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RIGIDITY OF THE HOLOMORPHIC AUTOMORPHISM OF THE GENERALIZED FOCK-BARGMANN-HARTOGS DOMAINS

Ting Guo, Qingdao, Zhiming Feng, Leshan, Enchao Bi, Qingdao

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Abstract. We study a class of typical Hartogs domains which is called a generalized Fock-Bargmann-Hartogs domain $D^p_{n,m}(\mu)$. The generalized Fock-Bargmann-Hartogs domain is defined by inequality $\mathrm{e}^{\mu\|z\|^2}\sum\limits_{j=1}^{m}|\omega_j|^{2p} < 1$, where $(z,\omega)\in\mathbb{C}^n\times\mathbb{C}^m$. In this paper, we will establish a rigidity of its holomorphic automorphism group. Our results imply that a holomorphic self-mapping of the generalized Fock-Bargmann-Hartogs domain $D^p_{n,m}(\mu)$ becomes a holomorphic automorphism if and only if it keeps the function $\sum\limits_{j=1}^{m}|\omega_j|^{2p}\mathrm{e}^{\mu\|z\|^2}$ invariant.

Keywords: generalized Fock-Bargmann-Hartogs domain; holomorphic automorphism group

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1. Introduction

Let Ω be a domain of $\mathbb{C}^n$ and let $\mathrm{Aut}(\Omega)$ be the set of all of the biholomorphic self-mappings of Ω. Obviously, $\mathrm{Aut}(\Omega)$ forms a group under the composition law. We call $\mathrm{Aut}(\Omega)$ the holomorphic automorphism group of Ω.

In one complex variable, the only case of the holomorphic automorphism group we need to study is the unit disk by the well-known Riemann mapping theorem. However, the classical Riemann mapping theorem is no longer valid for higher dimensional cases, which leads to more complicated discussions on determining the

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holomorphic automorphism group of Ω in $\mathbb{C}^n$. Although there are many difficulties in studying automorphism groups of domains in $\mathbb{C}^n$, great progress has been made in this field. In fact, the automorphism groups of various bounded domains had been deeply investigated in the past twenty years. In 1997, Dini-Primicerio in [4] gave the explicit form for the automorphism group of complex ellipsoid. Later on, Ahn-Byun-Park in [1] studied the automorphism group of a class Hartogs type domains over classical symmetric domains, namely Cartan-Hartogs domain. For more references, please refer to Ishi-Kai, see [5], Kodama, see [7], Tu, see [8] and references therein. However, there are still a lot of automorphism groups of domains in $\mathbb{C}^n$ (whether bounded or not) that are undetermined. So it is interesting to determine the automorphism groups of various domain explicitly or give the equivalent descriptions for the holomorphic automorphism group.

We notice that both complex ellipsoid and Cartan-Hartogs domain can be regarded as Hartogs domains. In general, Hartogs domains are nonhomogeneous domains, but many properties are similar to the homogeneous domains. Hence, it is much more convenient to discuss the automorphism group of Hartogs domains. Another typical example of Hartogs domains is called Fock-Bargmann-Hartogs domains which has been studied by researchers from the perspective of geometry and analysis.

For a given positive real number $\mu > 0$, the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ is defined by

$$D_{n,m}(\mu) = \{(z,\omega) \in \mathbb{C}^n \times \mathbb{C}^m\colon\ \|\omega\|^2 < \mathrm{e}^{-\mu\|z\|^2}\} \quad (\mu > 0),$$

where $\|\cdot\|$ is the standard Hermitian norm. Obviously, the Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ are strongly pseudoconvex domains in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary. Moreover, each Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ is an unbounded nonhyperbolic domain in the sense of Kobayashi in $\mathbb{C}^n \times \mathbb{C}^m$.

By verifying that the Bergman kernel ensures revised Cartan's theorem, Kim-Ninh-Yamamori in [6] determined the automorphism group of Fock-Bargmann-Hartogs domains. In 2015, Tu-Wang in [9] studied the rigidity of proper holomorphic mappings between two equidimensional Fock-Bargmann-Hartogs domains and proved that any proper holomorphic self-mapping on the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ with $m \geqslant 2$ must be an automorphism. In 2016, Bi-Feng-Tu in [2] proved the existence of balanced metric on the Fock-Bargmann-Hartogs domains.

