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MONOTONICITY OF FIRST EIGENVALUES ALONG THE YAMABE FLOW

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Abstract. We construct some nondecreasing quantities associated to the first eigenvalue of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{2}(n-2)/(n-1)$) along the Yamabe flow, where Δ_φ is the Witten-Laplacian operator with a C^2 function φ. We also prove a monotonic result on the first eigenvalue of $-\Delta_\varphi + \frac{1}{4}(n/(n-1))R$ along the Yamabe flow. Moreover, we establish some nondecreasing quantities for the first eigenvalue of $-\Delta_\varphi + cR^a$ with $a \in (0,1)$ along the Yamabe flow.

Keywords: monotonicity; first eigenvalue; Witten-Laplacian operator; Yamabe flow

MSC 2020: 58C40

1. Introduction

Eigenvalues of geometric operators are essential tools in understanding geometry and topology of Riemannian manifolds. In 2002, Perelman in [10] introduced the F-entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation. The nondecreasing of this functional implies the monotonicity for the first eigenvalue of $-4\Delta + R$ along the Ricci flow. Inspired by this work (see [10]), much efforts have been devoted to study the monotonicity for eigenvalues of geometric operators under different geometric flows, especially the Ricci flow and the Yamabe flow in recent years. In 2007, Cao in [1] proved that all eigenvalues of the operator $-\Delta + \frac{1}{2}R$ are nondecreasing under the Ricci flow. Moreover, Cao in [2] showed that the first eigenvalue of general operators $-\Delta + cR$ ($c \geqslant \frac{1}{4}$) is nondecreasing under the Ricci flow and obtained the monotonicity under the normalized Ricci flow for the case of $c = \frac{1}{4}$ and the nonpositive average scalar curvature. In 2018, Ho in [8] obtained that the first eigenvalue of $-\Delta + cR$ is

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nondecreasing if $0 \leqslant c < \frac{1}{4}(n-2)/(n-1)$ and $\min\limits_M R_{g(0)} \geqslant (n-2)n^{-1}\max\limits_M R_{g(0)} \geqslant 0$ or if $c \geqslant \frac{1}{4}(n-2)/(n-1)$ and $\min\limits_M R_{g(0)} \geqslant 0$.

Along the Ricci flow, Fang, Yang and Zhu in [6] and Fang, Xu and Zhu in [4] generalized Cao's results to eigenvalues of geometric operators related to the Witten-Laplacian operator

$$\Delta_\varphi := \Delta - \nabla_{\nabla\varphi}$$

for some C^2 function φ. Zhao in [13] obtained an evolution equation for the first eigenvalue of the Laplacian operator and gave some monotonic quantities under the Yamabe flow. Moreover, he proved that the first eigenvalue of the p-Laplace operator is increasing and differentiable almost everywhere along the unnormalized powers of the mth mean curvature flow (see [14]) and the unnormalized H^k-flow (see [15]). More generally, Guo, Philipowski and Thalmaier in [7] derived an explicit formula for the evolution of the lowest eigenvalue of the Laplace-Beltrami operator with potential in abstract geometric flows. Considering the geometric operator $-\Delta_\varphi + \frac{1}{2}R$, Fang and Yang in [5] established an evolution equation for the first eigenvalue and constructed some monotonic quantities under the Yamabe flow.

Let $(M^n, g(t))$ be a Riemannian manifold with the metric $g(t)$ that evolves under the Yamabe flow. Motivated by the works [2], [5], [8], we consider first eigenvalues of geometric operators related to the Witten-Laplacian operator Δ_φ with $\varphi \in C^2(M^n)$.

Throughout this paper, $\varrho(t)$ and $\sigma(t)$ are two solutions to the ordinate differential equation $y' = y^2$ with the initial values $\varrho(0) = \max\limits_{x\in M^n} R_{g(0)}(x)$ and $\sigma(0) = \min\limits_{x\in M^n} R_{g(0)}(x)$, respectively. The main theorems of this paper are the following.

First of all, we derive some monotonic quantities for the first eigenvalue of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{4}(n-2)/(n-1)$) under the Yamabe flow.

Theorem 1.1. *Let $g(t)$, $t \in [0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with*

$$\min_{M^n} R_{g(0)} \geqslant \max\Big\{\frac{2(n-1)}{n-2}\Delta\varphi, 0\Big\}. \tag{1.1}$$

Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{2}(n-2)/(n-1)$), then the quantity

$$\exp\Big\{\int_0^t \Big[2(n-1)c\varrho(\tau) - \frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\lambda(t) \tag{1.2}$$

is nondecreasing along the Yamabe flow.

