

A NEW APPROACH TO HOM-LEFT-SYMMETRIC BIALGEBRAS

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Abstract. The main purpose of this paper is to consider a new definition of Hom-left-symmetric bialgebra. The coboundary Hom-left-symmetric bialgebra is also studied. In particular, we give a necessary and sufficient condition that s -matrix is a solution of the Hom- S -equation by a cocycle condition.

Keywords: Hom-left-symmetric algebra; Hom- S -equation; Hom-left-symmetric bialgebra

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1. INTRODUCTION

Hom-left-symmetric algebras were first introduced by Makhlof and Silvestrov in [5]. And then it was further investigated in [10] and [13]. Recently, Sheng and Chen constructed strict Hom-Lie 2-algebras from Hom-left-symmetric algebras in [7].

Yau developed a generalization of the classical Yang-Baxter equation (CYBE), a twisted generalization of the CYBE and the closely related object of Hom-Lie bialgebra in [9], [11], [12]. Sheng and Bai introduced a new definition of a Hom-Lie bialgebra and investigated their properties in [6]. Bimodule theory of Hom-left-symmetric algebra and Hom-left symmetric bialgebra were first considered in [8], but the related Hom- S -equation was not studied.

Inspired by the work of Sheng and Bai (see [6]), we give a new definition of Hom-left symmetric bialgebra. The coboundary Hom-left symmetric bialgebra and Hom- S -equation are also considered.

The paper is organized as follows. In Section 2, we review some necessary results on Hom-Lie algebras and Hom-left symmetric algebras. In Section 3, we introduce the

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new definition of Hom-left symmetric bialgebras and list some equivalent conditions. Finally, we consider the s -matrices and Hom- S -equations. In particular, we describe the condition for r being a solution of the Hom- S -equation using a cocycle condition.

2. PRELIMINARIES

Let us begin with some results on Hom-Lie algebras and Hom-left-symmetric algebras. For more details, one can refer to [3], [5], [6], [7], [8] and the references therein.

A Hom-Lie algebra is a triple $(\mathcal{G}, [\cdot, \cdot], \varphi)$ consisting of a linear space \mathcal{G} , a skew-symmetric bilinear map $[\cdot, \cdot]: \wedge^2 \mathcal{G} \rightarrow \mathcal{G}$ and an algebra morphism φ satisfying

$$[\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0$$

for any $x, y, z \in \mathcal{G}$. The Hom-Lie algebra $(\mathcal{G}, [\cdot, \cdot], \varphi)$ is said to be *regular* (involutive) if φ is nondegenerate (satisfies $\varphi^2 = I$).

There is a more general notion of Hom-Lie algebras introduced by Makhlouf and Silvestrov in [5], in which φ is only a homomorphism of linear spaces. A Hom-Lie algebra in this paper is called a *multiplicative Hom-Lie algebra* in [2].

A regular Hom-Lie algebra $(\mathcal{G}, [\cdot, \cdot], \varphi)$ is called a *symplectic Hom-Lie algebra* if there is a nondegenerate skew-symmetric 2-Hom-cocycle ω (the symplectic form) on \mathcal{G} , that is for any $x, y, z \in \mathcal{G}$

$$\omega([x, y], \varphi(z)) + \omega([y, z], \varphi(x)) + \omega([z, x], \varphi(y)) = 0.$$

We denote it by $(\mathcal{G}, [\cdot, \cdot], \varphi, \omega)$.

A Hom-left-symmetric algebra is a triple (A, \cdot, φ) , where A is a vector space, $\cdot: A \times A \rightarrow A$ is a bilinear map, and $\varphi \in \text{gl}(A, A)$ satisfies

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \quad (x, y, z)_\varphi = (y, x, z)_\varphi$$

for any $x, y, z \in A$. We denote $(x, y, z)_\varphi = (x \cdot y) \cdot \varphi(z) - \varphi(x) \cdot (y \cdot z)$ and call it φ -*associator*.

Let (A, φ) be a Hom-left-symmetric algebra and V a vector space. Let $S, T: A \rightarrow \text{gl}(V)$ be two linear maps and $\psi \in \text{gl}(V)$. The quadruple (S, T, V, ψ) is called a *bimodule* of (A, φ) if for any $x, y \in A, v \in V$,

$$(2.1) \quad \psi(S(x)v) = S(\varphi(x))\psi(v), \quad \psi(T(x)v) = T(\varphi(x))\psi(v),$$

$$(2.2) \quad S(\varphi(x))S(y)v - S(xy)\psi(v) = S(\varphi(y))S(x)v - S(yx)\psi(v),$$

$$(2.3) \quad S(\varphi(x))T(y)v - T(\varphi(y))S(x)v = T(xy)\psi(v) - T(\varphi(y))T(x)v.$$

Let L (or R) be the left (or right) multiplication operator associated to (A, \cdot, φ) , i.e.

$$L(x_0)x_1 = R(x_1)x_0 = x_0x_1 \quad \forall x_0, x_1 \in A.$$

Then (A, L, R, φ) is a representation of (A, φ) called the *regular representation*.

