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RAMSEY NUMBERS FOR TREES II

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Abstract. Let $r(G_1, G_2)$ be the Ramsey number of the two graphs G_1 and G_2. For $n_1 \geqslant n_2 \geqslant 1$ let $S(n_1, n_2)$ be the double star given by $V(S(n_1, n_2)) = \{v_0, v_1, \ldots, v_{n_1}, w_0, w_1, \ldots, w_{n_2}\}$ and $E(S(n_1, n_2)) = \{v_0v_1, \ldots, v_0v_{n_1}, v_0w_0, w_0w_1, \ldots, w_0w_{n_2}\}$. We determine $r(K_{1,m-1}, S(n_1, n_2))$ under certain conditions. For $n \geqslant 6$ let $T_n^3 = S(n-5, 3)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$, where $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $E_2 = \{v_0v_1, \ldots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}$ and $E_3 = \{v_0v_1, \ldots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}$. We also obtain explicit formulas for $r(K_{1,m-1}, T_n)$, $r(T_m', T_n)$ $(n \geqslant m+3)$, $r(T_n, T_n)$, $r(T_n', T_n)$ and $r(P_n, T_n)$, where $T_n \in \{T_n'', T_n''', T_n^3\}$, P_n is the path on n vertices and T_n' is the unique tree with n vertices and maximal degree $n-2$.

Keywords: Ramsey number; tree; Turán's problem

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1. Introduction

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G, and let $\Delta(G)$ and $\delta(G)$ denote the maximal degree and minimal degree of G, respectively.

For a graph G, as usual $\overline{G}$ denotes the complement of G. Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2.

Let $\mathbb{N}$ be the set of positive integers. For $n \in \mathbb{N}$ with $n \geqslant 6$ let T_n be a tree on n vertices. As mentioned in [8], recently Zhao proved that $r(T_n, T_n) \leqslant 2n-2$, which was conjectured by Burr and Erdős, see [1].

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Let $m, n \in \mathbb{N}$. For $n \geqslant 3$ let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n-1$, and for $n \geqslant 4$ let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n-2$. In 1972, Harary in [6] showed that for $m, n \geqslant 3$,

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3 & \text{if } 2 \nmid mn, \\ m+n-2 & \text{if } 2 \mid mn. \end{cases} \tag{1.1}$$

From [2], page 72, if G is a graph with $\delta(G) \geqslant n-1$, then G contains every tree on n vertices. Using this fact, in 1995, Guo and Volkmann in [5] proved that for $n > m \geqslant 4$,

$$r(K_{1,m-1}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \mid m(n-1), \\ m+n-4 & \text{if } 2 \nmid m(n-1). \end{cases} \tag{1.2}$$

In 2012 the author in [9] evaluated the Ramsey number $r(T_m, T^*_n)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T^*_m\}$, where P_m is a path on m vertices and T^*_n is the tree on n vertices with $V(T^*_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(T^*_n) = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In particular, he proved that for $n > m \geqslant 7$,

$$r(K_{1,m-1}, T^*_n) = \begin{cases} m+n-3 & \text{if } m-1 \mid n-3, \\ m+n-4 & \text{if } m-1 \nmid n-3. \end{cases} \tag{1.3}$$

For $n \geqslant 5$ let $T^1_n = (V, E_1)$ and $T^2_n = (V, E_2)$ be the trees on n vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $E_1 = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\}$ and $E_2 = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-3}v_{n-1}\}$. Then $\Delta(T^1_n) = \Delta(T^2_n) = \Delta(T^*_n) = n-3$. In [12], Sun, Wang and Wu proved that

$$r(K_{1,m-1}, T^1_n) = r(K_{1,m-1}, T^2_n) = m+n-4 \quad \text{for } n > m \geqslant 7 \text{ and } 2 \mid mn. \tag{1.4}$$

For $n_1, n_2 \in \mathbb{N}$ with $n_1 \geqslant n_2$, let $S(n_1, n_2)$ be the double star given by

$$V(S(n_1, n_2)) = \{v_0, v_1, \ldots, v_{n_1}, w_0, w_1, \ldots, w_{n_2}\},$$
$$E(S(n_1, n_2)) = \{v_0v_1, \ldots, v_0v_{n_1}, v_0w_0, w_0w_1, \ldots, w_0w_{n_2}\}.$$

We say that v_0 and w_0 are centers of $S(n_1, n_2)$. In [4], Grossman, Harary and Klawe evaluated the Ramsey number $r(S(n_1, n_2), S(n_1, n_2))$ under certain conditions. In particular, they showed that for odd n_1 and $n_2 = 1, 2$,

$$r(S(n_1, n_2), S(n_1, n_2)) = \max\{2n_1 + 1, n_1 + 2n_2 + 2\}. \tag{1.5}$$

It is clear that $T'_n = S(n-3,1)$ and $T^2_n = S(n-4,2)$. In this paper, we prove the following general result:

$$(1.6)\qquad r(K_{1,m-1}, S(n_1,n_2)) = \begin{cases} m+n_1 & \text{if } 2 \mid mn_1,\ n_1 \geqslant m-2 \geqslant n_2 \geqslant 2 \\ & \text{and } n_1 > m-5+n_2+\dfrac{(n_2-1)(n_2-2)}{m-1-n_2}, \\ m-1+n_1 & \text{if } 2 \nmid mn_1,\ n_1 \geqslant m-2 > n_2 \\ & \text{and } n_1 > m-5+n_2+\dfrac{(n_2-1)^2}{m-2-n_2}. \end{cases}$$

Also,

$$(1.7)\qquad r(K_{1,m-1}, T^1_n) = m+n-5 \quad \text{for } n \geqslant m+2 \geqslant 7 \text{ and } 2 \nmid mn.$$

For $n \geqslant 6$ let $T^3_n = S(n-5,3)$, $T''_n = (V, E_2)$ and $T'''_n = (V, E_3)$, where

$$V = \{v_0, v_1, \ldots, v_{n-1}\}, \quad E_2 = \{v_0v_1, \ldots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\},$$
$$E_3 = \{v_0v_1, \ldots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}.$$

Then $\Delta(T^3_n) = \Delta(T''_n) = \Delta(T'''_n) = n-4$. In this paper, we evaluate $r(K_{1,m-1}, T_n)$ and $r(T'_m, T_n)$ for $T_n \in \{T''_n, T'''_n, T^3_n\}$. In particular, we show that

$$(1.8)\qquad r(K_{1,m-1}, T''_n) = r(K_{1,m-1}, T'''_n) = \begin{cases} m+n-5 & \text{if } 2 \mid m(n-1),\ m \geqslant 7,\ n \geqslant 15 \\ & \text{and } n > m+1+\dfrac{8}{m-6}, \\ m+n-6 & \text{if } 2 \nmid m(n-1) \text{ and } n \geqslant m+3 \geqslant 9, \end{cases}$$

and that for $m \geqslant 9$ and $n > m+2+\max\{0, (20-m)/(m-8)\}$,

$$(1.9)\qquad r(T'_m, T''_n) = r(T'_m, T'''_n) = r(T'_m, T^3_n) = \begin{cases} m+n-5 & \text{if } m-1 \mid n-5, \\ m+n-6 & \text{if } m-1 \nmid n-5. \end{cases}$$

We also prove that for $m \geqslant 11$, $n \geqslant (m-3)^2+4$ and $m-1 \nmid n-5$,

$$(1.10)\quad r(G_m, T_n) = m+n-6 \quad \text{for } G_m \in \{T^*_m, T^1_m, T^2_m\} \text{ and } T_n \in \{T''_n, T'''_n, T^3_n\}.$$

In addition, we establish the following results:

$$r(T''_n, T''_n) = r(T''_n, T'''_n) = r(T'''_n, T'''_n) = \begin{cases} 2n-9 & \text{if } 2 \mid n \text{ and } n > 29, \\ 2n-8 & \text{if } 2 \nmid n \text{ and } n > 22, \end{cases}$$
$$r(T^3_n, T''_n) = r(T^3_n, T'''_n) = r(T^3_n, T^3_n) = 2n-8 \quad \text{for } n > 22,$$
$$r(T''_n, T'_n) = r(T'''_n, T'_n) = r(T^3_n, T'_n) = 2n-5 \quad \text{for } n \geqslant 10,$$
$$r(T''_n, T^i_n) = r(T'''_n, T^i_n) = r(T^3_n, T^i_n) = 2n-7 \quad \text{for } n > 16 \text{ and } i = 1, 2,$$
$$r(P_n, T''_n) = r(P_n, T'''_n) = r(P_n, T^3_n) = 2n-9 \quad \text{for } n \geqslant 33.$$

In addition to the above notation, throughout this paper, we use the following notation: $[x]$—the greatest integer not exceeding x, $d(v)$—the degree of the vertex v in a graph, $d(u,v)$—the distance between the two vertices u and v in a graph, K_n—the complete graph on n vertices, $G[V_1]$—the subgraph of G induced by vertices in the set V_1, $G - V_1$—the subgraph of G obtained by deleting vertices in V_1 and all edges incident with them, $\Gamma(v)$—the set of vertices adjacent to the vertex v, $e(V_1 V'_1)$—the number of edges with one endpoint in V_1 and the other endpoint in V'_1.