As a direct corollary of Theorem 1.1, we have:

Corollary 1.1. *Let $g(t)$, $t \in [0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with $R_{g(0)} \geqslant 0$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR$ with $c \geqslant \frac{1}{2}(n-2)/(n-1)$ and $\varphi \in C^2(M^n)$ is concave, then the quantity*

$$\exp\Big\{\int_0^t \Big[2(n-1)c\varrho(\tau) - \frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\lambda(t) \tag{1.3}$$

is nondecreasing along the Yamabe flow.

In special dimensions, we have monotonicity result for the first eigenvalue of $-\Delta_\varphi + \frac{1}{4}(n/(n-1))R$ under the Yamabe flow.

Theorem 1.2. *Let $g(t)$, $t \in [0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with $n \in \{2,3,4\}$ and*

$$\min_{M^n} R_{g(0)} \geqslant \max\{(n-1)\Delta\varphi, 0\}. \tag{1.4}$$

Then the first eigenvalue of $-\Delta_\varphi + \frac{1}{4}(n/(n-1))R$ is nondecreasing along the Yamabe flow.

Moreover, we also consider monotonic quantities associated to first eigenvalues of $-\Delta_\varphi + cR^a$ $(a \neq 1)$ along the Yamabe flow.

Theorem 1.3. *Let $g(t)$, $t \in [0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with $R_{g(0)} \geqslant 0$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR^a$ with $0 < a < 1$ and $c \geqslant \frac{1}{2}(n-2)/(n-1)$. If*

$$\frac{\Delta\varphi}{c} \leqslant R^a_{g(t)} \leqslant 1$$

on $[0,T_1]$ for some $T_1 \in (0,T)$. Then the quantity

$$\exp\Big\{\int_0^t \Big[2c(n-1)\varrho^a(\tau) - \frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\lambda(t)$$

is nondecreasing along the Yamabe flow on $(0,T_1)$.

Theorem 1.4. *Let $g(t)$, $t \in [0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with $R_{g(0)} \geqslant 0$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR^a$ with $0 < a < 1$ and $c > 0$. If*

$$\max\Big\{\frac{\Delta\varphi}{c}, 0\Big\} \leqslant R^a_{g(t)} \leqslant \Big(\frac{2c(n-1)}{n-2a}\Big)^{a/(1-a)} \tag{1.5}$$

on $[0, T_1]$ *for some* $T_1 \in (0, T)$, *then the quantity*

$$\exp\left\{\int_0^t \left[2c(n-1)\varrho^a(\tau) - \frac{n}{2}\sigma(\tau)\right] \mathrm{d}\tau\right\}\lambda(t)$$

is nondecreasing along the Yamabe flow on $(0, T_1)$.

We arrange this note as follows. In Section 2, we derive an evolution equation for the first eigenvalue of geometric operators $-\Delta + cR$ under the Yamabe flow and we finish the proof of Theorems 1.1 and 1.2 as applications. In Section 3, we consider the evolution of the first eigenvalue of $-\Delta_\varphi + cR^a$ $(a \neq 1)$ and prove Theorems 1.3 and 1.4. Under the normalized Yamabe flow, we present evolution equations and monotonic results for first eigenvalues in Section 4.

2. First eigenvalues of $-\Delta_\varphi + cR$ under the Yamabe flow

Let $(M^n, g(t))$ be an n-dimensional closed Riemannian manifold with $g(t)$, $t \in [0, T)$ being a smooth solution to the Yamabe flow. Let $\lambda(t)$ be an eigenvalue of the operator $-\Delta_\varphi + cR$ at the time $t_0 \in [0, T)$ and f be the associated eigenfunction with the normalization $\int_{M^n} f^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V = 1$, i.e.,

$$-\Delta_\varphi f + cRf = \lambda f. \tag{2.1}$$

Since $\mathrm{d}[\int_{M^n} f^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V]/\mathrm{d}t = 0$, we obtain that

$$\int_{M^n} f\left[\frac{\partial}{\partial t} f \mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{\partial}{\partial t}(f \mathrm{e}^{-\varphi}\,\mathrm{d}V)\right] = 0. \tag{2.2}$$

For any function satisfying (2.2), we define the functional

$$\lambda(f, t) = \int_{M^n} (-\Delta_\varphi f + cRf) f \mathrm{e}^{-\varphi}\,\mathrm{d}V. \tag{2.3}$$

At time t, if f is the eigenfunction of $\lambda(t)$, then

$$\lambda(f, t) = \lambda(t).$$

Proceeding as in [5], we have the evolution equation for the functional $\lambda(f, t)$ under general geometric flows.