Suppose that (S, T, V, ψ) is a bimodule on Hom-left-symmetric algebra (A, \cdot, φ) . Let $S^*, T^* : A \rightarrow \text{gl}(V^*)$, $\varphi^* : A^* \rightarrow A^*$, $\psi^* : V^* \rightarrow V^*$ be the dual maps of φ and ψ , respectively, given by

$$\begin{aligned} \langle S^*(x)u^*, v \rangle &= -\langle S(x)v, u^* \rangle, & \langle T^*(x)u^*, v \rangle &= -\langle T(x)v, u^* \rangle, \\ \varphi^*(x^*)(y) &= x^*(\varphi(y)), & \psi^*(u^*)(v) &= u^*(\psi(v)). \end{aligned}$$

If in addition for all $x, y \in A$, $x^* \in A^*$, $u^* \in V^*$, $v \in V$, the following hold:

$$(2.4) \quad \psi(S(\varphi(x))v) = S(x)\psi(v), \quad \psi(T(\varphi(x))v) = T(x)\psi(v),$$

$$(2.5) \quad S(x)S(\varphi(y))v - \psi(S(xy)v) = S(y)S(\varphi(x))v - \psi(S(yx)v),$$

$$(2.6) \quad S(x)T(\varphi(y))v - T(y)S(\varphi(x))v = \psi(T(xy)v) - T(y)T(\varphi(x))v,$$

then $(S^* - T^*, -T^*, V^*, \psi^*)$ becomes a bimodule of (A, φ) .

Let (A, \cdot, φ) and (B, \circ, ψ) be two Hom-left-symmetric algebras. Suppose that there are linear maps $l_A, r_A : A \rightarrow \text{gl}(B)$ and $l_B, r_B : B \rightarrow \text{gl}(A)$ such that (l_A, r_A, ψ) is a bimodule of A and (l_B, r_B, φ) is a bimodule of B and for any $x, y \in A$, $a, b \in B$ they satisfy the following conditions:

$$\begin{aligned} r_A(\varphi(x))[a, b] &= r_A(l_B(b)x)\psi(a) - r_A(l_B(a)x)\psi(b) + \psi(a) \circ (r_A(x)b) \\ &\quad - \psi(b) \circ (r_A(x)a), \\ l_A(\varphi(x))(a \circ b) &= -l_A(l_B(a)x - r_B(a)x)\psi(b) + (l_A(x)a - r_A(x)a) \circ \psi(b) \\ &\quad + r_A(r_B(b)x)\psi(a) + \psi(a) \circ (l_A(x)b), \\ r_B(\psi(a))[x, y] &= r_B(l_A(y)a)\varphi(x) - r_B(l_A(x)a)\varphi(y) + \varphi(x) \cdot (r_B(a)y) \\ &\quad - \varphi(y) \cdot (r_B(a)x), \\ l_B(a)(x \cdot y) &= -l_B(l_A(x)a - r_A(x)a)\varphi(y) + (l_B(a)x - r_B(a)x) \cdot \varphi(y) \\ &\quad + r_B(r_A(y)a)\varphi(x) + \varphi(x) \cdot (l_B(a)y). \end{aligned}$$

Then there is a Hom-left-symmetric algebra structure on the vector space $A \oplus B$ given by

$$(x + a) * (y + b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a).$$

We denote this Hom-left-symmetric algebra by $(A \bowtie_{l_B, r_B}^{l_A, r_A} B, \varphi \oplus \psi)$ or simply $(A \bowtie B, \varphi \oplus \psi)$. And $(A, B, l_A, r_A, l_B, r_B, \varphi, \psi)$ satisfying the above conditions is called a *matched pair* of Hom-left-symmetric algebras. On the other hand, every Hom-left-symmetric algebra which is a direct sum of the underlying vector spaces of two subalgebras can be obtained in the above way.

3. HOM-LEFT-SYMMETRIC BIALGEBRAS

Let (S, T, V, ψ) be a bimodule of Hom-left-symmetric algebra (A, \cdot, φ) . In view of Section 2, we know that $(S^* - T^*, -T^*, V^*, \psi^*)$ is not a bimodule in general. If $(S^* - T^*, -T^*, V^*, \psi^*)$ is a bimodule of (A, \cdot, φ) , we say that (S, T, V, ψ) is an admissible bimodule. In this section, we mainly focus on the regular representation.

Proposition 3.1. *Let (A, \cdot, φ) be a Hom-left-symmetric algebra. The regular representation (A, φ, L, R) is admissible if and only if the following equations hold:*

$$(3.1) \quad (\varphi^2(x) - x)\varphi(y) = 0, \quad \varphi(y)(\varphi^2(x) - x) = 0,$$

$$(3.2) \quad x(\varphi(y)z) - \varphi((xy)z) = y(\varphi(x)z) - \varphi((yx)z),$$

$$(3.3) \quad x(z\varphi(y)) - (\varphi(x)z)y = \varphi(z(xy)) - (z\varphi(x))y.$$

Proof. Using (2.4)–(2.6), we can get the results. □

Definition 3.2. A Hom-left-symmetric algebra (A, \cdot, φ) is *admissible* if its regular representation is admissible.

Clearly, we have the following result:

Proposition 3.3. *If a Hom-left-symmetric algebra (A, \cdot, φ) is admissible, then the sub-adjacent Hom-Lie algebra $(\mathcal{G}(A), [\cdot, \cdot], \varphi)$ is also admissible.*

Corollary 3.4. *Let (A, \cdot, φ) be a regular admissible Hom-left-symmetric algebra. Then*

$$L^*(x)\varphi^{*2}(\xi) = L^*(x)(\xi), \quad R^*(x)\varphi^{*2}(\xi) = R^*(x)(\xi).$$