In this paper, we will consider a class of more widely Hartogs domain called generalized Fock-Bargmann-Hartogs domains which is defined by

$$D^p_{n,m}(\mu) = \left\{(z,\omega) \in \mathbb{C}^n \times \mathbb{C}^m\colon\ \sum_{j=1}^m |\omega_j|^{2p} < \mathrm{e}^{-\mu\|z\|^2}\right\} \quad (\mu > 0,\ p \in \mathbb{N}^+).$$

Obviously, each generalized Fock-Bargmann-Hartogs domain $D^p_{n,m}(\mu)$ is also an unbounded nonhyperbolic domain. In general, a generalized Fock-Bargmann-Hartogs domain is not a strongly pseudoconvex domain. When $p = 1$, this domain degenerates into Fock-Bargmann-Hartogs domain. Hence, this can be regarded as a natural generalization.

Recently, Bi-Tu in [3] has established the rigidity of the proper holomorphic mappings between two equidimensional generalized Fock-Bargmann-Hartogs domains and determined the holomorphic automorphism group of the generalized Fock-Bargmann-Hartogs domain. Inspired by this, in our paper we will give an equivalent description for the holomorphic automorphism group of the generalized Fock-Bargmann-Hartogs domain. More precisely, our main result is as follows.

Theorem 1.1. *For $p \in \mathbb{N}^+$, let F be a holomorphic self-mapping of the generalized Fock-Bargmann-Hartogs domains $D^p_{n,m}(\mu)$. Then F is an automorphism of $D^p_{n,m}(\mu)$ if and only if F keeps the function*

$$L(z,\omega) = \mathrm{e}^{\mu\|z\|^2} \sum_{j=1}^{m} |\omega_j|^{2p}, \quad (z,\omega) \in D^p_{n,m}(\mu)$$

invariant.

Therefore we easily have the following results.

Corollary 1.2. *Let F be a holomorphic self-mapping of Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$. Then F is an automorphism of $D_{n,m}(\mu)$ if and only if F keeps the function*

$$L(z,\omega) = \mathrm{e}^{\mu\|z\|^2} \|\omega\|^2, \quad (z,\omega) \in D_{n,m}(\mu)$$

invariant.

Remark 1.3. We remark that our results are not trivial. For example, when $\mu = 0$ and $m = 1$, the generalized Fock-Bargmann-Hartogs domain $D^p_{n,m}(\mu)$ and the function $L(z,\omega)$ in Theorem 1.1 reduce to the domain D and the function $M(z,\omega)$ given by

$$D = \{(z,\omega) \in \mathbb{C}^n \times \mathbb{C} \colon \ |\omega| < 1\} \quad \text{and} \quad M(z,\omega) = |\omega|^{2p}, \quad (z,\omega) \in D.$$

Hence, if we consider, for instance, a holomorphic mapping $F \colon D \to \mathbb{C}^{n+1}$ defined by

$$F(z,\omega) = \left(z, \frac{2\omega - 1}{2 - \omega}\right), \quad (z,\omega) \in D,$$

then F induces a holomorphic automorphism of D, while it is obvious that $M(F(z,\omega)) \neq M(z,\omega)$ on D.

2. Preliminaries

In order to prove our results, we firstly give some lemmas.

Lemma 2.1. *For $p > 0$ and $p \neq 1$ let $f\colon D^p_{n,m}(\mu) \to D^p_{n,m}(\mu)$ be a biholomorphic mapping with $f(0) = 0$. Then we have $f(z,\omega) = (zA, \omega U)$. Here $A \in U(n)$, $\theta_i \in \mathbb{R}$, $\sigma \in S_m$ is a permutation $(1 \leqslant j \leqslant m)$ and*

$$\omega U = (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)}) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\mathrm{i}\theta_m} \end{pmatrix}.$$

Proof. This is a direct result of Corollary 1.5 in [3]. □

Lemma 2.2 (Bi-Tu [3]). *Let $p > 0$ and $p \neq 1$. The automorphism group $\mathrm{Aut}(D^p_{n,m}(\mu))$ is generated by the following mappings:*

$$\begin{aligned} \varphi_A\colon\ & (z,\omega) \mapsto (zA, \omega), \\ \varphi_D\colon\ & (z,\omega) \mapsto (z, \omega U), \\ \varphi_v\colon\ & (z,\omega) \mapsto (z+v, \omega \mathrm{e}^{(-2\mu\langle z,v\rangle - \mu\|v\|^2)/2p}), \end{aligned}$$

where $v \in C^n$, $A \in U(n)$, $\sigma \in S_m$ is a permutation and

$$\omega U = (\omega_{\sigma(1)}, \ldots, \omega_{\sigma(m)}) \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta_1} & & \\ & \ddots & \\ & & \mathrm{e}^{\mathrm{i}\theta_m} \end{pmatrix}.$$