Lemma 2.1. *Assume that* $\lambda(t)$ *is the first eigenvalue of* $-\Delta_\varphi + cR$, f *is the eigenfunction of* λ *at time* t_0 *and the metric* $g(t)$ *evolves by*

$$\frac{\partial g_{ij}}{\partial t} = v_{ij},$$

where v_{ij} is a symmetric 2-tensor. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(f,t)\big|_{t=t_0} = \int_{M^n}\Big(v_{ij}\nabla_i\nabla_j f - v_{ij}\nabla_i\varphi\nabla_j f + c\frac{\partial R}{\partial t}f\Big)f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \int_{M^n}\Big(\nabla_i v_{ij} - \frac{\nabla_i tr(v)}{2}\Big)\nabla_j f\cdot f\mathrm{e}^{-\varphi}\,\mathrm{d}V. \tag{2.4}$$

Remark 2.1. By the eigenvalue perturbation theory, we may assume that the first eigenvalue and first eigenfunction are C^1 in time along the Yamabe flow (see [9], [11], [12] and the references therein). Since (2.4) does not depend on the evolution equation of f, $\mathrm{d}\lambda(t)/\mathrm{d}t = \mathrm{d}\lambda(f,t)/\mathrm{d}t$.

Now we are ready to prove the following evolution equation under the Yamabe flow for the first eigenvalue of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{4}(n-2)/(n-1)$).

Theorem 2.1. *Let $g(t)$, $t\in[0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR$, $f(x,t)>0$ satisfies $\int_{M^n} f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V = 0$ and*

$$-\Delta_\varphi f(x,t) + cRf(x,t) = \lambda(t)f(x,t). \tag{2.5}$$

Then along the Yamabe flow, $\lambda(t)$ evolves by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = {}&(n-1)c\int_{M^n} Rf^2\Big[\Big(2c - \frac{n-2}{2n-2}\Big)R - \Delta\varphi\Big]\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &- \Big[2(n-1)c - \frac{n}{2}\Big]\lambda(t)\int_{M^n} Rf^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+ \Big[(n-1)c - \frac{n-2}{2}\Big]\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+ (n-1)c\int_{M^n} R|f\nabla\varphi - \nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V.\end{aligned} \tag{2.6}$$

P r o o f. Note that

$$\frac{\partial R}{\partial t} = (n-1)\Delta R + R^2 \tag{2.7}$$

under the Yamabe flow.

Substituting $v_{ij} = -Rg_{ij}$ and (2.7) into (2.4), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) &= \int_{M^n}\Big(-R\Delta_\varphi f + cf\frac{\partial R}{\partial t}\Big)f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{n-2}{2}\int_{M^n} f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &= -\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + (n-1)c\int_{M^n}\Delta Rf^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &\quad + c\int_{M^n} R^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{n-2}{2}\int_{M^n} f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V.\end{aligned} \tag{2.8}$$

Integrating by parts, we get

$$\int_{M^n} f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V = -\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V - \int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \tag{2.9}$$

and

$$\begin{aligned}\int_{M^n} f^2\Delta R\mathrm{e}^{-\varphi}\,\mathrm{d}V =& -2\int_{M^n} f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V + \int_{M^n} f^2\nabla_{\nabla\varphi}R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ =& \,2\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + 2\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+\int_{M^n} Rf^2|\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V - \int_{M^n} f^2R\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &-2\int_{M^n} fR\langle\nabla\varphi,\nabla f\rangle\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ =& \,2\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+\int_{M^n} R|f\nabla\varphi - \nabla f|^2\mathrm{e}^{-\varphi} - \int_{M^n} f^2R\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V.\end{aligned} \tag{2.10}$$

Plugging (2.9) and (2.10) into (2.8), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) =& -\frac{n}{2}\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + 2(n-1)c\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+\Big[(n-1)c - \frac{n-2}{2}\Big]\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+(n-1)c\int_{M^n} R|f\nabla\varphi-\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &-(n-1)c\int_{M^n} f^2R\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V + c\int_{M^n} R^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ =& \Big[2(n-1)c - \frac{n}{2}\Big]\int_{M^n} Rf^2(cR-\lambda)\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+\Big[(n-1)c - \frac{n-2}{2}\Big]\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &+(n-1)c\int_{M^n} R|f\nabla\varphi-\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V + c\int_{M^n} R^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &-(n-1)c\int_{M^n} f^2R\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ =& (n-1)c\int_{M^n} Rf^2\Big[\Big(2c - \frac{n-2}{2n-2}\Big)R - \Delta\varphi\Big]\mathrm{e}^{-\varphi}\,\mathrm{d}V\\ &-\Big[2(n-1)c - \frac{n}{2}\Big]\lambda(t)\int_{M^n} Rf^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\end{aligned}$$

$$+\Big[(n-1)c-\frac{n-2}{2}\Big]\int_{M^n}R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V$$
$$+(n-1)c\int_{M^n}R|f\nabla\varphi-\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V,$$

where we used (2.5) in the second equality. □

As we obtained Theorem 2.1, we are ready to finish the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. As in [5], by applying the maximum principle to (2.7), we get

$$(2.11)\qquad R_{g(t)}\leqslant\varrho(t)=\bigg(\frac{1}{\max\limits_{M^n}R_{g(0)}}-t\bigg)^{-1}$$

and

$$(2.12)\qquad R_{g(t)}\geqslant\sigma(t)=\bigg(\frac{1}{\min\limits_{M^n}R_{g(0)}}-t\bigg)^{-1}.$$