Proof. If (A, \cdot, φ) is regular, then we have $\varphi^2(x) \cdot y = x \cdot y$. Hence,

$$\begin{aligned} \langle L^*(x)\varphi^{*2}(\xi), y \rangle &= -\langle \xi, \varphi^2(x \cdot y) \rangle = -\langle \xi, \varphi^2(x) \cdot \varphi^2(y) \rangle \\ &= \langle \xi, x \cdot \varphi^2(y) \rangle = -\langle \xi, x \cdot y \rangle = -\langle L^*(x)\xi, y \rangle. \end{aligned}$$

It follows that $L^*(x)\varphi^{*2}(\xi) = L^*(x)(\xi)$. Similarly, $R^*(x)\varphi^{*2}(\xi) = R^*(x)(\xi)$. □

Proposition 3.5. Let (A, \cdot, φ) and (A^*, \circ, φ^*) be two regular admissible Hom-left-symmetric algebras. Then we get

$$\xi \circ L^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ L^*(\varphi(x))\eta, \quad \xi \circ R^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ R^*(\varphi(x))\eta$$

for any $x \in A, \xi, \eta \in A^*$.

Proof. For any $x, y \in A, \xi, \eta \in A^*$,

$$\begin{aligned} \langle \xi \circ R^*(\varphi(x))\eta, y \rangle &= - \langle R^*(\varphi(x))\eta, L^*(\xi)y \rangle \\ &= \langle \eta, \varphi^2(L^*(\xi)y)\varphi(x) \rangle = \langle \xi \varphi^{*2} R^*(\varphi(x))\eta, y \rangle. \end{aligned}$$

It follows that

$$\xi \circ R^*(\varphi(x))\eta = \xi \circ \varphi^{*2} R^*(\varphi(x))\eta.$$

In view of (A^*, \circ, φ^*) also being admissible, we have

$$\xi \circ \varphi^{*2} R^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ \varphi^{*2} R^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ R^*(\varphi(x))\eta.$$

Therefore, we get

$$\xi \circ R^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ R^*(\varphi(x))\eta.$$

Analogously,

$$\xi \circ L^*(\varphi(x))\eta = \varphi^{*2}(\xi) \circ L^*(\varphi(x))\eta.$$

□

Definition 3.6. A pair of admissible Hom-left-symmetric algebras (A, \cdot, φ) and (A^*, \circ, φ^*) is called a *Hom-left-symmetric bialgebra* if

$$(3.4) \quad \langle \Delta[x, y], \varphi^*(\xi) \otimes \eta \rangle = \langle (L(\varphi(x)) \otimes \varphi + \varphi \otimes \text{ad}_{\varphi(x)})\Delta(y), \varphi^*(\xi) \otimes \eta \rangle \\ - \langle (L(\varphi(y)) \otimes \varphi + \varphi \otimes \text{ad}_{\varphi(y)})\Delta(x), \varphi^*(\xi) \otimes \eta \rangle,$$

$$(3.5) \quad \langle \Delta^*[x, y], \varphi(x) \otimes y \rangle = \langle (L_\circ(\varphi^*(\xi)) \otimes \varphi^* + \varphi^* \otimes \text{ad}_{\varphi^*(\xi)})\Delta^*(\eta), \varphi(x) \otimes y \rangle \\ - \langle (L_\circ(\varphi^*(\eta)) \otimes \varphi^* + \varphi^* \otimes \text{ad}_{\varphi^*(\eta)})\Delta^*(\xi), \varphi(x) \otimes y \rangle.$$

We denote a Hom-left-symmetric bialgebra by $(A, A^*, \varphi, \varphi^*)$.

Theorem 3.7. A pair of admissible Hom-left-symmetric algebras (A, \cdot, φ) and (A^*, \circ, φ^*) is a Hom-left-symmetric bialgebra if and only if $(\mathcal{G}(A), \mathcal{G}(A^*), L^*, L_\circ^*)$ is a matched pair of Hom-Lie algebras.

Proof. $(\mathcal{G}(A), \mathcal{G}(A^*), L^*, L^*_\circ)$ is a matched pair of Hom-Lie algebras if and only if

$$(3.6) \quad L^*_\circ(\varphi^*(\xi))[x, y] = [L^*_\circ(\xi)x, \varphi(y)] + [\varphi(x), L^*_\circ(\xi)y] + L^*_\circ(L.(y)\xi)\varphi(x) \\ - L^*_\circ(L.(x)\xi)\varphi(y)$$

and

$$(3.7) \quad L^*(\varphi(x))[\xi, \eta] = [L^*(x)\xi, \varphi^*(\eta)] + [\varphi^*(\xi), L^*(x)\eta] + L^*(L_\circ(\eta)x)\varphi^*(\xi) \\ - L^*(L_\circ(\xi)x)\varphi^*(\eta).$$