2. Basic lemmas

For a forbidden graph L let $\mathrm{ex}(p; L)$ be the maximal number of edges in a graph of order p not containing any copies of L. The corresponding Turán's problem is to evaluate $\mathrm{ex}(p; L)$. Let $p, n \in \mathbb{N}$ with $p \geqslant n \geqslant 2$. For a given tree T_n on n vertices, it is difficult to determine the value of $\mathrm{ex}(p; T_n)$. The famous Erdős-Sós conjecture asserts that $\mathrm{ex}(p; T_n) \leqslant \frac{1}{2}(n-2)p$. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. In 1975 Faudree and Schelp in [3] showed that

$$\mathrm{ex}(p; P_n) = k\binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}. \tag{2.1}$$

In [10], [11], [12], the author and his coauthors determined $\mathrm{ex}(p; T_n)$ for $T_n \in \{T'_n, T^*_n, T^1_n, T^2_n, T^3_n, T''_n, T'''_n\}$.

Lemma 2.1 ([9], Lemma 2.1)**.** *Let G_1 and G_2 be two graphs. Suppose that $p \in \mathbb{N}$, $p \geqslant \max\{|V(G_1)|, |V(G_2)|\}$ and $\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leqslant p$.*

P r o o f. Let G be a graph of order p. If $e(G) \leqslant \mathrm{ex}(p; G_1)$ and $e(\overline{G}) \leqslant \mathrm{ex}(p; G_2)$, then $\mathrm{ex}(p; G_1) + \mathrm{ex}(p; G_2) \geqslant e(G) + e(\overline{G}) = \binom{p}{2}$. This contradicts the assumption. Hence, either $e(G) > \mathrm{ex}(p; G_1)$ or $e(\overline{G}) > \mathrm{ex}(p; G_2)$. Therefore, G contains a copy of G_1 or $\overline{G}$ contains a copy of G_2. This shows that $r(G_1, G_2) \leqslant |V(G)| = p$. So the lemma is proved. □

Lemma 2.2. *Let $k, p \in \mathbb{N}$ with $p \geqslant k+1$. Then there exists a k-regular graph of order p if and only if $2 \mid kp$.*

This is a known result. See for example [11], Corollary 2.1.

Lemma 2.3 ([9], Lemma 2.3). *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geqslant 2$ and $\Delta(G_2) = d_2 \geqslant 2$. Then:*

(i) $r(G_1, G_2) \geqslant d_1 + d_2 - \frac{1}{2}(1 - (-1)^{(d_1-1)(d_2-1)})$.

(ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leqslant m$. Then $r(G_1, G_2) \geqslant 2d_2 - 1 \geqslant d_1 + d_2$.*

(iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*

(1) $2 \mid (d_1 + d_2 - m)$,

(2) $d_1 \neq m - 1$,

(3) *G_2 has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$*

holds, then $r(G_1, G_2) \geqslant d_1 + d_2$.

Lemma 2.4. *Let $p, n \in \mathbb{N}$ with $p \geqslant n - 1 \geqslant 1$. Then $\mathrm{ex}(p; K_{1,n-1}) = [\frac{1}{2}(n-2)p]$.*

This is a known result. See for example [11], Theorem 2.1.

Lemma 2.5 ([11], Theorem 3.1). *Let $p, n \in \mathbb{N}$ with $p \geqslant n \geqslant 5$, and let $r \in \{0, 1, \ldots, n-2\}$ be given by $p \equiv r \pmod{n-1}$. Then*

$$\mathrm{ex}(p; T_n') = \begin{cases} \left[\dfrac{(n-2)(p-1)-r-1}{2}\right] & \text{if } n \geqslant 7 \text{ and } 2 \leqslant r \leqslant n-4, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 ([12], Theorems 2.1 and 3.1). *Suppose that $p, n \in \mathbb{N}$, $p \geqslant n - 1 \geqslant 4$ and $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, n-2\}$. For $i = 1$ or 2,*

$$\begin{aligned} \mathrm{ex}(p; T_n^i) &= \max\Big\{\Big[\frac{(n-2)p}{2}\Big] - (n-1+r), \frac{(n-2)p - r(n-1-r)}{2}\Big\} \\ &= \begin{cases} \Big[\dfrac{(n-2)p}{2}\Big] - (n-1+r) & \text{if } n \geqslant 16 \text{ and } 3 \leqslant r \leqslant n-6 \\ & \text{or if } 13 \leqslant n \leqslant 15 \text{ and } 4 \leqslant r \leqslant n-7, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.7 ([10], Theorems 3.1 and 5.1). *Let $p, n \in \mathbb{N}$, $p \geqslant n \geqslant 10$, $p = k(n-1)+r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$\operatorname{ex}(p; T_n'') = \operatorname{ex}(p; T_n''') = \frac{(n-2)p - r(n-1-r)}{2} + \max\Big\{0, \Big[\frac{r(n-4-r)-3(n-1)}{2}\Big]\Big\}.$$

Lemma 2.8 ([10], Lemmas 4.6 and 4.7). *Let $n \in \mathbb{N}$ with $n \geqslant 15$. Then*

$$\operatorname{ex}(2n-9; T_n^3) = n^2 - 10n + 24 + \max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}$$

and

$$\operatorname{ex}(2n-8; T_n^3) = n^2 - 9n + 29 + \max\Big\{\Big[\frac{n-37}{4}\Big], 0\Big\}.$$

Lemma 2.9 ([10], Theorems 4.1–4.5). *Let $p, n \in \mathbb{N}$, $p \geqslant n \geqslant 10$, $p = k(n-1)+r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$.*

(i) *If $r \in \{0, 1, 2, n-6, n-5, n-4, n-3, n-2\}$, then*

$$\operatorname{ex}(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2}.$$

(ii) *If $n \geqslant 15$ and $r \in \{3, 4, \dots, n-9\}$, then*

$$\operatorname{ex}(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2} + \max\Big\{0, \Big[\frac{r(n-4-r)-3(n-1)}{2}\Big]\Big\}.$$

(iii) *If $n \geqslant 15$ and $r = n-8$, then*

$$\operatorname{ex}(p; T_n^3) = \frac{(n-2)p - 7n + 30}{2} + \max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}.$$

(iv) *If $n \geqslant 15$ and $r = n-7$, then*

$$\operatorname{ex}(p; T_n^3) = \frac{(n-2)p - 6(n-7)}{2} + \max\Big\{\Big[\frac{n-37}{4}\Big], 0\Big\}.$$

Lemma 2.10. *Let $n \in \mathbb{N}$, $n \geqslant 10$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $p = k(n-1)+r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then*

$$\operatorname{ex}(p; T_n) \leqslant \frac{(n-2)p}{2} - \min\Big\{n-1+r, \frac{r(n-1-r)}{2}\Big\}.$$

P r o o f. This is immediate from [10], Lemmas 2.8, 3.1, 4.1 and 5.1. □

Lemma 2.11 ([11], Theorems 4.1–4.3). *Let $p, n \in \mathbb{N}$, $p \geqslant n \geqslant 6$ and $p = k(n-1)+r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n-5, n-4, n-3, n-2\}$. Then*

$$\operatorname{ex}(p; T_n^*) = \begin{cases} \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n-5, \\ \dfrac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.12 ([11], Theorem 4.4). *Let $p, n \in \mathbb{N}$, $p \geqslant n \geqslant 11$, $r \in \{2, 3, \ldots, n-6\}$ and $p \equiv r \pmod{n-1}$. Let $t \in \{0, 1, \ldots, r+1\}$ be given by $n - 3 \equiv t \pmod{r+2}$. Then*

$$\operatorname{ex}(p; T_n^*) = \begin{cases} \Big[\dfrac{(n-2)(p-1) - 2r - t - 3}{2}\Big] & \text{if } r \geqslant 4 \text{ and } 2 \leqslant t \leqslant r-1, \\ \dfrac{(n-2)(p-1) - t(r+2-t) - r - 1}{2} & \text{otherwise.} \end{cases}$$