In particular, we have

$$(2.13)\qquad R_{g(t)}\geqslant\min_{M^n}R_{g(0)}.$$

Since $c\geqslant\frac{1}{2}(n-2)/(n-1)$ and $\min\limits_{M^n}R_{g(0)}\geqslant\max\{2((n-1)/(n-2))\Delta\varphi,0\}$, we have

$$(2.14)\qquad cR_{g(t)}f^2\Big[\Big(2c-\frac{n-2}{2n-2}\Big)R_{g(t)}-\Delta\varphi\Big]\geqslant 0$$

and

$$(2.15)\qquad \Big[(n-1)c-\frac{n-2}{2}\Big]R_{g(t)}\geqslant 0.$$

Therefore, we can derive from Theorem 2.1 that

$$(2.16)\quad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t)\geqslant-\Big[2(n-1)c-\frac{n}{2}\Big]\lambda(t)\int_{M^n}Rf^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\geqslant\Big[\frac{n}{2}\sigma(t)-2(n-1)c\varrho(t)\Big]\lambda(t).$$

Therefore, we have

$$(2.17)\quad \frac{\mathrm{d}}{\mathrm{dt}}\exp\bigg\{\int_0^t\Big[2(n-1)c\varrho(\tau)-\frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\bigg\}\lambda(t)$$
$$=\exp\bigg\{\int_0^t\Big[2(n-1)c\varrho(\tau)-\frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\bigg\}$$
$$\times\Big\{\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t)-\Big[\frac{n}{2}\sigma(t)-2(n-1)c\varrho(t)\Big]\lambda(t)\Big\}\geqslant 0.$$

This proves the theorem. □

P r o o f of Theorem 1.2. Assume $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi+\frac{1}{4}(n/(n-1))$ in this proof. Note that $R_{g(t)} \geqslant \min_{M^n} R_{g(0)}$ by (2.12).

Substituting $c=\frac{1}{4}(n/(n-1))$ into (2.6), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) &= \frac{n}{4}\int_{M^n} Rf^2\Big[\frac{R}{n-1}-\Delta\varphi\Big]\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{4-n}{4}\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\ &\quad + \frac{n}{4}\int_{M^n} R|f\nabla\varphi-\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \geqslant 0\end{aligned} \tag{2.18}$$

if $n\in\{2,3,4\}$ and $R\geqslant\max\{(n-1)\Delta\varphi,0\}$.

We conclude that $\lambda(t)$ is nondecreasing under the Yamabe flow. □

3. First eigenvalues of $-\Delta_\varphi + cR^a$ $(a\neq 1)$ under the Yamabe flow

It is not hard to derive the following evolution equation for the functional $\lambda(f,t)$ under general geometric flows.

Lemma 3.1. *Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi+cR^a$ $(a\neq 1)$, f is the eigenfunction of λ at time t_1 and the metric $g(t)$ evolves by*

$$\frac{\partial g_{ij}}{\partial t} = v_{ij},$$

where v_{ij} is a symmetric 2-tensor. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\lambda(f,t)\big|_{t=t_1} &= \int_{M^n}\Big(v_{ij}\nabla_i\nabla_j f - v_{ij}\nabla_i\varphi\nabla_j f + acR^{a-1}\frac{\partial R}{\partial t}f\Big)f\mathrm{e}^{-\varphi}\,\mathrm{d}V \\ &\quad + \int_{M^n}\Big(\nabla_i v_{ij} - \frac{\nabla_i tr(v)}{2}\Big)\nabla_j f\cdot f\mathrm{e}^{-\varphi}\,\mathrm{d}V.\end{aligned} \tag{3.1}$$

Remark 3.1. Since (3.1) does not depend on the evolution equation of f, $\mathrm{d}\lambda(t)/\mathrm{d}t=\mathrm{d}\lambda(f,t)/\mathrm{d}t$.

Now we are ready to prove the evolution equation under the Yamabe flow for the first eigenvalue of $-\Delta_\varphi+cR^a$ $(0<a<1)$.