According to (3.6), we get

$$\begin{aligned} & \langle -L^*_\circ(\varphi^*(\xi))[x, y] + [L^*_\circ(\xi)x, \varphi(y)] + [\varphi(x), L^*_\circ(\xi)y] \\ & \quad + L^*_\circ(L.(y)\xi)\varphi(x) - L^*_\circ(L.(x)\xi)\varphi(y), \eta \rangle \\ & = \langle [x, y], \varphi^*(\xi) \circ \eta \rangle - \langle \text{ad}_{\varphi(y)}L^*_\circ(\xi)x, \eta \rangle + \langle \text{ad}_{\varphi(x)}L^*_\circ(\xi)y, \eta \rangle \\ & \quad - \langle \varphi(x), (L.(y)\xi) \circ \eta \rangle + \langle \varphi(y), (L.(x)\xi) \circ \eta \rangle \\ & = \langle [x, y], \varphi^*(\xi) \circ \eta \rangle - \langle x, \xi \circ \text{ad}^*_{\varphi(y)}\eta \rangle + \langle y, \xi \circ \text{ad}^*_{\varphi(x)}\eta \rangle \\ & \quad - \langle x, \varphi^*((L.(y)\xi)) \circ \varphi^*(\eta) \rangle + \langle y, \varphi^*((L.(x)\xi)) \circ \varphi^*(\eta) \rangle \\ & = \langle [x, y], \varphi^*(\xi) \circ \eta \rangle - \langle x, \varphi^{*2}(\xi) \circ \text{ad}^*_{\varphi(y)}\eta \rangle + \langle y, \varphi^{*2}(\xi) \circ \text{ad}^*_{\varphi(x)}\eta \rangle \\ & \quad - \langle x, L^*(\varphi(y))\varphi^*(\xi) \circ \varphi^*(\eta) \rangle + \langle y, L^*(\varphi(x))\varphi^*(\xi) \circ \varphi^*(\eta) \rangle \\ & = \langle \Delta[x, y], \varphi^*(\xi) \otimes \eta \rangle - \langle \Delta(x), \varphi^{*2}(\xi) \otimes \text{ad}^*_{\varphi(y)}\eta \rangle \\ & \quad + \langle \Delta(y), \varphi^{*2}(\xi) \otimes \text{ad}^*_{\varphi(x)}\eta \rangle - \langle \Delta(x), L^*(\varphi(y))\varphi^*(\xi) \otimes \varphi^*(\eta) \rangle \\ & \quad + \langle \Delta(y), L^*(\varphi(x))\varphi^*(\xi) \otimes \varphi^*(\eta) \rangle \\ & = \langle \Delta[x, y], \varphi^*(\xi) \otimes \eta \rangle - \langle (\varphi \otimes \text{ad}_{\varphi(y)})\Delta(x), \varphi^*(\xi) \otimes \eta \rangle \\ & \quad + \langle (\varphi \otimes \text{ad}_{\varphi(x)})\Delta(y), \varphi^*(\xi) \otimes \eta \rangle \\ & \quad - \langle (L.(\varphi(y)) \otimes \varphi)\Delta(x), \varphi^*(\xi) \otimes \eta \rangle \\ & \quad + \langle (L.(\varphi(x)) \otimes \varphi)\Delta(y), \varphi^*(\xi) \otimes \eta \rangle \\ & = 0, \end{aligned}$$

which implies that (3.4) is equivalent to (3.6). Similarly, we can verify that (3.5) is equivalent to (3.7). \square

Theorem 3.8. *Let (A, \cdot, φ) and (A^*, \circ, φ^*) be two admissible Hom-left-symmetric algebras. Then the following conditions are equivalent.*

- (i) $(A, A^*, \varphi, \varphi^*)$ is a Hom-left-symmetric bialgebra.

(ii) $(\mathcal{G}(A) \bowtie \mathcal{G}(A)^*, \mathcal{G}(A), \mathcal{G}(A^*), \omega_p)$ is a parakähler Hom-Lie algebra, where ω_p is given by

$$\omega_p(x + \xi, y + \eta) = \langle \xi, y \rangle - \langle x, \eta \rangle$$

for any $x, y \in A, \xi, \eta \in A^*$.

(iii) $(\mathcal{G}(A), \mathcal{G}(A^*), L^*, L_o^*, \varphi, \varphi^*)$ is a matched pair of Hom-Lie algebras.

(iv) $(A, A^*, L^* - R^*, -R_o^*, L_o^* - R_o^*, -R_o^*, \varphi, \varphi^*)$ is a matched pair of Hom-left-symmetric algebras.

Proof. According to Theorem 3.7, (i) \Leftrightarrow (iii). Due to Theorem 3.9 in [8], (iv) \Leftrightarrow (iii). According to Theorem 2.13 in [8], (ii) \Leftrightarrow (iii). \square

Example 3.9. Let $(A, A^*, \varphi, \varphi^*)$ be a Hom-left-symmetric bialgebra. Then its dual $(A^*, A, \varphi^*, \varphi)$ is also a Hom-left-symmetric bialgebra.

Example 3.10. Let $(\mathcal{G}, \varphi, \omega)$ be a symplectic Hom-Lie algebra. Suppose that $r \in \wedge^2 \mathcal{G}$ is a nondegenerate classical r -matrix satisfying $(\varphi \otimes \varphi)r = r$, and $r^\sharp: \mathcal{G}^* \rightarrow \mathcal{G}$ is an induced linear map given by

$$\langle r^\sharp(\xi), \eta \rangle = \langle r, \xi \wedge \eta \rangle$$

and satisfying

$$\omega(x, y) = \langle r^{\sharp^{-1}}(x), y \rangle.$$

According to [6], $(\mathcal{G}, \mathcal{G}^*)$ is a Hom-Lie bialgebra with

$$\langle \Delta(x), \varphi^*(\xi) \otimes \eta \rangle = \langle (\text{ad}_x \otimes \varphi + \varphi \otimes \text{ad}_x)r, \varphi^*(\xi) \otimes \eta \rangle$$

for any $\xi, \eta \in \mathcal{G}^*$.

On the other hand, there exists a Hom-left-symmetric algebra structure \cdot on \mathcal{G} given by $\omega(x \cdot y, \varphi(z)) = \omega(\varphi(y), [x, z])$ for all $x, y, z \in \mathcal{G}$. Furthermore, there is a compatible left-symmetric algebra structure on the Lie algebra \mathcal{G}^* given by $a \circ b = r^{\sharp^{-1}}(r^\sharp(a) \cdot r^\sharp(b))$ for any $a, b \in \mathcal{G}^*$.