Lemma 2.3. *For $p, N \in \mathbb{N}^+$ and $p > 1$, let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_m)$ be tuples of nonnegative integers. For $z \in \mathbb{C}^n$, $\omega \in \mathbb{C}^m$, set*

$$P_N(z) = \sum_{|\alpha|=N} d_\alpha z^\alpha, \quad Q_N(z,\omega) = \sum_{\substack{|\alpha|+|\beta|=N \\ |\alpha|\geqslant 1, |\beta|\geqslant 1}} e_{\alpha\beta} z^\alpha \omega^\beta,$$
$$R_N(z,\omega) = \sum_{\substack{|\alpha|+|\beta|=N \\ |\alpha|\geqslant 1, |\beta|\geqslant 1}} f_{\alpha\beta} z^\alpha \omega^\beta,$$

where d_α and $e_{\alpha\beta}$ are n-dimensional row vectors, $f_{\alpha\beta}$ is an m-dimensional row vector, $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$. Assume that A is an invertible matrix of order n, $\langle\cdot,\cdot\rangle$ denotes the *standard Hermitian inner product on $\mathbb{C}^n$ or $\mathbb{C}^m$, $P_{N,j}$, $Q_{N,j}$ and $R_{N,j}$ denote the jth component of P_N, Q_N and R_N, respectively.*

(i) *If*

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2\,\mathrm{Re}(\overline{\omega_j} R_{2,j}(z,\omega)) \equiv 0 \tag{2.1}$$

and

$$\begin{aligned} &\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} (2\,\mathrm{Re}(\overline{\omega_j} R_{3,j}(z,\omega)) + |R_{2,j}(z,\omega)|^2) \\ &\quad + \sum_{j=1}^{m} \binom{p}{2} |\omega_j|^{2(p-2)} (2\,\mathrm{Re}(\overline{\omega_j} R_{2,j}(z,\omega)))^2 \\ &\quad + \mu \sum_{j=1}^{m} |\omega_j|^{2p} (\|zA\|^2 - \|z\|^2) \equiv 0, \end{aligned} \tag{2.2}$$

then

$$R_2(z,\omega) \equiv 0, \quad R_3(z,\omega) \equiv 0, \quad \|z\|^2 \equiv \|zA\|^2.$$

(ii) *Suppose that $N \geqslant 4$ and*

$$\begin{aligned} &\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2\,\mathrm{Re}(\overline{\omega_j} R_{N,j}(z,\omega)) \\ &\quad + \mu \sum_{j=1}^{m} |\omega_j|^{2p} 2\,\mathrm{Re}\langle zA, P_{N-2}(z) + Q_{N-2}(z,\omega)\rangle \equiv 0. \end{aligned} \tag{2.3}$$

Then

$$R_N(z,\omega) \equiv 0, \quad P_{N-2}(z) \equiv 0, \quad Q_{N-2}(z,\omega) \equiv 0.$$

P r o o f. Let $zA = \sum\limits_{k=1}^{n} E_k z_k$, where $\{E_k\colon 1 \leqslant k \leqslant n\}$ is the bases of $\mathbb{C}^n$.

(i) For any arbitrarily fixed ω, let us put

$$g(z) = \sum_{j=1}^{m} \omega_j^{p-1} \overline{\omega}_j^{p} R_{N,j}(z,\omega).$$

If

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2\,\mathrm{Re}(\overline{\omega_j} R_{N,j}(z,\omega)) \equiv 0,$$

then

$$\operatorname{Re}(g(z)) \equiv 0.$$

Accordingly we obtain that

$$\sum_{j=1}^{m} \omega_j^{p-1} \overline{\omega}_j^{p} R_{N,j}(z,\omega) \equiv 0$$

for all $z \in \mathbb{C}^n$ and all $\omega \in \mathbb{C}^m$. Thus, by applying the differential operators $\partial^p/\partial\overline{\omega_j}^p$ $(1 \leqslant j \leqslant m)$ to both sides of the above equation, it follows that

$$\omega_j^{p-1} R_{N,j}(z,\omega) \equiv 0, \quad 1 \leqslant j \leqslant m,$$

which implies

$$R_{N,j}(z,\omega) \equiv 0, \quad 1 \leqslant j \leqslant m.$$

Hence

$$R_N(z,\omega) \equiv 0.$$

In particular, we conclude by (2.1) that $R_2(z,\omega) = 0$.

Since $R_2(z,\omega) = 0$, then (2.2) can be rewritten as

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2 \operatorname{Re}\left(\overline{\omega_j}\left(\sum_{\substack{|\alpha|+|\beta|=3\\ |\alpha|\geqslant 1, |\beta|\geqslant 1}} f_{\alpha\beta} z^\alpha \omega^\beta\right)_j\right) + \mu \sum_{j=1}^{m} |\omega_j|^{2p}(\|zA\|^2 - \|z\|^2) \equiv 0. \tag{2.4}$$

Take z, $\overline{z}$ as independent variables. Since the sets $\{z^\alpha, \overline{z^\alpha}\colon 1 \leqslant |\alpha| \leqslant 2\}$ and $\{z_i\overline{z_j}\colon 1 \leqslant i,\ j \leqslant n\}$ are disjoint, we have

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2 \operatorname{Re}(\overline{\omega_j} R_{3,j}(z,\omega)) \equiv 0$$

and

$$\|z\|^2 \equiv \|zA\|^2 \tag{2.5}$$

by (2.4).