Theorem 3.1. *Let $g(t)$, $t\in[0,T)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi+cR^a$ $(a\neq 1)$, $f(x,t)>0$ satisfies $\int_{M^n} f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V=0$ and*

$$-\Delta_\varphi f(x,t)+cR^a f(x,t)=\lambda(t)f(x,t). \tag{3.2}$$

Then along the Yamabe flow, $\lambda(t)$ evolves by

$$
\begin{aligned}
(3.3)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) = {} & \lambda(t)\int_{M^n}\Big[\frac{n}{2}R - 2c(n-1)R^a\Big]f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + \int_{M^n}\Big[2c(n-1)R^a - \frac{n-2a}{2}R - (n-1)\Delta\varphi\Big]cR^af^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + \int_{M^n}\Big[c(n-1)R^{a-1} - \frac{n-2}{2}\Big]R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + a(1-a)c(n-1)\int_{M^n}R^{a-2}|\nabla R|^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + c(n-1)\int_{M^n}R^a|\nabla f - f\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V.
\end{aligned}
$$

Proof. Note that the scalar curvature R evolves under the Yamabe flow by

$$
(3.4)\qquad \frac{\partial R}{\partial t} = (n-1)\Delta R + R^2.
$$

Substituting $v_{ij} = -Rg_{ij}$ and (3.4) into (3.1), we obtain that

$$
\begin{aligned}
(3.5)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) = {} & \int_{M^n}\Big(-R\Delta_\varphi f + acR^{a-1}f\frac{\partial R}{\partial t}\Big)f\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + \frac{n-2}{2}\int_{M^n}f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
= {} & -\int_{M^n}Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + ac(n-1)\int_{M^n}f^2R^{a-1}\Delta R\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + ac\int_{M^n}R^{a+1}f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{n-2}{2}\int_{M^n}f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
(3.6)\qquad \int_{M^n}f^2R^{a-1}\Delta R\mathrm{e}^{-\varphi}\,\mathrm{d}V = {} & -2\int_{M^n}fR^{a-1}\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + (1-a)\int_{M^n}R^{a-2}|\nabla R|^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& + \int_{M^n}f^2R^{a-1}\nabla_{\nabla\varphi}R\mathrm{e}^{-\varphi}\,\mathrm{d}V
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{M^n}fR^{a-1}\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V = {} & (1-a)\int_{M^n}fR^{a-1}\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& - \int_{M^n}R^af\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V \\
& - \int_{M^n}R^a|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V,
\end{aligned}
$$

that is

$$\int_{M^n} fR^{a-1}\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V = -\frac{1}{a}\int_{M^n} R^a f\Delta_\varphi f\mathrm{e}^{-\varphi}dV - \frac{1}{a}\int_{M^n} R^a|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V. \tag{3.7}$$

In particular, we get

$$\int_{M^n} f\nabla_{\nabla f}R\mathrm{e}^{-\varphi}\,\mathrm{d}V = -\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V - \int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V. \tag{3.8}$$

Moreover, we have

$$\begin{aligned}
&\int_{M^n} f^2R^{a-1}\nabla_{\nabla\varphi}R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad = -\int_{M^n} f^2R^a\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V + (1-a)\int_{M^n} f^2R^{a-1}\nabla_{\nabla\varphi}R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad - 2\int_{M^n} fR^a\nabla_{\nabla\varphi}f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \int_{M^n} f^2R^a|\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V,
\end{aligned}$$

that is

$$\begin{aligned}
&\int_{M^n} f^2R^{a-1}\nabla_{\nabla\varphi}R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad = -\frac{1}{a}\int_{M^n} f^2R^a\Delta\varphi\mathrm{e}^{-\varphi}dV - \frac{2}{a}\int_{M^n} fR^a\nabla_{\nabla\varphi}f\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad + \frac{1}{a}\int_{M^n} f^2R^a|\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V.
\end{aligned}\tag{3.9}$$

Applying (3.7) and (3.9) to (3.6), we obtain

$$\begin{aligned}
&\int_{M^n} f^2R^{a-1}\Delta R\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad = \frac{2}{a}\int_{M^n} R^a f\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{2}{a}\int_{M^n} R^a|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad + (1-a)\int_{M^n} R^{a-2}|\nabla R|^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V - \frac{1}{a}\int_{M^n} f^2R^a\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad - \frac{2}{a}\int_{M^n} fR^a\nabla_{\nabla\varphi}f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{1}{a}\int_{M^n} f^2R^a|\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad = \frac{2}{a}\int_{M^n} R^a f\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{1}{a}\int_{M^n} R^a|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad + (1-a)\int_{M^n} R^{a-2}|\nabla R|^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V - \frac{1}{a}\int_{M^n} f^2R^a\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&\qquad\quad + \frac{1}{a}\int_{M^n} R^a|\nabla f - f\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V.
\end{aligned}\tag{3.10}$$