3. Formulas for $r(T_n, T_n'')$, $r(T_n, T_n''')$ and $r(T_n, T_n^3)$

Theorem 3.1. *Let $n \in \mathbb{N}$. Then*

$$r(T_n'', T_n'') = r(T_n'', T_n''') = r(T_n''', T_n''') = \begin{cases} 2n-9 & \text{if } 2 \mid n \text{ and } n > 29, \\ 2n-8 & \text{if } 2 \nmid n \text{ and } n > 22. \end{cases}$$

P r o o f. Suppose that $T_n, T_n^0 \in \{T_n'', T_n'''\}$. By Lemma 2.7,

$$\begin{aligned} \operatorname{ex}(2n-9; T_n) &= \frac{(2n-9)(n-5) - (n-29)}{2} + \max\Big\{0, \Big[\frac{n-29}{2}\Big]\Big\} \\ &= \Big[\frac{(2n-9)(n-5)}{2}\Big] \quad \text{for } n \geqslant 29. \end{aligned}$$

Hence, for $n \in \{30, 32, 34, \ldots\}$,

$$\operatorname{ex}(2n-9; T_n) + \operatorname{ex}(2n-9; T_n^0) = 2\Big[\frac{(2n-9)(n-5)}{2}\Big] = (2n-9)(n-5) - 1 < \binom{2n-9}{2}.$$

Applying Lemma 2.1 yields $r(T_n, T_n^0) \leqslant 2n-9$. On the other hand, appealing to Lemma 2.3 (i),

$$r(T_n, T_n^0) \geqslant n-4+n-4-\frac{1-(-1)^{(n-5)(n-5)}}{2}=2n-9.$$

Therefore $r(T_n, T_n^0)=2n-9$ for $n \in \{30, 32, 34, \ldots\}$.

Now assume that $2 \nmid n$ and $n>22$. By Lemma 2.7,

$$\operatorname{ex}(2n-8; T_n)=\frac{(n-2)(2n-8)-6(n-7)}{2}=n^2-9n+29.$$

Thus,

$$\operatorname{ex}(2n-8; T_n)+\operatorname{ex}(2n-8; T_n^0)=2(n^2-9n+29)<2n^2-17n+36=\binom{2n-8}{2}.$$

Hence, $r(T_n, T_n^0) \leqslant 2n-8$ by Lemma 2.1. By Lemma 2.2, we may construct a regular graph G of order $2n-9$ with degree $n-5$. Clearly $\overline{G}$ is also a regular graph with degree $n-5$. Since $\Delta(T_n)=\Delta(T_n^0)=n-4$, both G and $\overline{G}$ do not contain any copies of T_n and T_n^0. Therefore, $r(T_n, T_n^0)>2n-9$ and so $r(T_n, T_n^0)=2n-8$. This completes the proof. □

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n>22$. Then*

$$r(T_n^3, T_n'')=r(T_n^3, T_n''')=r(T_n^3, T_n^3)=2n-8.$$

P r o o f. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. When n is odd, using Lemma 2.3 (i) we see that $r(T_n^3, T_n) \geqslant n-4+n-4=2n-8$. When n is even, we may construct a regular graph H with degree $n-10$ and $V(H)=\{v_1, \ldots, v_{n-6}\}$. Let G_0 be a graph given by

$$V(G_0)=\{v_0, v_1, \ldots, v_{n-4}, u_1, \ldots, u_{n-6}\}$$

and

$$\begin{aligned}E(G_0)=E(H)\cup\{&v_0v_1,\ldots,v_0v_{n-4},v_1v_{n-5},\ldots,v_{n-6}v_{n-5},v_1v_{n-4},\ldots,v_{n-5}v_{n-4},v_1u_1,\\&v_1u_2,v_2u_1,v_2u_2,\ldots,v_{n-7}u_{n-7},v_{n-7}u_{n-6},v_{n-6}u_{n-7},v_{n-6}u_{n-6},\\&u_1u_2,\ldots,u_1u_{n-6},u_2u_3,\ldots,u_2u_{n-6},u_3u_{n-6},\ldots,u_{n-7}u_{n-6}\}.\end{aligned}$$

Then $d(v_0)=d(v_{n-5})=d(v_{n-4})=n-4$ and $d(v_1)=\ldots=d(v_{n-6})=d(u_1)=\ldots=d(u_{n-6})=n-5$. Clearly $|V(G_0)|=2n-9$ and G_0 does not contain any copies of T_n^3. Since $\Delta(\overline{G}_0)=n-5$ and $\Delta(T_n)=n-4$, $\overline{G}_0$ does not contain any copies of T_n. Thus, $r(T_n^3, T_n) \geqslant |V(G_0)|+1=2n-8$.

From Lemma 2.7, $\mathrm{ex}(2n-8;T_n'') = \mathrm{ex}(2n-8;T_n''') = n^2-9n+29$. By Lemma 2.8, $\mathrm{ex}(2n-8;T_n^3) = n^2-9n+29+\max\{0,[\frac{1}{4}(n-37)]\}$. Thus,

$$\begin{aligned}\mathrm{ex}(2n-8;T_n^3)+\mathrm{ex}(2n-8;T_n) &\leqslant 2n^2-18n+58+2\max\Big\{0,\Big[\frac{n-37}{4}\Big]\Big\}\\ &< 2n^2-18n+58+n-22 = \binom{2n-8}{2}.\end{aligned}$$

Hence, applying Lemma 2.1 gives $r(T_n^3,T_n) \leqslant 2n-8$ and so $r(T_n^3,T_n) = 2n-8$ as claimed. □

Theorem 3.3. *Let $n \in \mathbb{N}$ with $n \geqslant 10$. Then*

$$r(T_n'',T_n') = r(T_n''',T_n') = r(T_n^3,T_n') = 2n-5.$$

P r o o f. Let $T_n \in \{T_n'',T_n''',T_n^3\}$. Since $\Delta(T_n) = n-4$ and $\Delta(T_n') = n-2$, using Lemma 2.3 (ii) we see that $r(T_n,T_n') \geqslant 2(n-2)-1 = 2n-5$. By Lemmas 2.5, 2.7 and 2.9,

$$\begin{aligned}&\mathrm{ex}(2n-5;T_n)+\mathrm{ex}(2n-5;T_n')\\ &\quad= \frac{(n-2)(2n-5)-3(n-4)}{2}+\Big[\frac{(n-2)(2n-6)-(n-3)}{2}\Big]\\ &\quad= \Big[\frac{4n^2-23n+37}{2}\Big] < \frac{4n^2-22n+30}{2} = \binom{2n-5}{2}.\end{aligned}$$

Hence, $r(T_n,T_n') \leqslant 2n-5$ by Lemma 2.1. Therefore, $r(T_n,T_n') = 2n-5$ as claimed. □

Theorem 3.4. *Let $n \in \mathbb{N}$, $n > 16$ and $i \in \{1,2\}$. Then*

$$r(T_n'',T_n^i) = r(T_n''',T_n^i) = r(T_n^3,T_n^i) = 2n-7.$$

P r o o f. Let $T_n \in \{T_n'',T_n''',T_n^3\}$. Since $\Delta(T_n) = n-4$ and $\Delta(T_n^i) = n-3$, using Lemma 2.3 (ii) we see that $r(T_n,T_n^i) \geqslant 2(n-3)-1 = 2n-7$. From Lemmas 2.6, 2.7 and 2.9,

$$\begin{aligned}&\mathrm{ex}(2n-7;T_n)+\mathrm{ex}(2n-7;T_n^i)\\ &\quad= \frac{(n-2)(2n-7)-5(n-6)}{2}+\Big[\frac{(n-2)(2n-7)}{2}\Big]-(2n-7)\\ &\quad= \Big[\frac{4n^2-31n+72}{2}\Big] < \frac{4n^2-30n+56}{2} = \binom{2n-7}{2}.\end{aligned}$$

Hence, $r(T_n,T_n^i) \leqslant 2n-7$ by Lemma 2.1. Therefore, $r(T_n,T_n^i) = 2n-7$ as claimed. □

Theorem 3.5. *Let $n \in \mathbb{N}$ with $n \geqslant 10$. Then $r(T_n, T_n^*) = 2n - 5$ for $T_n \in \{T_n'', T_n''', T_n^3\}$.*

P r o o f. By Lemmas 2.7 and 2.9, $\mathrm{ex}(2n-5; T_n) = \frac{1}{2}((n-2)(2n-5) - 3(n-4)) = n^2 - 6n + 11 < n^2 - 5n + 4$. Thus the result follows from [9], Lemma 3.1. $\square$

Remark 3.1. By [9], Theorem 6.3 with $m = n$ and $a = 2$, $r(T_n, K_{1,n-1}) = 2n-3$ for $n \geqslant 6$ and $T_n \in \{T_n'', T_n''', T_n^3\}$.