Moreover, by direct calculation, we have

$$L^*(x)a = r^{\sharp^{-1}}[x, r^\sharp(a)], \quad R_o^*(x)a = -r^{\sharp^{-1}}[r^\sharp(a) \cdot x]$$

and

$$L_o^*(a)x = [r^\sharp(a), x], \quad R_o^*(a)x = -x \cdot r^\sharp(a)$$

for all $x \in \mathcal{G}, a \in \mathcal{G}^*$. Hence, according to Theorem 3.8, as left-symmetric algebras, $(\mathcal{G}, \mathcal{G}^*)$ is a left-symmetric bialgebra if and only if $[[x, y], \varphi(z)] = 0$ for any $x, y, z \in \mathcal{G}$.

4. s -MATRICES AND HOM- S -EQUATION

For any $r \in \text{Sym}^2(A)$, the induced linear map $r^\sharp: A^* \rightarrow A$ is given by $\langle r^\sharp(\xi), \eta \rangle = \langle r, \xi \otimes \eta \rangle$ for any $\xi, \eta \in A^*$. We say that r is invertible if the linear map r^\sharp is an isomorphism.

Definition 4.1. A *coboundary Hom-left-symmetric bialgebra* is a Hom-left-symmetric bialgebra (A, A^*) such that

$$(4.1) \quad \langle \Delta(x), \varphi^*(\xi) \otimes \eta \rangle = \langle (L(x) \otimes \varphi + \varphi \otimes \text{ad}_x)r, \varphi^*(\xi) \otimes \eta \rangle,$$

where $r \in \text{Sym}^2(A)$ satisfies

$$(4.2) \quad \varphi r^\sharp \varphi^* = r^\sharp.$$

Obviously, $\varphi r^\sharp \varphi^* = r^\sharp$ is equivalent to $(\varphi \otimes \varphi)r = r$.

Proposition 4.2. Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. Then for any $\xi \in \text{Im}(\varphi^*)$ and $\eta \in A^*$ we get

$$(4.3) \quad \xi \circ \eta = \text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi.$$

Proof. Suppose $r = X \otimes Y$, we obtain

$$\begin{aligned} \langle x, \xi \circ \eta \rangle &= \langle \Delta(x), \xi \otimes \eta \rangle \\ &= \langle (L(x) \otimes \varphi + \varphi \otimes \text{ad}_x)(X \otimes Y), \xi \otimes \eta \rangle \\ &= \langle xX \otimes \varphi(Y), \xi \otimes \eta \rangle + \langle \varphi(X) \otimes [x, Y], \xi \otimes \eta \rangle \\ &= \langle xX, \xi \rangle \langle \varphi(Y), \eta \rangle + \langle xX, \eta \rangle \langle \varphi(Y), \xi \rangle + \langle \varphi(X), \xi \rangle \langle [x, Y], \eta \rangle \\ &\quad + \langle \varphi(X), \eta \rangle \langle [x, Y], \xi \rangle \\ &= \langle x \langle Y, \varphi^*(\eta) \rangle X, \xi \rangle + \langle x \langle Y, \varphi^*(\xi) \rangle X, \eta \rangle + \langle [x, \langle X, \varphi^*(\xi) \rangle Y], \eta \rangle \\ &\quad + \langle [x, \langle X, \varphi^*(\eta) \rangle Y], \xi \rangle \\ &= \langle x \cdot r^\sharp \varphi^*(\eta), \xi \rangle + \langle [x, r^\sharp \varphi^*(\xi)], \eta \rangle \\ &= - \langle x, R^*(r^\sharp \varphi^*(\eta))\xi \rangle + \langle x, \text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta \rangle. \end{aligned}$$

It follows that

$$\xi \circ \eta = \text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi.$$

□

Corollary 4.3. *Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. Then we get*

$$(4.4) \quad \varphi^*(\xi \circ \eta) = \varphi^*(\text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi)$$

for any $\xi, \eta \in A^*$.

Proof. For any $x \in A$, since A is admissible, using (3.1) and (4.2),

$$\begin{aligned} \langle R^*(r^\sharp \varphi^{*2}(\xi))\varphi^*(\eta), x \rangle &= -\langle \varphi^*(\eta), x \cdot r^\sharp \varphi^{*2}(\xi) \rangle = -\langle \eta, \varphi(x) \cdot \varphi r^\sharp \varphi^{*2}(\xi) \rangle \\ &= -\langle \eta, \varphi(x) \cdot r^\sharp \varphi(\xi) \rangle = -\langle \eta, \varphi(x) \cdot \varphi^2 r^\sharp \varphi(\xi) \rangle \\ &= -\langle \varphi^* \eta, x \cdot \varphi r^\sharp \varphi(\xi) \rangle = \langle R^*(\varphi r^\sharp \varphi(\xi))\varphi^* \eta, x \rangle \\ &= \langle \varphi^* R^*(r^\sharp \varphi(\xi))\eta, x \rangle. \end{aligned}$$

Hence,

$$(4.5) \quad \varphi^* R^*(r^\sharp \varphi(\xi))\eta = R^*(r^\sharp \varphi^{*2}(\xi))\varphi^*(\eta).$$

Analogously,

$$\varphi^* L^*(r^\sharp \varphi(\xi))\eta = L^*(r^\sharp \varphi^{*2}(\xi))\varphi^*(\eta).$$

Therefore,

$$(4.6) \quad \varphi^* \text{ad}_{r^\sharp \varphi(\xi)}^* \eta = \text{ad}_{r^\sharp \varphi^{*2}(\xi)}^* \varphi^*(\eta).$$