Plugging (3.8) and (3.10) into (3.5), we obtain

$$
\begin{aligned}
(3.11)\quad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) = &-\int_{M^n} Rf\Delta_\varphi f \mathrm{e}^{-\varphi}\,\mathrm{d}V + 2c(n-1)\int_{M^n} R^a f\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+ c(n-1)\int_{M^n} R^a|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+ a(1-a)c(n-1)\int_{M^n} R^{a-2}|\nabla R|^2 f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&- c(n-1)\int_{M^n} f^2R^a\Delta\varphi\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+ c(n-1)\int_{M^n} R^a|\nabla f - f\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V + ac\int_{M^n} R^{a+1}f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&- \frac{n-2}{2}\int_{M^n} Rf\Delta_\varphi f\mathrm{e}^{-\varphi}\,\mathrm{d}V - \frac{n-2}{2}\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V.
\end{aligned}
$$

Now (3.3) follows immediately by applying (3.2) to (3.11) and rearranging. □

As we obtained Theorem 3.1, we are ready to finish the proof of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. Since $0 < a < 1$, $c \geqslant \frac{1}{2}(n-2)/(n-1)$ and (2.13), it is easy to verify from condition (1.5) that

$$(3.12)\qquad R_{g(t)} \geqslant 0,$$

$$(3.13)\qquad c(n-1)R^{a-1}_{g(t)} - \frac{n-2}{2} \geqslant 0,$$

$$(3.14)\qquad 2c(n-1)R^a_{g(t)} - \frac{n-2a}{2}R - (n-1)\Delta\varphi \geqslant 0.$$

Therefore, we can derive the following inequality from Theorem 3.1, (2.11) and (2.12).

$$
\begin{aligned}
(3.15)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) &\geqslant \lambda(t)\int_{M^n}\Big[\frac{n}{2}R_{g(t)} - 2c(n-1)R^a_{g(t)}\Big]f^2(t)\,\mathrm{d}V\\
&\geqslant \Big[\frac{n}{2}\sigma(t) - 2c(n-1)\varrho^a(t)\Big]\lambda(t).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
(3.16)\qquad &\frac{\mathrm{d}}{\mathrm{dt}}\exp\Big\{\int_0^t\Big[2c(n-1)\varrho^a(\tau) - \frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\lambda(t)\\
&\quad = \exp\Big\{\int_0^t\Big[2c(n-1)\varrho^a(\tau) - \frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\\
&\qquad \times\Big\{\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) - \Big[\frac{n}{2}\sigma(t) - 2c(n-1)\varrho^a(t)\Big]\lambda(t)\Big\} \geqslant 0.
\end{aligned}
$$

This completes the proof of this theorem. □

P r o o f of Theorem 1.4. Since $0 < a < 1$, $c > 0$ and (2.13) holds, it is easy to verify from condition (1.6) that the inequalities (3.12) to (3.14) still apply. We conclude this theorem by the rest arguments as in the proof of Theorem 1.3. $\square$

4. Results under the normalized Yamabe flow

In this section, we consider an evolution equation of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{4}(n-2)/(n-1)$) and $-\Delta_\varphi + cR^a$ ($0 < a < 1$ and $c > 0$) under the normalized Yamabe flow of

$$\frac{\partial}{\partial t} g_{ij} = -(R-r)g_{ij}, \tag{4.1}$$

where $r = \int_{M^n} R\,\mathrm{d}V / \int_{M^n} \mathrm{d}V$ is the average scalar curvature.

Note that the evolution formula for the scalar curvature (see, e.g., [3]) is

$$\frac{\partial}{\partial t} R = (n-1)\Delta R + R(R-r). \tag{4.2}$$

By similar arguments as in the proof of Theorem 2.1, we can obtain the following result.

Theorem 4.1. *Let $g(t)$, $t \in [0,\infty)$, be a solution of the normalized Yamabe flow on a closed Riemannian manifold M^n. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR$, $f(x,t) > 0$ satisfies $\int_{M^n} f^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V = 0$ and*

$$-\Delta_\varphi f(x,t) + cRf(x,t) = \lambda(t) f(x,t). \tag{4.3}$$

Then along the normalized Yamabe flow, $\lambda(t)$ evolves by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) &= (n-1)c \int_{M^n} Rf^2 \Big[\Big(2c - \frac{n-2}{2n-2}\Big)R - \Delta\varphi\Big] \mathrm{e}^{-\varphi}\,\mathrm{d}V \\ &\quad + \Big[(n-1)c - \frac{n-2}{2}\Big] \int_{M^n} R|\nabla f|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V \\ &\quad + (n-1)c \int_{M^n} R|f\nabla\varphi - \nabla f|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V \\ &\quad - \Big[2(n-1)c - \frac{n}{2}\Big]\lambda(t) \int_{M^n} Rf^2 \mathrm{e}^{-\varphi}\,\mathrm{d}V - r\lambda(t). \end{aligned} \tag{4.4}$$

As we obtained (4.4), the similar argument as in the proof of Theorem 1.1 implies:

Theorem 4.2. *Let $g(t)$, $t \in [0,\infty)$, be a solution of the Yamabe flow on a closed Riemannian manifold M^n with $R_{g(t)} \geqslant \max\{2((n-1)/(n-2))\Delta\varphi, 0\}$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR$ ($c \geqslant \frac{1}{2}(n-2)/(n-1)$), then the quantity*

$$\exp\biggl\{\int_0^t \Bigl[r + 2(n-1)c\varrho(\tau) - \frac{n}{2}\sigma(\tau)\Bigr]\,\mathrm{d}\tau\biggr\}\lambda(t)$$

is nondecreasing along the normalized Yamabe flow.