Theorem 3.6. *Let $n \in \mathbb{N}$. Then $r(P_n, T_n'') = r(P_n, T_n''') = 2n - 9$ for $n \geqslant 30$ and $r(P_n, T_n^3) = 2n - 9$ for $n \geqslant 33$.*

P r o o f. Suppose that $n \geqslant 30$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T_n) = n - 4$ and $\Delta(P_n) = 2$, appealing to Lemma 2.3 (ii) we obtain $r(P_n, T_n) \geqslant 2(n-4) - 1 = 2n - 9$. By (2.1) and Lemma 2.7 for $T_n \in \{T_n'', T_n'''\}$,

$$\begin{aligned}\mathrm{ex}(2n-9; P_n) &+ \mathrm{ex}(2n-9; T_n)\\ &= \frac{(n-2)(2n-9) - 7(n-8)}{2} + \left[\frac{(2n-9)(n-5)}{2}\right]\\ &= \left[\frac{4n^2 - 39n + 119}{2}\right] < \frac{4n^2 - 38n + 90}{2} = \binom{2n-9}{2}.\end{aligned}$$

Hence, applying Lemma 2.1 gives $r(P_n, T_n) \leqslant 2n - 9$ and so $r(P_n, T_n) = 2n - 9$.

Now assume that $n \geqslant 33$. From (2.1) and Lemma 2.8,

$$\begin{aligned}\mathrm{ex}(2n-9; P_n) &+ \mathrm{ex}(2n-9; T_n^3)\\ &= \frac{(n-2)(2n-9) - 7(n-8)}{2} + n^2 - 10n + 24 + \left[\frac{n}{2}\right]\\ &= 2n^2 - 20n + 61 + \left[\frac{n}{2}\right] < 2n^2 - 19n + 45 = \binom{2n-9}{2}.\end{aligned}$$

Hence, $r(P_n, T_n^3) \leqslant 2n - 9$ by Lemma 2.1 and so $r(P_n, T_n^3) = 2n - 9$. $\square$

4. Formulas for $r(K_{1,m-1}, S(n_1, n_2))$, $r(K_{1,m-1}, T_n^1)$, $r(K_{1,m-1}, T_n'')$ and $r(K_{1,m-1}, T_n{}''')$

Theorem 4.1. *Let $m, n_1, n_2 \in \mathbb{N}$ with $n_1 \geqslant m - 2 \geqslant n_2 \geqslant 2$ and $2 \mid mn_1$. If $n_1 > m - 5 + n_2 + (n_2 - 1)(n_2 - 2)/(m - 1 - n_2)$, then $r(K_{1,m-1}, S(n_1, n_2)) = m + n_1$.*

P r o o f. Since $\Delta(S(n_1, n_2)) = n_1 + 1$, from Lemma 2.3 (i) we see that

$$r(K_{1,m-1}, S(n_1, n_2)) \geqslant m - 1 + n_1 + 1 - \frac{1 - (-1)^{(m-2)n_1}}{2} = m + n_1.$$

Now we show that $r(K_{1,m-1}, S(n_1,n_2)) \leqslant m+n_1$. Let G be a graph of order $m+n_1$ such that $\overline{G}$ does not contain any copies of $K_{1,m-1}$. That is, $\Delta(\overline{G}) \leqslant m-2$. We show that G contains a copy of $S(n_1,n_2)$. Clearly

$$\delta(G) = m+n_1-1-\Delta(\overline{G}) \geqslant m+n_1-1-(m-2) = n_1+1.$$

Suppose that $\Delta(G) = n_1+1+s$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \ldots, v_{n_1+1+s}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1$. Then $|V_1'| = m-2-s$. For $i = 1, 2, \ldots, n_1 + 1 + s$, we have

$$|V_1'| + 1 + |\Gamma(v_i) \cap \Gamma(v_0)| \geqslant d(v_i) \geqslant \delta(G) \geqslant n_1 + 1$$

and so

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant n_1 - |V_1'| = n_1 - (m-2) + s \geqslant s.$$

For $s \geqslant n_2$ we have $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant s \geqslant n_2$ and $|\Gamma(v_0)| - n_2 = n_1+1+s-n_2 \geqslant n_1+1$. Hence $G[V_1]$ contains a copy of $S(n_1,n_2)$ with centers v_0 and v_i.

Now assume that $s < n_2$ and $V_1' = V(G) - V_1 = \{u_1, \ldots, u_{m-2-s}\}$. It is clear that for $i = 1, 2, \ldots, m-2-s$,

$$m-3-s+|\Gamma(u_i) \cap \Gamma(v_0)| = |V_1'| - 1 + |\Gamma(u_i) \cap \Gamma(v_0)| \geqslant d(u_i) \geqslant \delta(G) \geqslant n_1 + 1$$

and so $|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant n_1 - (m-4-s)$. It then follows that $e(V_1 V_1') \geqslant (m-2-s) \times (n_1 - (m-4-s))$. By the assumption,

$$n_1 > m-5+n_2+\frac{(n_2-2)(n_2-1)}{m-1-n_2} \geqslant m-5+n_2-2s+\frac{(n_2-2)(n_2-1-s)}{m-1-n_2}.$$

Thus, $(m-1-n_2)n_1 > (m-2-s)(m-4-s) + (s+1)(n_2-s-1)$ and so $e(V_1V_1') \geqslant (m-2-s)(n_1-(m-4-s)) > (n_1+1+s)(n_2-s-1)$. Therefore, $|\Gamma(v_i) \cap V_1'| \geqslant n_2 - s$ for some $v_i \in \Gamma(v_0)$. From the above, $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant s$. Thus, G contains a copy of $S(n_1,n_2)$ with centers v_0 and v_i. Therefore $r(K_{1,m-1}, S(n_1,n_2)) \leqslant m+n_1$ and so the theorem is proved. □

Corollary 4.1. *Let $m, n \in \mathbb{N}$, $n-2 \geqslant m \geqslant 4$ and $2 \mid mn$. Then $r(K_{1,m-1}, T_n^2) = m+n-4$.*

Proof. Since $T_n^2 = S(n-4, 2)$, putting $n_1 = n-4$ and $n_2 = 2$ in Theorem 4.1 yields the result. □

Corollary 4.2. *Let $m, n \in \mathbb{N}$, $m \geqslant 5$, $n > m+3+2/(m-4)$ and $2 \mid m(n-1)$. Then $r(K_{1,m-1}, T_n^3) = m+n-5$.*

P r o o f. Since $T_n^3 = S(n-5,3)$, taking $n_1 = n-5$ and $n_2 = 3$ in Theorem 4.1 gives the result. $\square$

Theorem 4.2. *Let $m, n_1, n_2 \in \mathbb{N}$, $n_1 \geqslant m-2 > n_2$ and $2 \nmid mn_1$. If $n_1 > m-5+n_2+(n_2-1)^2/(m-2-n_2)$, then $r(K_{1,m-1}, S(n_1,n_2)) = m-1+n_1$.*

P r o o f. Since $\Delta(S(n_1,n_2)) = n_1+1$, from Lemma 2.3 (i) we see that

$$r(K_{1,m-1}, S(n_1,n_2)) \geqslant m-1+n_1+1-\frac{1-(-1)^{(m-2)n_1}}{2} = m-1+n_1.$$

Now we show that $r(K_{1,m-1}, S(n_1,n_2)) \leqslant m-1+n_1$. Let G be a graph of order $m-1+n_1$ such that $\overline{G}$ does not contain any copies of $K_{1,m-1}$. We need to show that G contains a copy of $S(n_1,n_2)$. Clearly $\Delta(\overline{G}) \leqslant m-2$ and so $\delta(G) = m-2+n_1-\Delta(\overline{G}) \geqslant n_1$. Since $2 \nmid mn_1$, there is no regular graph of order $m-1+n_1$ with degree n_1 by Euler's theorem. Hence $\Delta(G) \geqslant \delta(G)+1 \geqslant n_1+1$. Suppose that $\Delta(G) = n_1+1+s$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \ldots, v_{n_1+1+s}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1$. Then $|V_1'| = m-3-s$. For $v_i \in \Gamma(v_0)$, $d(v_i) \geqslant \delta(G) \geqslant n_1$ and so $|\Gamma(v_i) \cap \Gamma(v_0)| + 1 + |V_1'| \geqslant d(v_i) \geqslant n_1$. Thus,

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant n_1 - 1 - |V_1'| = n_1 - (m-2) + s \geqslant s.$$

Hence, $G[V_1]$ contains a copy of $S(n_1,n_2)$ with centers v_0 and v_i for $s \geqslant n_2$.