By formula (2.5), we can see that A is an n-dimensional unitary matrix. Similarly to the above discussion, we can get

$$R_3(z,\omega) \equiv 0.$$

(ii) It is easy to show

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2\operatorname{Re}(\overline{\omega_j} R_{N,j}(z,\omega)) = \sum_{j=1}^{m} p|\omega_j|^{2(p-1)} 2\operatorname{Re}\left(\overline{\omega_j}\left(\sum_{\substack{|\alpha|+|\beta|=N\\ |\alpha|\geqslant 1, |\beta|\geqslant 1}} f_{\alpha\beta} z^\alpha \omega^\beta\right)_j\right)$$

and

$$\begin{aligned} 2\sum_{j=1}^{m} |\omega_j|^{2p} \operatorname{Re}\langle zA, P_{N-2}(z) + Q_{N-2}(z,\omega)\rangle &= \sum_{j=1}^{m} |\omega_j|^{2p} \sum_{1\leqslant k\leqslant n} \sum_{|\alpha|=N-2} \{\langle E_k, d_\alpha\rangle z_k \overline{z^\alpha} + \langle d_\alpha, E_k\rangle \overline{z_k} z^\alpha\} \\ &\quad + \sum_{j=1}^{m} |\omega_j|^{2p} \sum_{1\leqslant k\leqslant n} \sum_{\substack{|\alpha|+|\beta|=N-2\\ |\alpha|\geqslant 1, |\beta|\geqslant 1}} \{\langle E_k, e_{\alpha\beta}\rangle z_k \overline{z^\alpha \omega^\beta} + \langle e_{\alpha\beta}, E_k\rangle z^\alpha \omega^\beta \overline{z_k}\}. \end{aligned}$$

Think of $z, \overline{z}$ as independent variables and $\omega, \overline{\omega}$ as constants. The sets $\{z^\alpha, \overline{z^\alpha}\colon 1 \leqslant |\alpha| \leqslant N-1\}$, $\{z_k\overline{z^\alpha}, \overline{z_k}z^\alpha\colon 1 \leqslant k \leqslant n,\ |\alpha| = N-2\}$ and $\{z_k\overline{z^\alpha}, \overline{z_k}z^\alpha\colon 1 \leqslant k \leqslant n,\ 1 \leqslant |\alpha| \leqslant N-3\}$ are pairwise disjoint. By (2.3), we have

$$\sum_{j=1}^{m} p|\omega_j|^{2(p-1)} (2\operatorname{Re}(\overline{\omega_j} R_{N,j}(z,\omega))) \equiv 0, \tag{2.6}$$

$$\langle E_k, d_\alpha\rangle = 0 \quad (1 \leqslant k \leqslant n,\ |\alpha| = N-2) \tag{2.7}$$

and

$$\sum_{|\beta|=N-2-|\alpha|} \langle E_k, e_{\alpha\beta}\rangle \overline{\omega}^\beta \equiv 0 \quad (1 \leqslant k \leqslant n,\ 1 \leqslant |\alpha| \leqslant N-1). \tag{2.8}$$

Therefore (2.6) implies

$$R_N(z,w) \equiv 0.$$

Since $\{E_k\colon 1 \leqslant k \leqslant n\}$ is a basis of $\mathbb{C}^n$, we must have

$$d_\alpha = 0 \Rightarrow P_{N-2}(z) \equiv 0$$

by (2.7).

Formula (2.8) yields that

$$\langle E_k, e_{\alpha\beta}\rangle = 0.$$

It follows that

$$e_{\alpha\beta} = 0 \Rightarrow Q_{N-2}(z,\omega) \equiv 0.$$

The proof is finished. □

3. Proof of the main theorem

P r o o f. Firstly, by Lemma 2.2 and Theorem 10 in [6], it is not hard to check that F keeps the function $L(z,\omega)$ invariant if F is an automorphism of generalized Fock-Bargmann-Hartogs domain.

Secondly, we will prove that the converse also holds. In this part, our proof is divided into four steps.