In particular, we can show a monotonicity quantity associated to the first eigenvalue of $-\Delta_\varphi + \frac{1}{4}(n/(n-1))R$.

Theorem 4.3. *Let $g(t)$, $t \in [0,\infty)$, be a solution of the normalized Yamabe flow on a closed Riemannian manifold with $n \in \{2,3,4\}$ and $R_{g(t)} \geqslant \max\{(n-1)\Delta\varphi, 0\}$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + \frac{1}{4}(n/(n-1))R$, then $\mathrm{e}^{rt}\lambda(t)$ is nondecreasing along the normalized Yamabe flow.*

P r o o f. Substituting $c = \frac{1}{4}(n/(n-1))$ into (4.4), we have

$$\begin{aligned}(4.5)\quad \frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = \frac{n}{4}\int_{M^n} Rf^2\Bigl[\Bigl(\frac{1}{n-1}\Bigr)R - \Delta\varphi\Bigr]\mathrm{e}^{-\varphi}\,\mathrm{d}V + \frac{4-n}{4}\int_{M^n} R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V \\ + \frac{n}{4}\int_{M^n} R|f\nabla\varphi - \nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V - r\lambda(t) \geqslant -r\lambda(t)\end{aligned}$$

if $n \in \{2,3,4\}$, $R \geqslant \max\{(n-1)\Delta\varphi, 0\}$.

Therefore, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{rt}\lambda(t)) = \mathrm{e}^{rt}\Bigl(\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) + r\lambda(t)\Bigr) \geqslant 0.$$

This completes the proof. $\square$

Consider an evolution equation of $-\Delta_\varphi + cR^a$ ($0 < a < 1$ and $c > 0$) under the normalized Yamabe flow. By similar arguments as in the proof of Theorem 3.1, we obtain the following evolution equation.

Theorem 4.4. *Let $g(t)$, $t \in [0,\infty)$, be a solution of the normalized Yamabe flow on a closed Riemannian manifold M^n. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi + cR^a$ ($a \neq 1$), $f(x,t) > 0$ satisfies $\int_{M^n} f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V = 0$ and*

$$(4.6)\qquad -\Delta_\varphi f(x,t) + cR^a f(x,t) = \lambda(t)f(x,t).$$

Then along the Yamabe flow, $\lambda(t)$ evolves by

$$
\begin{aligned}
(4.7)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t) = \lambda(t)&\int_{M^n}\Big[\frac{n}{2}R-2c(n-1)R^a\Big]f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+\int_{M^n}\Big[2c(n-1)R^a-\frac{n-2a}{2}R-(n-1)\Delta\varphi\Big]cR^af^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+\int_{M^n}\Big[c(n-1)R^{a-1}-\frac{n-2}{2}\Big]R|\nabla f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+a(1-a)c(n-1)\int_{M^n}R^{a-2}|\nabla R|^2f^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+c(n-1)\int_{M^n}R^a|\nabla f-f\nabla\varphi|^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\\
&+(1-a)cr\int_{M^n}R^af^2\mathrm{e}^{-\varphi}\,\mathrm{d}V-r\lambda(t).
\end{aligned}
$$

Similarly, we can derive a monotonic quantity on the first eigenvalue of $-\Delta_\varphi+cR^a$ with $0<a<1$ and $c>0$.

Theorem 4.5. *Let $g(t)$, $t\in[0,\infty)$, be a solution of the normalized Yamabe flow on a closed Riemannian manifold M^n with $R_{g(0)}\geqslant 0$. Assume that $\lambda(t)$ is the first eigenvalue of $-\Delta_\varphi+cR^a$ with $0<a<1$ and $c>0$. If*

$$(4.8)\qquad \max\Big\{\frac{\Delta\varphi}{c},0\Big\}\leqslant R^a_{g(t)}\leqslant\Big(\frac{2c(n-1)}{n-2a}\Big)^{a/(1-a)}$$

on $[0,T_2]$ for some $T_2\in(0,\infty)$, then the quantity

$$\exp\Big\{\int_0^t\Big[r+2c(n-1)\varrho^a(\tau)-\frac{n}{2}\sigma(\tau)\Big]\,\mathrm{d}\tau\Big\}\lambda(t)$$

is nondecreasing along the normalized Yamabe flow on $(0,T_2)$.