Now assume that $s < n_2$ and $V_1' = \{u_1, \ldots, u_{m-3-s}\}$. As $d(u_i) \geqslant n_1$, we see that $|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant n_1-(m-4-s)$ and so $e(V_1V_1') \geqslant (m-3-s)(n_1-(m-4-s))$. Since

$$n_1 > m-5+n_2+\frac{(n_2-1)^2}{m-2-n_2} \geqslant m-5+n_2-2s+\frac{(n_2-1)(n_2-1-s)}{m-2-n_2},$$

we get $(m-2-n_2)n_1 > (m-3-s)(m-4-s)+(s+1)(n_2-s-1)$. Hence,

$$\begin{aligned} e(V_1V_1') &\geqslant (m-3-s)(n_1-(m-4-s)) \\ &> (m-3-s)n_1-(m-2-n_2)n_1+(s+1)(n_2-s-1) \\ &= (n_1+1+s)(n_2-s-1). \end{aligned}$$

Therefore, we have $|\Gamma(v_i) \cap V_1'| \geqslant n_2 - s$ for some $v_i \in \Gamma(v_0)$. From the above, $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant s$. Thus, G contains a copy of $S(n_1,n_2)$ with centers v_0 and v_i. Consequently, $r(K_{1,m-1}, S(n_1,n_2)) \leqslant m-1+n_1$ and so the theorem is proved. $\square$

Corollary 4.3. *Let $m, n \in \mathbb{N}$, $m \geqslant 5$, $n > m+1+1/(m-4)$ and $2 \nmid mn$. Then $r(K_{1,m-1}, T_n^2) = m+n-5$.*

P r o o f. Since $T_n^2 = S(n-4,2)$, putting $n_1 = n-4$ and $n_2 = 2$ in Theorem 4.2 yields the result. $\square$

Corollary 4.4. *Let $m,n \in \mathbb{N}$, $m \geqslant 6$, $n > m+3+4/(m-5)$ and $2 \nmid m(n-1)$. Then $r(K_{1,m-1}, T_n^3) = m+n-6$.*

P r o o f. Since $T_n^3 = S(n-5,3)$, putting $n_1 = n-5$ and $n_2 = 3$ in Theorem 4.2 we deduce the result. $\square$

Theorem 4.3. *Let $m,n \in \mathbb{N}$, $n \geqslant m+2 \geqslant 7$ and $2 \nmid mn$. Then $r(K_{1,m-1}, T_n^1) = m+n-5$.*

P r o o f. Since $n > m$ and $2 \nmid mn$, we have $n \geqslant m+2$. Let G be a graph of order $m+n-5$ such that $\overline{G}$ does not contain any copies of $K_{1,m-1}$. We show that G contains a copy of T_n^1. Clearly $\Delta(\overline{G}) \leqslant m-2$ and so $\delta(G) = m+n-6-\Delta(\overline{G}) \geqslant n-4$. If $\Delta(G) = n-4$, then G is a regular graph of order $m+n-5$ with degree $n-4$ and so $(m+n-5)(n-4) = 2e(G)$. Since $m+n-5$ and $n-4$ are odd, we get a contradiction. Thus, $\Delta(G) \geqslant n-3$. Assume that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n-3+c$, $\Gamma(v_0) = \{v_1, \ldots, v_{n-3+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1 = \{u_1, u_2, \ldots, u_{m-3-c}\}$. Since $\delta(G) \geqslant n-4$ for $v_i \in \Gamma(v_0)$ we have $1 + |\Gamma(v_i) \cap \Gamma(v_0)| + |V_1'| \geqslant d(v_i) \geqslant n-4$ and so $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant n-5-(m-3-c) = n-m-2+c \geqslant c$.

We first assume that $|V_1'| = m-3-c \geqslant 2$. For $i = 1,2$ we have $|\Gamma(u_i) \cap \Gamma(v_0)| + |V_1'| - 1 \geqslant d(u_i) \geqslant \delta(G) \geqslant n-4$ and so $|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant n-4+1-(m-3-c) = n-m+c \geqslant 2$. Hence G contains a copy of T_n^1. If $|V_1'| = 1$, then $c = m-4 \geqslant 1$. Since $d(u_1) \geqslant n-4 > 1$, we have $u_1v_j \in E(G)$ for some $v_j \in \Gamma(v_0)$. Recall that $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant c \geqslant 1$ for $v_i \in \Gamma(v_0)$. G must contain a copy of T_n^1. Now assume that $|V_1'| = 0$. That is, $c = m-3$ and $G = G[V_1]$. Since $d(v_0) = n-3+m-3 \geqslant n-3+2$ and $d(v_i) \geqslant n-4 \geqslant 3$ for $v_i \in \Gamma(v_0)$, we see that $G[\Gamma(v_0)]$ contains a copy of $2K_2$ and so G contains a copy of T_n^1.

By the above, G contains a copy of T_n^1. Therefore $r(K_{1,m-1}, T_n^1) \leqslant m+n-5$. From Lemma 2.3,

$$r(K_{1,m-1}, T_n^j) \geqslant m-1+n-3-\frac{1-(-1)^{(m-2)(n-4)}}{2} = m+n-5.$$

Hence $r(K_{1,m-1}, T_n^1) = m+n-5$ as claimed. $\square$

Lemma 4.1. *Let $m,n \in \mathbb{N}$, $n \geqslant 15$, $m \geqslant 7$, $n > m+1+8/(m-6)$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Let G_m be a connected graph of order m such that $\operatorname{ex}(m+n-5; G_m) \leqslant \frac{1}{2}(m-2)(m+n-5)$. Then $r(G_m, T_n) \leqslant m+n-5$. Moreover, if $m-1 \mid n-5$, then $r(G_m, T_n) = m+n-5$.*

P r o o f. If $T_n \neq T_n^3$ or $m \notin \{n-3, n-4\}$, appealing to Lemmas 2.7 and 2.9 we have

$$\begin{aligned}\mathrm{ex}(m+n-5; T_n) &= \frac{(n-2)(m+n-5)-(m-4)(n-m+3)}{2} \\ &\quad + \max\Big\{0, \Big[\frac{(m-4)(n-m)-3(n-1)}{2}\Big]\Big\} \\ &= \frac{(n-2)(m+n-5)-(m-4)(n-m+3)}{2} \\ &\quad + \max\Big\{0, \Big[\frac{(m-7)(n-m-3)-18}{2}\Big]\Big\}.\end{aligned}$$

Thus, if $(m-7)(n-m-3) \geqslant 18$, then

$$\begin{aligned}&\mathrm{ex}(m+n-5; G_m) + \mathrm{ex}(m+n-5; T_n) \\ &\quad \leqslant \frac{(m-2)(m+n-5)}{2} + \frac{(n-2)(m+n-5)-(m-4)(n-m+3)}{2} \\ &\qquad + \frac{(m-7)(n-m-3)-18}{2} \\ &\quad = \frac{(m+n-5)(m+n-7)}{2} \\ &\quad < \binom{m+n-5}{2}.\end{aligned}$$

If $(m-7)(n-m-3) < 18$, since $n > m+1+8/(m-6)$ we see that $(m-6)n > m^2-5m+2$, $(m-4)(n-m+3) > 2(m+n-5)$ and so

$$\begin{aligned}&\mathrm{ex}(m+n-5; G_m) + \mathrm{ex}(m+n-5; T_n) \\ &\quad \leqslant \frac{(m-2+n-2)(m+n-5)-(m-4)(n-m+3)}{2} < \binom{m+n-5}{2}.\end{aligned}$$

Hence, $r(G_m, T_n) \leqslant m+n-5$ by Lemma 2.1.

For $m = n-3$, using Lemma 2.8 we see that

$$\begin{aligned}\mathrm{ex}(m+n-5; G_m) + \mathrm{ex}(m+n-5; T_n^3) &= \mathrm{ex}(2n-8; G_{n-3}) + \mathrm{ex}(2n-8; T_n^3) \\ &\leqslant \frac{(2n-8)(n-5)}{2} + n^2 - 9n + 29 + \max\Big\{0, \Big[\frac{n-37}{4}\Big]\Big\} \\ &= 2n^2 - 18n + 49 + \max\Big\{0, \Big[\frac{n-37}{4}\Big]\Big\} \\ &< 2n^2 - 17n + 36 = \binom{m+n-5}{2}.\end{aligned}$$

For $m = n-4$, appealing to Lemma 2.8,

$$\begin{aligned}\mathrm{ex}(m+n-5;G_m)+\mathrm{ex}(m+n-5;T_n^3) &= \mathrm{ex}(2n-9;G_{n-4})+\mathrm{ex}(2n-9;T_n^3)\\ &\leqslant \frac{(2n-9)(n-6)}{2}+n^2-10n+24+\max\Bigl\{\Bigl[\frac{n}{2}\Bigr],13\Bigr\}\\ &= 2n^2-20n+51-\frac{n}{2}+\max\Bigl\{\Bigl[\frac{n}{2}\Bigr],13\Bigr\}\\ &< 2n^2-19n+45=\binom{m+n-5}{2}.\end{aligned}$$

Thus, $r(G_m, T_n^3) \leqslant m+n-5$ for $m = n-4, n-3$ by Lemma 2.1.