Step 1. Let

$$\begin{aligned} F(z,\omega) &= (F_1(z,\omega), F_2(z,\omega)) \\ &= (F_{1,1}(z,\omega),\dots,F_{1,n}(z,\omega),F_{2,1}(z,\omega),\dots,F_{2,m}(z,\omega)). \end{aligned}$$

Since F keeps the function $\mathrm{e}^{\mu\|z\|^2}\sum\limits_{j=1}^{m}|\omega_j|^{2p}$ invariant, we have

$$\sum_{j=1}^{m}|F_{2,j}(z,0)|^{2p}\mathrm{e}^{\mu\|F_1(z,0)\|^2} = 0.$$

Therefore

$$\sum_{j=1}^{m}|F_{2,j}(z,0)|^{2p} = 0.$$

Then we can see that $F_2(z,0) = 0$. It follows that $F(0,0) = (v,0)$. Consider a biholomorphic self-mapping of $D^p_{n,m}(\mu)$ which is defined by

$$G(z,\omega) = (z+v, \mathrm{e}^{-\mu\langle z,v\rangle/p-\mu\|v\|^2/2p}\omega).$$

Then $G(z,\omega)$ maps $(0,0)$ to $(v,0)$.

In fact, by Lemma 2.2, we know that G is an automorphism of $D^p_{n,m}(\mu)$. Let $H = G^{-1}\circ F$. Then we obtain that H keeps the function $\sum\limits_{j=1}^{m}|\omega_j|^{2p}\mathrm{e}^{\mu\|z\|^2}$ invariant and $H(0,0) = (0,0)$.

Let $H(0,\omega) = (h_1(\omega), h_2(\omega))$, in which

$$h_1(\omega) = \omega V + \sum_{i\geqslant 2} f_i(\omega), \quad h_2(\omega) = \omega U + \sum_{i\geqslant 2} g_i(\omega),$$

where $f_i(\omega)$ and $g_i(\omega)$ are homogeneous polynomial mappings of degree i. For any $(0,\omega) \in D^p_{n,m}(\mu)$ we get

$$\sum_{j=1}^m |\omega_j|^{2p} < 1.$$

For any $t \in [0,1]$ we can see

$$\sum_{j=1}^m |t\omega_j|^{2p} = t^{2p}\sum_{j=1}^m |\omega_j|^{2p} \leqslant \sum_{j=1}^m |\omega_j|^{2p} < 1.$$

Hence we conclude $(0, t\omega) \in D^p_{n,m}(\mu)$ for all $t \in [0,1]$.

Since H keeps the function $\mathrm{e}^{\mu\|z\|^2} \sum\limits_{j=1}^m |\omega_j|^{2p}$ invariant, we have

$$\sum_{j=1}^m |(h_2(t\omega))_j|^{2p} \mathrm{e}^{\mu\|h_1(t\omega)\|^2} = \sum_{j=1}^m |(t\omega)_j|^{2p}. \tag{3.1}$$

Substituting $h_1(t\omega)$ and $h_2(t\omega)$ into (3.1), we obtain

$$\sum_{j=1}^m \left|\left(t\omega U + \sum_{i\geqslant 2} g_i(t\omega)\right)_j\right|^{2p} \mathrm{e}^{\mu\|t\omega V + \sum_{i\geqslant 2} f_i(t\omega)\|^2} = \sum_{j=1}^m |(t\omega)_j|^{2p}.$$

Then we have

$$\sum_{j=1}^m \left|\left(\omega U + \sum_{i\geqslant 2} t^{i-1} g_i(\omega)\right)_j\right|^{2p} \mathrm{e}^{\mu\|t\omega V + \sum_{i\geqslant 2} f_i(t\omega)\|^2} = \sum_{j=1}^m |\omega_j|^{2p}.$$

Taking $t \to 0^+$, we get

$$\sum_{j=1}^m |(\omega U)_j|^{2p} = \sum_{j=1}^m |\omega_j|^{2p}. \tag{3.2}$$

Hence, the correspondence $\omega \mapsto \omega U$ gives rise to a holomorphic automorphism of complex ellipsoid $\left\{\omega \in \mathbb{C}^m \colon \sum\limits_{j=1}^m |\omega|^{2p} < 1\right\}$ and the self-mapping Φ of $\mathbb{C}^n \times \mathbb{C}^m$ defined by

$$\Phi(z,\omega) = (z, \omega U), \quad (z,\omega) \in \mathbb{C}^n \times \mathbb{C}^m,$$

induces a holomorphic automorphism of $D^p_{n,m}(\mu)$ by Lemma 2.1. Therefore, taking the composite mapping $H\circ\Phi^{-1}$ if necessary, we may assume that

$$H(0,\omega)=\Bigl(\omega V+\sum_{i\geqslant 2} f_i(\omega),\omega+\sum_{i\geqslant 2} g_i(\omega)\Bigr),$$

where $f_i(\omega)$ and $g_i(\omega)$ are homogeneous polynomial mappings of degree i.