By Proposition 4.2, we obtain

$$(4.7) \quad \varphi^*(\xi \circ \eta) = \varphi^*(\xi) \circ \varphi^*(\eta) = \text{ad}_{r^\sharp \varphi^{*2}(\xi)}^* \varphi^*(\eta) - R^*(r^\sharp \varphi^{*2}(\eta))\varphi^*(\xi).$$

Combining this with (4.5)–(4.7), we get the conclusion. \square

Corollary 4.4. *Let (A, A^*) be a coboundary Hom-left-symmetric bialgebra. If A is regular, then for any $\xi \in A^*$, we have*

$$(4.8) \quad R^*(r^\sharp \varphi^*(\xi)) = R^*(\varphi r^\sharp(\xi)), \quad \text{ad}_{r^\sharp \varphi^*(\xi)}^* = \text{ad}_{\varphi r^\sharp \xi}, \quad L^*(r^\sharp \varphi^*(\xi)) = L^*(\varphi r^\sharp(\xi)).$$

Proof. Due to (2.1) and (4.5),

$$R^*(r^\sharp \varphi^{*2}(\xi))\varphi^*(\eta) = \varphi^* R^*(r^\sharp \varphi(\xi))\eta = R^*(\varphi r^\sharp \varphi(\xi))\varphi^*(\eta).$$

If A is regular, then we have

$$R^*(r^\sharp \varphi^*(\xi)) = R^*(\varphi r^\sharp(\xi)).$$

Similarly,

$$L^*(r^\sharp \varphi^*(\xi)) = L^*(\varphi r^\sharp(\xi)) \quad \text{and} \quad \text{ad}_{r^\sharp \varphi^*(\xi)}^* = \text{ad}_{\varphi r^\sharp(\xi)}^*.$$

\square

Definition 4.5. Let (A, \cdot, φ) be a Hom-left-symmetric algebra. For any $r = \sum_i a_i \otimes b_i \in \text{Sym}^2(A)$, $[[r, r]]$ is called *Hom-S-equation* in (A, \cdot, φ) , where

$$[[r, r]] = \sum_{i,j} -a_i \cdot a_j \otimes \varphi(b_i) \otimes \varphi(b_j) + \varphi(a_i) \otimes b_i \cdot a_j \otimes \varphi(b_j) + \varphi(a_i) \otimes \varphi(a_i) \otimes [b_i, b_j].$$

The Hom-S-equation in a Hom-left-symmetric algebra is an analogue of the Hom-Yang-Baxter equation in a Hom-Lie algebra, see [11]. When $\varphi = I$, Hom-S-equation becomes the S-equation discussed in [1].

Lemma 4.6. Let (A, \cdot, φ) be a regular admissible Hom-left-symmetric algebra. If $r \in \text{Sym}^2(A)$ satisfies $\varphi r^\sharp \varphi^* = r^\sharp$ and $\xi \circ \eta = \text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi$, then we get

$$(4.9) \quad [[r, r]](\xi, \eta) = [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)] + r^\sharp \varphi^*(\eta \circ \xi) - r^\sharp \varphi^*(\xi \circ \eta).$$

Proof. For any $\gamma \in A^*$, using Corollaries 4.3 and 4.4,

$$(4.10) \quad \begin{aligned} \langle r^\sharp \varphi^*(\xi \circ \eta), \gamma \rangle &= \langle \text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi, \varphi r^\sharp(\gamma) \rangle \\ &= \langle \xi, \varphi r^\sharp(\gamma) \cdot r^\sharp \varphi^*(\eta) \rangle - \langle \eta, [r^\sharp \varphi^*(\xi), \varphi r^\sharp(\gamma)] \rangle \\ &= \langle \xi, r^\sharp \varphi^*(\gamma) \cdot r^\sharp \varphi^*(\eta) \rangle - \langle \eta, [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\gamma)] \rangle. \end{aligned}$$

On the other hand,

$$(4.11) \quad \begin{aligned} [[r, r]](\xi, \eta, \gamma) &= -\langle \xi, r^\sharp \varphi^*(\eta) \cdot r^\sharp \varphi^*(\gamma) \rangle + \langle \eta, r^\sharp \varphi^*(\xi) \cdot r^\sharp \varphi^*(\gamma) \rangle \\ &\quad + \langle \gamma, [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)] \rangle. \end{aligned}$$

Combining this with (4.10) and (4.11), we obtain

$$[[r, r]](\xi, \eta) = [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)] + r^\sharp \varphi^*(\eta \circ \xi) - r^\sharp \varphi^*(\xi \circ \eta).$$

□

If A is regular and (A, A^*) is a coboundary Hom-left-symmetric bialgebra, then (4.1)

$$\Delta(x) = (L.(x) \otimes \varphi + \varphi \otimes \text{ad}_x)r$$

and for any $\xi, \eta \in A^*$, $\xi \circ \eta$ is given by (4.3).

Theorem 4.7. Let A be a regular admissible Hom-left-symmetric algebra. Define a bilinear map $\circ: A^* \otimes A^* \rightarrow A^*$ by (4.3) for some $r \in \text{Sym}^2(A)$ satisfying (4.2). Then (A^*, \circ, φ^*) is a Hom-left-symmetric algebra if and only if

$$(L(x) \otimes \varphi \otimes \varphi + \varphi \otimes L(x) \otimes \varphi + \varphi \otimes \varphi \otimes \text{ad}_x)[[r, r]] = 0.$$

Under this condition, (A, A^*) is a coboundary Hom-left-symmetric bialgebra.