Now assume that $m-1 \mid n-5$. Then $m+n-6 = k(m-1)$ for $k \in \{2,3,\ldots\}$. Since $\Delta(\overline{kK_{m-1}}) = n-5$ we see that kK_{m-1} does not contain G_m as a subgraph and $\overline{kK_{m-1}}$ does not contain T_n as a subgraph. Hence $r(G_m,T_n) > k(m-1) = m+n-6$ and so $r(G_m,T_n) = m+n-5$. The proof is now complete. □

Theorem 4.4. *Let $m,n \in \mathbb{N}$, $n \geqslant 15$, $m \geqslant 7$, $n > m+1+8/(m-6)$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. If $2 \mid m(n-1)$, then $r(K_{1,m-1},T_n) = m+n-5$.*

P r o o f. By Euler's theorem or Lemma 2.4, $\mathrm{ex}(m+n-5;K_{1,m-1}) \leqslant \frac{1}{2}(m-2)\times(m+n-5)$. Thus, applying Lemma 4.1 we obtain $r(K_{1,m-1},T_n) \leqslant m+n-5$. Suppose that $2 \mid m(n-1)$. By Lemma 2.3,

$$r(K_{1,m-1},T_n) \geqslant m-1+n-4-\frac{1-(-1)^{(m-2)(n-5)}}{2} = m+n-5.$$

Thus the result follows. □

Corollary 4.5. *Let $n \in \mathbb{N}$, $n \geqslant 17$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Then $r(K_{1,n-3},T_n) = 2n-7$.*

P r o o f. Taking $m = n-2$ in Theorem 4.4 gives the result. □

Theorem 4.5. *Let $m,n \in \mathbb{N}$, $m \geqslant 6$, $n \geqslant m+3$ and $2 \nmid m(n-1)$. Then*

$$r(K_{1,m-1},T_n'') = r(K_{1,m-1},T_n''') = m+n-6.$$

P r o o f. Let G be a graph of order $m+n-6$ such that $\overline{G}$ does not contain any copies of $K_{1,m-1}$. That is, $\Delta(\overline{G}) \leqslant m-2$. Thus, $\delta(G) = m+n-7-\Delta(\overline{G}) \geqslant n-5$. If $\Delta(G) = n-5$, then G is a regular graph of order $m+n-6$ with degree $n-5$ and so $(m+n-6)(n-5) = 2e(G)$. Since $m+n-6$ and $n-5$ are odd, we get a contradiction. Thus, $\Delta(G) \geqslant n-4$. Assume that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n-4+c$, $\Gamma(v_0) =$

$\{v_1, \ldots, v_{n-4+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1 = \{u_1, \ldots, u_{m-3-c}\}$. Since $\delta(G) \geqslant n-5$, we see that for $v_i \in \Gamma(v_0)$, $|\Gamma(v_i) \cap \Gamma(v_0)| + 1 + |V_1'| \geqslant d(v_i) \geqslant n-5$ and so

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant n-5-1-(m-3-c) = n-m-3+c \geqslant c.$$

For $u_i \in V_1'$, we see that $|\Gamma(u_i) \cap \Gamma(v_0)| + |V_1'| - 1 \geqslant d(u_i) \geqslant n-5$ and so

$$|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant n-5-(m-4-c) = n-m-1+c \geqslant 2+c.$$

We first assume that $c = 0$. Since $|V_1'| = m-3 \geqslant 3$ and $\delta(G) \geqslant n-5$, we see that $|\Gamma(u_i) \cap \{v_1, \ldots, v_{n-4}\}| \geqslant n-5-(m-4) = n-m-1 \geqslant 2$ for $u_i \in V_1'$ and so $e(V_1V_1') \geqslant (m-3)(n-m-1)$. Since $n \geqslant m+3$ we see that $(m-4)n \geqslant (m-4)(m+3) = m^2-m-12 > m^2-2m-7$ and so $e(V_1V_1') \geqslant (m-3)(n-m-1) > n-4$. Therefore, $|\Gamma(v_i) \cap V_1'| \geqslant 2$ for some $i \in \{1, 2, \ldots, n-4\}$. With no loss of generality, we may suppose that $u_1v_i, u_2v_i, u_2v_j, u_3v_k \in E(G)$, where v_i, v_j, v_k are distinct vertices in $\Gamma(v_0)$. Thus G contains a copy of T_n'' and a copy of T_n'''.

Next we assume that $|V_1'| = m-3-c \geqslant 3$ and $c \geqslant 1$. Then $|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant 3$ for $i = 1, 2, 3$. Hence there are distinct vertices $v_j, v_k, v_l \in \Gamma(v_0)$ such that $u_1v_j, u_2v_k, u_3v_l \in E(G)$ and so G contains a copy of T_n'''. Since $d(v_j) \geqslant n-5 > 2$, v_j is adjacent to some vertex w different from v_0 and u_1. Hence, G contains a copy of T_n''.

Now assume that $|V_1'| = 2$. That is, $c = m-5$. Since $|\Gamma(u_i) \cap \Gamma(v_0)| \geqslant \delta(G) - 1 \geqslant n-6 \geqslant 3$ for $i = 1, 2$, and $|\Gamma(v_i) \cap \Gamma(v_0)| \geqslant n-m-3+c = n-8 \geqslant 1$ for $v_i \in \Gamma(v_0)$, it is easy to see that G contains a copy of T_n'' and a copy of T_n'''.

Suppose that $|V_1'| = 1$. Then $c = m-4 \geqslant 2$, $d(u_1) \geqslant \delta(G) \geqslant n-5 \geqslant 4$ and $d(v_i) \geqslant \delta(G) \geqslant n-5 \geqslant 4$ for $i = 1, 2, \ldots, n-4+m-4$. Hence G contains a copy of T_n'' and a copy of T_n'''.

Finally we assume that $|V_1'| = 0$. That is, $c = m-3$. Since $d(v_i) \geqslant \delta(G) \geqslant n-5 \geqslant 4$ for $i = 1, 2, \ldots, n-4+m-3$, it is easy to see that G contains a copy of T_n'' and a copy of T_n'''.

Suppose that $T_n \in \{T_n'', T_n'''\}$. By the above, G contains a copy of T_n. Hence $r(K_{1,m-1}, T_n) \leqslant m+n-6$. By Lemma 2.3, $r(K_{1,m-1}, T_n) \geqslant m-1+n-4-\frac{1}{2}(1-(-1)^{(m-2)(n-5)}) = m+n-6$. Thus $r(K_{1,m-1}, T_n) = m+n-6$ as asserted. □

Theorem 4.6. *Let $n \in \mathbb{N}$ with $n \geqslant 15$. Then $r(K_{1,n-4}, T_n^3) = 2n-8$.*

Proof. By Euler's theorem, $\mathrm{ex}(2n-8; K_{1,n-4}) \leqslant \frac{1}{2}(n-5)(2n-8)$. Thus, $r(K_{1,n-4}, T_n^3) \leqslant 2n-8$ by taking $G_m = K_{1,n-4}$ in Lemma 4.1. If $2 \nmid n$, from Lemma 2.3 we have $r(K_{1,n-4}, T_n^3) \geqslant n-4+n-4 = 2n-8$. Thus the result is true for odd n. Now assume that $2 \mid n$. Let G_0 be the graph of order $2n-9$ constructed in Theorem 3.2. Then G_0 does not contain T_n^3 as a subgraph. As $\delta(G_0) = n-5$, we have $\Delta(\overline{G}_0) = 2n-10-(n-5) = n-5$ and so $\overline{G}_0$ does not contain $K_{1,n-4}$ as a subgraph. Hence $r(K_{1,n-4}, T_n^3) > |V(G_0)| = 2n-9$ and so $r(K_{1,n-4}, T_n^3) = 2n-8$ as claimed. □