On the other hand, $h_2(\omega)$ induces a holomorphic mapping on a bounded domain $\Bigl\{\omega\in\mathbb{C}^m\colon \sum\limits_{j=1}^{m}|\omega_j|^{2p}<1\Bigr\}$ satisfying $h_2(0)=0$ and $h_2'(0)=I$, where h_2' is the derivative of h_2 and I is the identity matrix. Hence, by Cartan lemma, we must have $h_2(\omega)=\omega$.

Consider $(0,\omega)\in D^p_{n,m}(\mu)$, then $H(0,\omega)=(h_1(\omega),\omega)$. Since H keeps the function $\sum\limits_{j=1}^{m}|\omega_j|^{2p}\mathrm{e}^{\mu\|z\|^2}$ invariant, it follows that

$$\sum_{j=1}^{m}|\omega_j|^{2p}\mathrm{e}^{\mu\|h_1(\omega)\|^2}=\sum_{j=1}^{m}|\omega_j|^{2p}. \tag{3.3}$$

Therefore by (3.3), we must have $\mathrm{e}^{\mu\|h_1(\omega)\|^2}=1$, which implies that $h_1(\omega)=0$.

Let $H(z,\omega)=(H_1(z,\omega),H_2(z,\omega))$. Then we have $H(0,0)=(0,0)$, $H_2(z,0)=0$ and $H(0,\omega)=(0,\omega)$ by the above discussions. Therefore we obtain

$$H_1(z,\omega)=zA+\sum_{i\geqslant 2}(P_i(z)+Q_i(z,\omega)),\quad H_2(z,\omega)=\omega+\sum_{i\geqslant 2}R_i(z,\omega),$$

where P_j, Q_j and R_j are homogeneous polynomial mappings of degree j, which are given by Lemma 2.3.

Step 2. Let $(z,\omega)\in D^p_{n,m}(\mu)$, which means $\sum\limits_{j=1}^{m}|\omega_j|^{2p}<\mathrm{e}^{-\mu\|z\|^2}$. It is not hard to see that

$$(tz,t\omega)\in D^p_{n,m}(\mu)\quad\forall\, t\in[0,1].$$

Since $H(tz,t\omega)=(H_1(tz,t\omega),H_2(tz,t\omega))$ and H keeps the function

$$\sum_{j=1}^{m}|\omega_j|^{2p}\mathrm{e}^{\mu\|z\|^2}$$

invariant, it follows

$$\sum_{j=1}^{m}|(H_2(tz,t\omega))_j|^{2p}\mathrm{e}^{\mu\|H_1(tz,t\omega)\|^2}=\sum_{j=1}^{m}|t\omega_j|^{2p}\mathrm{e}^{\mu\|tz\|^2},$$

which means

$$\frac{\sum_{j=1}^{m}\left|(H_2(tz,t\omega))_j\right|^{2p}}{\sum_{j=1}^{m}\left|t\omega_j\right|^{2p}}=\mathrm{e}^{\mu\|tz\|^2-\mu\|H_1(tz,t\omega)\|^2}. \tag{3.4}$$

Substituting $H_1(tz,t\omega)$ and $H_2(tz,t\omega)$ into (3.4), we obtain that for all $t\in[0,1]$

$$\frac{\sum_{j=1}^{m}\left|\left(t\omega+\sum_{i\geqslant 2}R_i(tz,t\omega)\right)_j\right|^{2p}}{\sum_{j=1}^{m}\left|t\omega_j\right|^{2p}}=\mathrm{e}^{\mu\|tz\|^2-\mu\|tzA+\sum_{i\geqslant 2}(P_i(tz)+Q_i(tz,t\omega))\|^2}. \tag{3.5}$$

Therefore we get

$$\begin{aligned}\frac{\sum_{j=1}^{m}\left|\omega_j+\left(\sum_{i\geqslant 2}t^{i-1}R_i(z,\omega)\right)_j\right|^{2p}}{\sum_{j=1}^{m}\left|\omega_j\right|^{2p}}&\\ =\mathrm{e}^{\mu t^2(\|z\|^2-\|zA+\sum_{i\geqslant 2}t^{i-1}(P_i(z)+Q_i(z,\omega))\|^2)}&.\end{aligned} \tag{3.6}$$

For the sake of simplicity, we write

$$H_{2,j}(t,z,\omega)=\left(\sum_{i\geqslant 2}t^{i-1}R_i(z,\omega)\right)_j\quad(1\leqslant j\leqslant m).$$