Proof. Clearly, φ^* is an algebra homomorphism. For any $\xi, \eta, \gamma \in A^*$, in view of Lemma 4.6, we have

$$\begin{aligned}
& (\xi, \eta, \gamma)_{\varphi^*} - (\eta, \xi, \gamma)_{\varphi^*} \\
&= (\xi \circ \eta) \circ \varphi^*(\gamma) - \varphi^*(\xi) \circ (\eta \circ \gamma) - \varphi^*(\xi) \circ (\eta \circ \gamma) - (\eta \circ \xi) \circ \varphi^*(\gamma) \\
&\quad + \varphi^*(\eta) \circ (\xi \circ \gamma) \\
&= \text{ad}_{r^\sharp \varphi^*(\xi \circ \eta)} \varphi^*(\gamma) - R^*(r^\sharp \varphi^{*2}(\gamma))(\xi \circ \eta) - \text{ad}_{r^\sharp \varphi^*(\eta \circ \xi)} \varphi^*(\gamma) \\
&\quad + R^*(r^\sharp \varphi^{*2}(\gamma))(\eta \circ \xi) - \text{ad}_{r^\sharp \varphi^{*2}(\xi)}(\eta \circ \gamma) + R^*(r^\sharp \varphi^*(\eta \circ \gamma))\varphi^*(\xi) \\
&\quad + \text{ad}_{r^\sharp \varphi^{*2}(\eta)}(\xi \circ \gamma) - R^*(r^\sharp \varphi^*(\xi \circ \gamma))\varphi^*(\eta) \\
&= \text{ad}_{r^\sharp \varphi^*(\xi \circ \eta)} \varphi^*(\gamma) - R^*(r^\sharp \varphi^{*2}(\gamma))(\text{ad}_{r^\sharp \varphi^*(\xi)} \eta - R^*(r^\sharp \varphi^*(\eta))\xi) \\
&\quad - \text{ad}_{r^\sharp \varphi^*(\eta \circ \xi)} \varphi^*(\gamma) + R^*(r^\sharp \varphi^{*2}(\gamma))(\text{ad}_{r^\sharp \varphi^*(\eta)} \xi - R^*(r^\sharp \varphi^*(\xi))\eta) \\
&\quad - \text{ad}_{r^\sharp \varphi^{*2}(\xi)}(\text{ad}_{r^\sharp \varphi^*(\eta)} \gamma - R^*(r^\sharp \varphi^*(\gamma))\eta) + R^*(r^\sharp \varphi^*(\eta \circ \gamma))\varphi^*(\xi) \\
&\quad + \text{ad}_{r^\sharp \varphi^{*2}(\eta)}(\text{ad}_{r^\sharp \varphi^*(\xi)} \gamma - R^*(r^\sharp \varphi^*(\gamma))\xi) - R^*(r^\sharp \varphi^*(\xi \circ \gamma))\varphi^*(\eta) \\
&= R^*(r^\sharp \varphi^*(\eta \circ \gamma) - r^\sharp \varphi^*(\eta)r^\sharp \varphi^*(\gamma))\varphi^*(\xi) \\
&\quad + R^*(r^\sharp \varphi^*(\xi \circ \gamma) - r^\sharp \varphi^*(\xi)r^\sharp \varphi^*(\gamma))\varphi^*(\eta) \\
&\quad + \text{ad}_{r^\sharp \varphi^*([\xi, \eta]) - [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)]} \varphi^*(\gamma) \\
&= R^*([r, r](\eta, \gamma)\varphi^*(\xi) + R^*([r, r](\xi, \gamma)\varphi^*(\eta) + \text{ad}_{[[r, r]](\xi, \eta)}^* \varphi^*(\gamma)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \langle (\xi, \eta, \gamma)_{\varphi^*} - (\eta, \xi, \gamma)_{\varphi^*}, x \rangle \\
&= \langle R^*([r, r](\eta, \gamma))\varphi^*(\xi) + R^*([r, r](\xi, \gamma))\varphi^*(\eta) - \text{ad}_{[[r, r]](\xi, \eta)}^* \varphi^*(\gamma), x \rangle \\
&= -\langle \varphi^*(\xi), x \cdot [[r, r](\eta, \gamma)] \rangle - \langle \varphi^*(\eta), x \cdot [[r, r](\xi, \gamma)] \rangle \\
&\quad - \langle \varphi^*(\gamma), [x, [[r, r](\xi, \eta)]] \rangle \\
&= [[r, r]](L^*(x)\varphi^*(\xi), \eta, \gamma) + [[r, r]](\xi, L^*(x)\varphi^*(\eta), \gamma) + [[r, r]](\xi, \eta, \text{ad}_x^* \varphi^*(\gamma)) \\
&= [[r, r]](L^*(x)\varphi^{*-1}(\xi), \eta, \gamma) + [[r, r]](\xi, L^*(x)\varphi^{*-1}(\eta), \gamma) \\
&\quad + [[r, r]](\xi, \eta, \text{ad}_x^* \varphi^{*-1}(\gamma)) \\
&= \langle [[r, r]], (L^*(x) \otimes \varphi \otimes \varphi)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma)) \rangle \\
&\quad + \langle [[r, r]], (\varphi \otimes L^*(x) \otimes \varphi)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma)) \rangle \\
&\quad + \langle [[r, r]], (\varphi \otimes \varphi \otimes \text{ad}_x^*)(\varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma)) \rangle \\
&= \langle (L(x) \otimes \varphi \otimes \varphi + \varphi \otimes L(x) \otimes \varphi + \varphi \otimes \varphi \otimes \text{ad}_x)[r, r], \\
&\quad \varphi^{*-1}(\xi) \otimes \varphi^{*-1}(\eta) \otimes \varphi^{*-1}(\gamma) \rangle.
\end{aligned}$$