Theorem 4.7. *Let $n \in \mathbb{N}$ with $n \geqslant 10$. Then*

$$r(K_{1,n-2}, T_n^3) = r(K_{1,n-2}, T_n'') = r(K_{1,n-2}, T_n''') = 2n-5.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(K_{1,n-2}) = n-2$ and $\Delta(T_n) = n-4$, we have $r(K_{1,n-2}, T_n) \geqslant 2(n-2)-1 = 2n-5$ by Lemma 2.3 (ii). By Lemmas 2.4, 2.7 and 2.9,

$$\begin{aligned} \mathrm{ex}(2n-5; K_{1,n-2}) &= \left[\frac{(n-3)(2n-5)}{2}\right] = n^2-6n+8+\left[\frac{n-1}{2}\right], \\ \mathrm{ex}(2n-5; T_n) &= \frac{(n-2)(2n-5)-3(n-4)}{2} = n^2-6n+11. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{ex}(2n-5; K_{1,n-2}) + \mathrm{ex}(2n-5; T_n) &= n^2-6n+8+\left[\frac{n-1}{2}\right]+n^2-6n+11 \\ &< 2n^2-11n+15 = \binom{2n-5}{2}. \end{aligned}$$

Now, applying Lemma 2.1 yields $r(K_{1,n-2}, T_n) \leqslant 2n-5$ and so $r(K_{1,n-2}, T_n) = 2n-5$, which proves the theorem. □

5. Formulas for $r(T_m', T_n'')$, $r(T_m', T_n''')$ and $r(T_m', T_n^3)$

Theorem 5.1. *Let $m, n \in \mathbb{N}$, $n \geqslant 15$, $m \geqslant 7$ and $m-1 \mid n-5$. Suppose that $G_m \in \{P_m, T_m', T_m^*, T_m^1, T_m^2, T_m^3, T_m'', T_m'''\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $m \geqslant 10$ or $G_m \notin \{T_m^3, T_m'', T_m'''\}$. Then $r(G_m, T_n) = m+n-5$.*

Proof. Note that $m+n-5 \equiv 1 \pmod{m-1}$. By (2.1) and Lemmas 2.5, 2.6, 2.10 and 2.11, $\mathrm{ex}(m+n-5; G_m) \leqslant \frac{1}{2}(m-2)(m+n-5)$. Thus, applying Lemma 4.1 and the fact $n \geqslant m+4$ gives the result. □

Theorem 5.2. *Let $m, n \in \mathbb{N}$, $m \geqslant 9$, $n > m+2+\max\{0, (20-m)/(m-8)\}$ and $m-1 \nmid n-5$. Then*

$$r(T_m', T_n'') = r(T_m', T_n''') = r(T_m', T_n^3) = m+n-6.$$

P r o o f. Let $T_n \in \{T''_n, T'''_n, T^3_n\}$. Since $\Delta(T'_m) = m-2 < m-1$ and $\Delta(T_n) = n-4 > m-2$, we have $r(T'_m, T_n) \geqslant m-2+n-4 = m+n-6$ by Lemma 2.3 (ii)–(iii). Note that $m \geqslant 9$ and so $n \geqslant 15$. Since $n > m+2+(20-m)/(m-8)$, we see that $(m-8)n > m^2-7m+4$ and so $(m-5)(n-m+4) > 3(m+n-6)-(m-2)$.

Suppose that $T_n \neq T^3_n$ or $n \neq m+3$. From Lemmas 2.7 and 2.9, if $(m-5)(n-m+1) \geqslant 3(n-1)$, then

$$\begin{aligned}\operatorname{ex}(m+n-6; T_n) \leqslant &\frac{(n-2)(m+n-6)-(m-5)(n-m+4)}{2}\\ &+\frac{(m-5)(n-m+1)-3(n-1)}{2} = \frac{(n-5)(m+n-6)}{2};\end{aligned}$$

if $(m-5)(n-m+1) < 3(n-1)$, then

$$\begin{aligned}\operatorname{ex}(m+n-6; T_n) &= \frac{(n-2)(m+n-6)-(m-5)(n-m+4)}{2}\\ &< \frac{(n-2)(m+n-6)-3(m+n-6)+m-2}{2}\\ &= \frac{(n-5)(m+n-6)+m-2}{2}.\end{aligned}$$

Recall that $m-1 \nmid n-5$. By Lemma 2.5, $\operatorname{ex}(m+n-6; T'_m) \leqslant \frac{1}{2}((m-2)(m+n-6)-(m-2))$. Thus,

$$\begin{aligned}&\operatorname{ex}(m+n-6; T'_m) + \operatorname{ex}(m+n-6; T_n)\\ &\quad< \frac{(m-2)(m+n-6)-(m-2)}{2} + \frac{(n-5)(m+n-6)+m-2}{2} = \binom{m+n-6}{2}.\end{aligned}$$

Now applying Lemma 2.1 yields $r(T'_m, T_n) \leqslant m+n-6$ and so $r(T'_m, T_n) = m+n-6$.

Now assume that $T_n = T^3_n$ and $n = m+3$. Then $\max\{0, (20-m)/(m-8)\} < 1$ and so $m = n-3 \geqslant 15$. Also, $m+n-6 = 2n-9 = n-1+n-8 = 2m-3 = m-1+m-2$. From Lemma 2.9 (iii),

$$\operatorname{ex}(2n-9; T^3_n) = n^2 - 10n + 24 + \max\Bigl\{\Bigl[\frac{n}{2}\Bigr], 13\Bigr\}.$$

By Lemma 2.5, $\operatorname{ex}(2m-3; T'_m) = \frac{1}{2}((m-2)(2m-3)-(m-2)) = (m-2)^2 = (n-5)^2$. Thus,

$$\begin{aligned}&\operatorname{ex}(m+n-6; T'_m) + \operatorname{ex}(m+n-6; T^3_n)\\ &\qquad= (n-5)^2 + n^2 - 10n + 24 + \max\Bigl\{\Bigl[\frac{n}{2}\Bigr], 13\Bigr\}\\ &\qquad= 2n^2 - 20n + 49 + \max\Bigl\{\Bigl[\frac{n}{2}\Bigr], 13\Bigr\} < 2n^2 - 19n + 45 = \binom{2n-9}{2}.\end{aligned}$$

Applying Lemma 2.1 gives $r(T'_m, T^3_n) \leqslant m+n-6$ and so $r(T'_m, T^3_n) = m+n-6$ for $n = m+3$. This completes the proof. $\square$

Theorem 5.3. *Let $n \in \mathbb{N}$ with $n \geqslant 18$. Then*

$$r(T'_{n-3}, T''_n) = r(T'_{n-3}, T'''_n) = r(T'_{n-3}, T^3_n) = 2n - 9.$$

Proof. Suppose that $T_n \in \{T''_n, T'''_n, T^3_n\}$. Since $\Delta(T_n) = n - 4 > n - 5 = \Delta(T'_{n-3})$, from Lemma 2.3 (ii) we have $r(T'_{n-3}, T_n) \geqslant 2(n-4) - 1 = 2n - 9$. By Lemma 2.5, $\operatorname{ex}(2n-9; T'_{n-3}) = \frac{1}{2}(n-5)(2n-10) = n^2 - 10n + 25$. From Lemma 2.7 for $T_n \in \{T''_n, T'''_n\}$,

$$\begin{aligned}\operatorname{ex}(2n-9; T_n) &= \frac{(n-2)(2n-9) - 7(n-8)}{2} + \max\Big\{0, \Big\lceil \frac{4(n-8) - 3(n-1)}{2} \Big\rceil\Big\} \\ &= n^2 - 10n + 37 + \max\Big\{0, \Big\lceil \frac{n-29}{2} \Big\rceil\Big\} < n^2 - 9n + 20\end{aligned}$$

and so

$$\operatorname{ex}(2n-9; T'_{n-3}) + \operatorname{ex}(2n-9; T_n) < n^2 - 10n + 25 + n^2 - 9n + 20 = \binom{2n-9}{2}.$$

Now, applying Lemma 2.1 yields $r(T'_{n-3}, T_n) \leqslant 2n - 9$ and so $r(T'_{n-3}, T_n) = 2n - 9$. On the other hand, from Lemma 2.8 we have

$$\operatorname{ex}(2n-9; T^3_n) = n^2 - 10n + 24 + \max\Big\{\Big\lceil \frac{n}{2} \Big\rceil, 13\Big\} < n^2 - 9n + 20.$$

Thus,

$$\operatorname{ex}(2n-9; T'_{n-3}) + \operatorname{ex}(2n-9; T^3_n) < n^2 - 10n + 25 + n^2 - 9n + 20 = \binom{2n-9}{2}.$$

Applying Lemma 2.1, $r(T'_{n-3}, T^3_n) \leqslant 2n - 9$ and so $r(T'_{n-3}, T^3_n) = 2n - 9$, which completes the proof. □

Theorem 5.4. *Let $m, n \in \mathbb{N}$ with $n > m \geqslant 10$, and $T_m \in \{T''_m, T'''_m, T^3_m\}$. Then*

$$r(T_m, T'_n) = r(T_m, T^*_n) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid n - 3, \\ m + n - 4 & \text{if } m - 1 \nmid n - 3 \text{ and } n \geqslant (m-3)^2 + 2. \end{cases}$$

Proof. If $m - 1 \mid n - 3$, then $\operatorname{ex}(m+n-3; T_m) = \frac{1}{2}((m-2)(m+n-3) - (m-2))$ by Lemmas 2.7 and 2.9. Thus, the result follows from [9], Theorems 4.1 and 5.1.