P r o o f. From $0<a<1$, $c>0$ and $R^a\geqslant 0$, we know that

$$(4.9)\qquad (1-a)cr\int_{M^n}R^af^2\mathrm{e}^{-\varphi}\,\mathrm{d}V\geqslant 0.$$

Since $0<a<1$ and $c>0$ and (2.13) applies, it is easy to verify from condition (4.8) that (3.12) to (3.14) still hold.

Applying (2.11), (2.12), (3.12), (3.13), (3.14) and (4.9) to Theorem 4.4, we get

$$
\begin{aligned}
(4.10)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\lambda(t)&\geqslant\lambda(t)\int_{M^n}\Big[\frac{n}{2}R_{g(t)}-2c(n-1)R^a_{g(t)}\Big]f^2(t)\,\mathrm{d}V-r\lambda(t)\\
&\geqslant\Big[\frac{n}{2}\sigma(t)-2c(n-1)\varrho^a(t)-r\Big]\lambda(t).
\end{aligned}
$$

Moreover, we have

$$\begin{aligned}(4.11)\qquad \frac{\mathrm{d}}{\mathrm{dt}}\exp\biggl\{\int_0^t\Bigl[r+2c(n-1)\varrho^a(\tau)-\frac{n}{2}\sigma(\tau)\Bigr]\,\mathrm{d}\tau\biggr\}\lambda(t)\\ = \exp\biggl\{\int_0^t\Bigl[r+2c(n-1)\varrho^a(\tau)-\frac{n}{2}\sigma(\tau)\Bigr]\,\mathrm{d}\tau\biggr\}\\ \times\Bigl\{\frac{\mathrm{d}}{\mathrm{dt}}\lambda(t)+\Bigl[r+2c(n-1)\varrho^a(t)-\frac{n}{2}\sigma(t)\Bigr]\lambda(t)\Bigr\}\geqslant 0.\end{aligned}$$

This completes the proof. □

References

[1] *X. Cao*: Eigenvalues of $(-\Delta+\frac{R}{2})$ on manifolds with nonnegative curvature operator. Math. Ann. *337* (2007), 435–441. zbl MR doi
[2] *X. Cao*: First eigenvalues of geometric operators under the Ricci flow. Proc. Am. Math. Soc. *136* (2008), 4075–4078. zbl MR doi
[3] *B. Chow, P. Lu, L. Ni*: Hamilton's Ricci Flow. Graduate Studies in Mathematics 77. American Mathematical Society, Providence, 2006. zbl MR doi
[4] *S. Fang, H. Xu, P. Zhu*: Evolution and monotonicity of eigenvalues under the Ricci flow. Sci. China, Math. *58* (2015), 1737–1744. zbl MR doi
[5] *S. Fang, F. Yang*: First eigenvalues of geometric operators under the Yamabe flow. Bull. Korean Math. Soc. *53* (2016), 1113–1122. zbl MR doi
[6] *S. Fang, F. Yang, P. Zhu*: Eigenvalues of geometric operators related to the Witten Laplacian under the Ricci flow. Glasg. Math. J. *59* (2017), 743–751. zbl MR doi
[7] *H. Guo, R. Philipowski, A. Thalmaier*: Entropy and lowest eigenvalue on evolving manifolds. Pac. J. Math. *264* (2013), 61–81. zbl MR doi
[8] *P. T. Ho*: First eigenvalues of geometric operators under the Yamabe flow. Ann. Glob. Anal. Geom. *54* (2018), 449–472. zbl MR doi
[9] *B. Kleiner, J. Lott*: Notes on Perelman's papers. Geom. Topol. *12* (2008), 2587–2858. zbl MR doi
[10] *G. Perelman*: The entropy formula for the Ricci flow and its geometric applications. Available at `https://arxiv.org/abs/math/0211159` (2002), 39 pages.
[11] *M. Reed, B. Simon*: Methods of Modern Mathematical Physics. IV: Analysis of Operators. Academic Press, New York, 1978. zbl MR
[12] *P. Topping*: Lectures on the Ricci Flow. London Mathematical Society Lecture Note Series 325. Cambridge University Press, Cambridge, 2006. zbl MR doi
[13] *L. Zhao*: The first eigenvalue of the Laplace operator under Yamabe flow. Math. Appl. *24* (2011), 274–278. MR
[14] *L. Zhao*: The first eigenvalue of the p-Laplace operator under powers of the mth mean curvature flow. Result. Math. *63* (2013), 937–948. zbl MR doi
[15] *L. Zhao*: The first eigenvalue of the p-Laplace operator under powers of mean curvature flow. Math. Methods Appl. Sci. *37* (2014), 744–751. zbl MR doi

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