One can prove that

$$H_{2,j}(t,z,\omega)=t(R_{2,j}(z,\omega)+tR_{3,j}(z,\omega)+O(t^2)).$$

Then by (3.6) we obtain

$$\frac{\sum_{j=1}^{m}\left|\omega_j+H_{2,j}(t,z,\omega)\right|^{2p}}{\sum_{j=1}^{m}\left|\omega_j\right|^{2p}}=\mathrm{e}^{\mu t^2(\|z\|^2-\|zA+\sum_{i\geqslant 2}t^{i-1}(P_i(z)+Q_i(z,\omega))\|^2)}. \tag{3.7}$$

It follows

$$\begin{aligned}\frac{\sum_{j=1}^{m}\left|\left|\omega_j\right|^2+2\,\mathrm{Re}(\overline{\omega_j}H_{2,j}(t,z,\omega))+\left|H_{2,j}(t,z,\omega)\right|^2\right|^{p}}{\sum_{j=1}^{m}\left|\omega_j\right|^{2p}}&\\ =\mathrm{e}^{\mu t^2(\|z\|^2-\|zA+\sum_{i\geqslant 2}t^{i-1}(P_i(z)+Q_i(z,\omega))\|^2)}&.\end{aligned} \tag{3.8}$$

Let $T_j(t,z,\omega)=2\,\mathrm{Re}(\overline{\omega_j}H_{2,j}(t,z,\omega))+|H_{2,j}(t,z,\omega)|^2$. Then one can see that

$$T_j(t,z,\omega)=t(2\,\mathrm{Re}(\overline{\omega_j}R_{2,j}(z,\omega))+t(2\,\mathrm{Re}(\overline{\omega_j}R_{3,j}(z,\omega))+|R_{2,j}(z,\omega)|^2)+o(t^2)).$$

Owing to (3.8), we have

$$\ln\left(1+\frac{\sum_{j=1}^{m}\sum_{k=0}^{p-1}\binom{p}{k}|\omega_j|^{2k}(T_j(t,z,\omega))^{p-k}}{\sum_{j=1}^{m}|\omega_j|^{2p}}\right) = \mu t^2\left(\|z\|^2-\left\|zA+\sum_{i\geqslant 2}t^{i-1}(P_i(z)+Q_i(z,\omega))\right\|^2\right). \tag{3.9}$$

Direct computations imply that

$$1+\frac{\sum_{j=1}^{m}\sum_{k=0}^{p-1}\binom{p}{k}|\omega_j|^{2k}(T_j(t,z,\omega))^{p-k}}{\sum_{j=1}^{m}|\omega_j|^{2p}} = 1+tA(z,\omega)+t^2(B(z,\omega)+C(z,\omega))+o(t^2), \tag{3.10}$$

where

$$A(z,\omega)=\frac{\sum_{j=1}^{m}p|\omega_j|^{2(p-1)}2\operatorname{Re}(\overline{\omega_j}R_{2,j}(z,\omega))}{\sum_{j=1}^{m}|\omega_j|^{2p}},$$

$$B(z,\omega)=\frac{\sum_{j=1}^{m}p|\omega_j|^{2(p-1)}\big(2\operatorname{Re}(\overline{\omega_j}R_{3,j}(z,\omega))+|R_{2,j}(z,\omega)|^2\big)}{\sum_{j=1}^{m}|\omega_j|^{2p}}$$

$$C(z,\omega)=\frac{\sum_{j=1}^{m}\binom{p}{2}|\omega_j|^{2(p-2)}(2\operatorname{Re}(\overline{\omega_j}R_{2,j}(z,\omega)))^2}{\sum_{j=1}^{m}|\omega_j|^{2p}}.$$

Hence we get

$$\ln(1+tA(z,\omega)+t^2(B(z,\omega)+C(z,\omega))+o(t^2)) = \mu t^2\left(\|z\|^2-\left\|zA+\sum_{i\geqslant 2}t^{i-1}(P_i(z)+Q_i(z,\omega))\right\|^2\right).$$

Dividing the two sides of the above equation by t^2 and taking $t\to 0^+$, we obtain

$$\sum_{j=1}^{m}p|\omega_j|^{2(p-1)}2\operatorname{Re}(\overline{\omega_j}R_{2,j}(z,\omega))=0$$

and

$$\sum_{j=1}^{m}p|\omega_j|^{2(p-1)}\big(2\operatorname{Re}(\overline{\omega_j}R_{3,j}(z,\omega))+|R_{2,j}(z,\omega)|^2\big) + \sum_{j=1}^{m}\binom{p}{2}|\omega_j|^{2(p-2)}(2\operatorname{Re}(\overline{\omega_j}R_{2,j}(z,\omega)))^2+\mu\sum_{j=1}^{m}|\omega_j|^{2p}(\|zA\|^2-\|z\|^2)=0.$$

By Lemma 2.3, we have

$$R_2(z,\omega)=0,\quad R_3(z,\omega)=0,\quad \|z\|^2=\|zA\|^2. \tag{3.11}$$

Since $\|z\|^2=\|zA\|^2$, we get that A is a unitary matrix of order n.

