' \bigoplus T_Y \in \mathcal{C}_{n-1}^{GT}$, $Y_1 := Y'' \bigoplus T_Y$. Consider the two G-exact sequences $0 \to Y' \to T_Y \xrightarrow{\beta} Y \to 0$ and $0 \to Y'' \to T_X \bigoplus T_Y \to Y \to 0$, then of Lemma 2.3 in [9], we have that $Y_1 := Y'' \bigoplus T_Y \cong Y' \bigoplus T_X \bigoplus T_Y \in \mathcal{C}_{n-1}^{GT}$. It follows from the induction assumption that $Z_1 \in \mathrm{Pres}_G^{n+1} T$. Moveover, if the sequence is T-exact, it is not difficult to verify that $0 \to X_1 \to Y_1 \to Z_1 \to 0$ is also T-exact, then $Z_1 \in \mathcal{C}_n^{GT}$ by the induction assumption.

(2) By the assumption, Y, Z are in $\mathcal{C}_n^{GT}$, then there are two G-exact sequences $0 \to Y' \to T_Y \to Y \to 0$ and $0 \to Z'' \to T_Z \to Z \to 0$ with $T_Y, T_Z \in \mathrm{Add}\, T$ and $Y', Z'' \in \mathcal{C}_{n-1}^{GT}$, such that both the sequences are T-exact. We consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y' \arrow[r, equal] \arrow[d] & Y' \arrow[d] & & \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & T_Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

It is easy to verify that all the sequences in the diagram above are G-exact and T-exact. Consider the two G-exact sequences $0 \to Z' \to T_Y \to Z \to 0$ and $0 \to$

$Z'' \to T_Z \to Z \to 0$, then by Lemma 2.3 of [9], we have that $Z_1 := Z' \bigoplus T_Z \cong Z'' \bigoplus T_Y \in \mathcal{C}_{n-1}^{GT}$, and then we get a G-exact sequence $0 \to Y_1 \to Z_1 \to X \to 0$ with $Y_1 = Y' \bigoplus T_Z \in \mathcal{C}_{n-1}^{GT}$ such that the G-exact sequence is also T-exact. Consequently, X is in $\mathcal{C}_{n-1}^{GT}$ by (1). $\square$

In order to give the proof of Theorem 1.2, we also need the following lemma.

Lemma 3.4. *If T is an n-Gorenstein tilting module, then $\mathcal{C}_n^{GT} = \mathrm{Pres}_G^n T$.*

Proof. Note that $\mathcal{C}_n^{GT} \subseteq \mathrm{Pres}_G^{n+1} T = \mathrm{Pres}_G^n T$. For any $X \in \mathrm{Pres}_G^{n+1} T$, there is an infinite G-exact sequence

$$\ldots \longrightarrow T_n \xrightarrow{f_n} \ldots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $T_i \in \mathrm{Add}\, T$ for all i. Set $\Im f_i = X_i \in \mathrm{Pres}_G^{n+1} T$. Consider the G-exact sequence $0 \to X_1 \to T_0 \to X \to 0$. We have that it is also T-exact since T is n-Gorenstein quasi-projective. Consequently, all the sequences $0 \to X_i \to T_{i-1} \to X_{i-1} \to 0$ are T-exact, i.e., $X \in \mathcal{C}_n^{GT}$. Then $\mathcal{C}_n^{GT} = \mathrm{Pres}_G^{n+1} T = \mathrm{Pres}_G^n T$. $\square$

Theorem 3.5. *Let T be an R-module. Then T is an n-Gorenstein tilting module if and only if $\mathrm{Hom}(T, -)$ preserves exactness in $\mathcal{C}_n^{GT}$, $\mathcal{GI} \subseteq \mathcal{C}_n^{GT}$ and $\mathcal{C}_n^{GT}$ is closed under n-G-images.*

Proof. ($\Leftarrow$) Note that $\mathcal{C}_n^{GT} \subseteq \mathrm{Pres}_G^{n+1} T \subseteq \mathrm{Pres}_G^n T$. Since $\mathcal{C}_n^{GT}$ is closed under n-G-images, we have that $\mathrm{Pres}_G^n T \subseteq \mathcal{C}_n^{GT}$. So $\mathcal{C}_n^{GT} = \mathrm{Pres}_G^{n+1} T = \mathrm{Pres}_G^n T$. For any $X \in \mathcal{C}_n^{GT}$, since R is an n-Gorenstein ring, the Gorenstein injective dimension of X is finite. By [8], Theorem 2.15, there is a G-exact sequence $0 \longrightarrow X \longrightarrow I_0 \xrightarrow{f_1} I_1 \longrightarrow \ldots \longrightarrow I_{n-1} \xrightarrow{f_n} I_n \longrightarrow 0$ with $I_i \in \mathcal{GI}$ for any i. Set $\mathrm{Ker}\, f_i = X_{i-1}$ for $1 \leqslant i \leqslant n$, where $X_0 = X$. We have that $X_1 \in \mathrm{Pres}_G^{n+1} T = \mathcal{C}_n^{GT}$ by Lemma 3.3. Since the sequence $0 \to X \to I_0 \to X_1 \to 0$ is T-exact, we have that $X \in T^{G\perp_1}$, i.e., $\mathcal{C}_n^{GT} \subseteq T^{G\perp_1}$. Consequently, T is an n-Gorenstein tilting module by Theorem 3.2.

($\Rightarrow$) By Propositions 2.9 and 2.10, Theorem 3.2 and Lemma 3.4. $\square$

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