Now assume that $m - 1 \nmid n - 3$. By Lemma 2.10, $\operatorname{ex}(m+n-4; T_m) < \frac{1}{2}(m-2) \times (m + n - 4)$. Applying [9], Theorems 4.4 and 5.4 deduces the result. The proof is now complete. □

6. Evaluation of $r(T_m^0, T_n)$ with $T_m^0 \in \{T_m^*, T_m^1, T_m^2\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$

Lemma 6.1 ([7], Theorem 8.3, pages 11–12)**.** *Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geqslant (a-1)(b-1)$, then there are two nonnegative integers x and y such that $n = ax + by$.*

Theorem 6.1. *Let $m, n \in \mathbb{N}$ with $m \geqslant 9$, $n > m+1+12/(m-8)$ and $m-1 \nmid n-5$. Suppose that $T_m^0 \in \{T_m^*, T_m^1, T_m^2\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $T_m^0 \neq T_m^*$ or $m \geqslant 11$. Then $r(T_m^0, T_n) = m+n-7$ or $m+n-6$. If $n \geqslant (m-3)^2+4$ or $m+n-7 = (m-1)x+(m-2)y$ for some nonnegative integers x and y, then $r(T_m^0, T_n) = m+n-6$.*

Proof. Note that $\Delta(T_m^0) = m-3 < n-4 = \Delta(T_n)$. Using Lemma 2.3 (ii)–(iii), $r(T_m^0, T_n) \geqslant m-3+n-4 = m+n-7$. Since $m-1 \nmid n-5$, from Lemmas 2.6, 2.11 and 2.12 we have $\operatorname{ex}(m+n-6; T_m^0) \leqslant \frac{1}{2}((m-2)(m+n-6)-(m-2))$.

We first assume that $T_n \neq T_n^3$ or $n \neq m+2, m+3$. By the proof of Theorem 5.2, $\operatorname{ex}(m+n-6; T_n) < \frac{1}{2}((n-5)(m+n-6)+m-2)$. Thus,

$$\begin{aligned}&\operatorname{ex}(m+n-6; T_m^0) + \operatorname{ex}(m+n-6; T_n)\\ &\quad< \frac{(m-2)(m+n-6)-(m-2)}{2} + \frac{(n-5)(m+n-6)+m-2}{2} = \binom{m+n-6}{2}.\end{aligned}$$

Hence, $r(T_m^0, T_n) \leqslant m+n-6$ by Lemma 2.1 and so $r(T_m^0, T_n) = m+n-6$ or $m+n-7$.

We next assume that $T_n = T_n^3$ and $n = m+2$. Then $m+n-6 = 2n-8 = n-1+n-7$, $m+2 > m+1+12/(m-8)$ and so $n-2 = m > 20$. By Lemma 2.9 (iv),

$$\begin{aligned}\operatorname{ex}(m+n-6; T_n^3) &= \operatorname{ex}(2n-8; T_n^3)\\ &= \frac{(n-2)(2n-8)-6(n-7)}{2} + \max\Big\{\Big[\frac{n-37}{4}\Big], 0\Big\}\\ &= n^2-9n+29+\max\Big\{\Big[\frac{n-37}{4}\Big], 0\Big\} < n^2-9n+29+\frac{n-22}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{ex}(m+n-6; T_m^0) + \operatorname{ex}(m+n-6; T_n) &< \frac{(n-4)(2n-9)}{2} + n^2-9n+29+\frac{n-22}{2}\\ &= (n-4)(2n-9) = \binom{2n-8}{2}.\end{aligned}$$

Hence $r(T_m^0, T_n^3) \leqslant m+n-6$ by Lemma 2.1 and so $r(T_m^0, T_n^3) = m+n-6$ or $m+n-7$.

Finally, we assume that $T_n = T_n^3$ and $n = m+3$. Then $m+n-6 = 2n-9 = n-1+n-8$, $m+3 > m+1+12/(m-8)$ and so $n-3 = m \geqslant 15$. From Lemma 2.9 (iii),

$$\begin{aligned}\operatorname{ex}(m+n-6; T_n^3) = \operatorname{ex}(2n-9; T_n^3) &= \frac{(n-2)(2n-9)-7n+30}{2} + \max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}\\ &= n^2-10n+24+\max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}.\end{aligned}$$

Recall that

$$\begin{aligned}\operatorname{ex}(m+n-6; T_m^0) = \operatorname{ex}(2m-3; T_m^0) &\leqslant \frac{(m-2)(2m-3)-(m-2)}{2}\\ &= (m-2)^2 = (n-5)^2.\end{aligned}$$

We then obtain

$$\begin{aligned}\operatorname{ex}(m+n-6; T_m^0) + \operatorname{ex}(m+n-6; T_n^3) &= (n-5)^2 + n^2 - 10n + 24 + \max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}\\ &= 2n^2 - 20n + 49 + \max\Big\{\Big[\frac{n}{2}\Big], 13\Big\}\\ &< 2n^2 - 19n + 45 = \binom{2n-9}{2}.\end{aligned}$$

Applying Lemma 2.1 gives $r(T_m^0, T_n^3) \leqslant m+n-6$ and so $r(T_m^0, T_n^3) = m+n-6$ or $m+n-7$ for $n = m+3$.

If $m+n-7 = (m-1)x + (m-2)y$ for some nonnegative integers x and y, setting $G = xK_{m-1} \cup yK_{m-2}$ we find that G does not contain any copies of T_m^0. Observe that $\Delta(\overline{G}) = n-5$ or $n-6$. We see that $\overline{G}$ does not contain any copies of T_n. Hence $r(T_m^0, T_n) > |V(G)| = m+n-7$ and so $r(T_m^0, T_n) = m+n-6$. If $n \geqslant (m-3)^2+4$, then $m+n-7 \geqslant (m-2)(m-3)$. By Lemma 6.1, $m+n-7 = (m-1)x + (m-2)y$ for some nonnegative integers x and y and so $r(T_m^0, T_n) = m+n-6$ as claimed.

Summarizing the above proves the theorem. □

References

[1] *S. A. Burr, P. Erdős*: Extremal Ramsey theory for graphs. Util. Math. *9* (1976), 247–258. zbl MR

[2] *G. Chartrand, L. Lesniak*: Graphs and Digraphs. Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, 1986. zbl MR doi

[3] *R. J. Faudree, R. H. Schelp*: Path Ramsey numbers in multicolorings. J. Comb. Theory, Ser. B *19* (1975), 150–160. zbl MR doi

[4] *J. W. Grossman, F. Harary, M. Klawe*: Generalized Ramsey theory for graphs, X: Double stars. Discrete Math. *28* (1979), 247–254. zbl MR doi

[5] *Y. Guo, L. Volkmann*: Tree-Ramsey numbers. Australas. J. Comb. *11* (1995), 169–175. zbl MR

[6] *F. Harary*: Recent results on generalized Ramsey theory for graphs. Graph Theory and Applications. Lecture Notes in Mathematics 303. Springer, Berlin, 1972, pp. 125–138. zbl MR doi

[7] *L. K. Hua*: Introduction to Number Theory. Springer, Berlin, 1982. zbl MR doi
[8] *S. P. Radziszowski*: Small Ramsey numbers. Electron. J. Comb. *2017* (2017), Article ID DS1, 104 pages. doi
[9] *Z.-H. Sun*: Ramsey numbers for trees. Bull. Aust. Math. Soc. *86* (2012), 164–176. zbl MR doi
[10] *Z.-H. Sun, Y.-Y. Tu*: Turán's problem for trees T_n with maximal degree $n-4$. Available at https://arxiv.org/abs/1410.7282 (2014), 28 pages.
[11] *Z.-H. Sun, L.-L. Wang*: Turán's problem for trees. J. Comb. Number Theory *3* (2011), 51–69. zbl MR
[12] *Z.-H. Sun, L.-L. Wang, Y.-L. Wu*: Turán's problem and Ramsey numbers for trees. Colloq. Math. *139* (2015), 273–298. zbl MR doi

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