Step 3. Now we would like to prove that $P_{i-2}(z)=0$, $Q_{i-2}(z,\omega)=0$ and $R_i(z,\omega)=0$ for all $(z,\omega)\in D^p_{n,m}(\mu)$ and all $i>3$. To this end, assuming not, we define a positive integer N by

$$N:=\min\{i\colon\ (P_{i-2}(z),\, Q_{i-2}(z,\omega),\, R_i(z,\omega))\neq(0,0,0)\}. \tag{3.12}$$

From (3.11) we know $N\geqslant 4$. Using (3.4) again, we obtain

$$\begin{aligned}\frac{\sum_{j=1}^m\left|\omega_j+\left(\sum_{i\geqslant N}t^{i-1}R_i(z,\omega)\right)_j\right|^{2p}}{\sum_{j=1}^m|\omega_j|^{2p}}&\\ &=\mathrm{e}^{\mu t^2\|z\|^2-\mu\|tzA+\sum_{i\geqslant N-2}(P_i(tz)+Q_i(tz,t\omega))\|^2}.\end{aligned} \tag{3.13}$$

Let

$$\begin{aligned}H'_{2,j}(t,z,\omega)&=\left(\sum_{i\geqslant N}t^{i-1}R_i(z,\omega)\right)_j,\\ T'_j(t,z,\omega)&=2\operatorname{Re}(\overline{\omega_j}H'_{2,j}(t,z,\omega))+|H'_{2,j}(t,z,\omega)|^2.\end{aligned}$$

By (3.13) we have

$$\begin{aligned}\ln\left(1+\frac{\sum_{j=1}^m\sum_{k=0}^{p-1}\binom{p}{k}|\omega_j|^{2k}(T'_j(t,z,\omega))^{p-k}}{\sum_{j=1}^m|\omega_j|^{2p}}\right)&\\ =\mu t^2\|z\|^2-\mu\left\|tzA+\sum_{i\geqslant N-2}(P_i(tz)+Q_i(tz,t\omega))\right\|^2&.\end{aligned} \tag{3.14}$$

Equation (3.14) yields that

$$\begin{aligned}\ln\left(1+\frac{t^{N-1}\sum_{j=1}^m p|\omega_j|^{2(p-1)}2\operatorname{Re}(\overline{\omega_j}R_{N,j}(z,\omega))+o(t^{N-1})}{\sum_{j=1}^m|\omega_j|^{2p}}\right)&\\ =-\mu t^{N-1}2\operatorname{Re}\langle zA,P_{N-2}(z)+Q_{N-2}(z,\omega)\rangle+o(t^{N-1})&.\end{aligned} \tag{3.15}$$

Dividing the two sides of equation (3.15) by t^{N-1} and taking $t\to0^+$, we get

$$\begin{aligned}&\sum_{j=1}^m p|\omega_j|^{2(p-1)}2\operatorname{Re}(\overline{\omega_j}R_{N,j}(z,\omega))\\ &\qquad+\mu\sum_{j=1}^m|\omega_j|^{2p}2\operatorname{Re}\langle zA,P_{N-2}(z)+Q_{N-2}(z,\omega)\rangle=0.\end{aligned}$$

From Lemma 2.3 we have $R_N(z,\omega)=0$, $P_{N-2}(z)=0$, $Q_{N-2}(z,\omega)=0$. This is also a contradiction with (3.12).

Step 4. From Steps 1, 2 and 3, we get $G^{-1}\circ F(z,\omega)=H(z,\omega)=(zA,\omega U)$, where $U\in U(m)$. Then by Lemma 2.2 and Theorem 10 in [6], we conclude that H is an automorphism of $D^p_{n,m}(\mu)$. Therefore

$$F(z,\omega)=(zA+v,\omega U\mathrm{e}^{-(\mu\langle z,v\rangle)/p-(\mu\|v\|^2)/2p}).$$

The proof is completed. □

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Authors' addresses: Ting Guo, School of Mathematics and Statistics, Qingdao University, 308 Ningxia Road, Qingdao, Shandong 266071, P. R. China, e-mail: `2018020209@qdu.edu.cn`; Zhiming Feng, School of Mathematical and Information Sciences, Leshan Normal University, 778 Binhe Rd, Shizhong District, Leshan, Sichuan 614000, P. R. China, e-mail: `fengzm2008@163.com`; Enchao Bi (corresponding author), School of Mathematics and Statistics, Qingdao University, 308 Ningxia Road, Qingdao, Shandong 266071, P. R. China, e-mail: `bienchao@whu.edu.cn`.