Therefore, (A^*, \circ) is a Hom-left-symmetric algebra if and only if $(L(x) \otimes \varphi \otimes \varphi + \varphi \otimes L(x) \otimes \varphi + \varphi \otimes \varphi \otimes \text{ad}_x)[r, r] = 0$. Finally, if $\Delta(x) = (L(x) \otimes \varphi + \varphi \otimes \text{ad}_x)r$, obviously, the compatibility conditions in Definition 3.6 hold. \square

Let (A, φ) be a regular admissible Hom-left-symmetric algebra and $r \in \text{Sym}^2(A)$ be invertible (that is, r^\sharp is invertible). Define $B \in \text{Sym}^2(A^*)$ by

$$B(x, y) = \langle x, r^{\sharp-1}(y) \rangle$$

for any $x, y \in A$.

Proposition 4.8. *The s -matrix r satisfies the Hom- S -equation $[[r, r]] = 0$ if and only if*

$$(4.12) \quad B(\varphi(x \cdot y), z) - B(x, \varphi(y \cdot z)) - B(\varphi(y \cdot x), z) + B(y, \varphi(x \cdot z)) = 0$$

for any $x, y, z \in A$.

P r o o f. If r satisfies the Hom- S -equation $[[r, r]] = 0$, then for any $\xi, \eta \in A^*$, by Lemma 4.5, we obtain

$$(4.13) \quad [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)] + r^\sharp \varphi^*(\eta \circ \xi) - r^\sharp \varphi^*(\xi \circ \eta) = 0.$$

For any $x, y, z \in A$, put $x = r^\sharp(\xi)$, $y = r^\sharp(\eta)$ and $z = r^\sharp(\gamma)$. According to Corollary 4.4 and (4.13),

$$\begin{aligned} & B(\varphi(x \cdot y), z) - B(\varphi(y \cdot x), z) \\ &= B(\varphi(r^\sharp(\xi) \cdot r^\sharp(\eta)), r^\sharp(\gamma)) - B(\varphi(r^\sharp(\eta) \cdot r^\sharp(\xi)), r^\sharp(\gamma)) \\ &= \langle \varphi(r^\sharp(\xi) \cdot r^\sharp(\eta)), \gamma \rangle - \langle \varphi(r^\sharp(\eta) \cdot r^\sharp(\xi)), \gamma \rangle \\ &= \langle r^\sharp \varphi^*(\xi) \cdot r^\sharp \varphi^*(\eta), \gamma \rangle - \langle r^\sharp \varphi^*(\eta) \cdot r^\sharp \varphi^*(\xi), \gamma \rangle \\ &= \langle [r^\sharp \varphi^*(\xi), r^\sharp \varphi^*(\eta)], \gamma \rangle \\ &= \langle r^\sharp \varphi^*(\xi \circ \eta) - r^\sharp \varphi^*(\eta \circ \xi), \gamma \rangle \\ &= \langle r^\sharp \varphi^*(\text{ad}_{r^\sharp \varphi^*(\xi)}^* \eta - R^*(r^\sharp \varphi^*(\eta))\xi) - r^\sharp \varphi^*(\text{ad}_{r^\sharp \varphi^*(\eta)}^* \xi - R^*(r^\sharp \varphi^*(\xi))\eta), \gamma \rangle \\ &= \langle r^\sharp \varphi^*(L^*(r^\sharp \varphi^*(\xi))\eta) - r^\sharp \varphi^*(L^*(r^\sharp \varphi^*(\eta))\xi), \gamma \rangle \\ &= \langle L^*(r^\sharp \varphi^*(\xi))\eta - L^*(r^\sharp \varphi^*(\eta))\xi, \varphi r^\sharp(\gamma) \rangle \\ &= \langle L^*(\varphi r^\sharp(\xi))\eta - L^*(\varphi r^\sharp(\eta))\xi, \varphi r^\sharp(\gamma) \rangle \\ &= -\langle \eta, \varphi r^\sharp(\xi) \cdot \varphi r^\sharp(\gamma) \rangle + \langle \xi, \varphi r^\sharp(\eta) \cdot \varphi r^\sharp(\gamma) \rangle \\ &= -\langle r^\sharp \varphi^*(\xi) \cdot r^\sharp \varphi^*(\gamma), \eta \rangle + \langle r^\sharp \varphi^*(\eta) \cdot r^\sharp \varphi^*(\gamma), \xi \rangle \\ &= -\langle \varphi(r^\sharp(\xi) \cdot r^\sharp(\gamma)), \eta \rangle + \langle \varphi(r^\sharp(\eta) \cdot r^\sharp(\gamma)), \xi \rangle \\ &= -B(\varphi(r^\sharp(\xi) \cdot r^\sharp(\gamma)), r^\sharp(\eta)) + B(\varphi(r^\sharp(\eta) \cdot r^\sharp(\gamma)), r^\sharp(\xi)) \\ &= B(\varphi(y \cdot z), x) - B(\varphi(x \cdot z), y) \\ &= B(x, \varphi(y \cdot z)) - B(y, \varphi(x \cdot z)). \end{aligned}$$

□

Remark 4.9. If φ is orthogonal or the center of A is zero, by Corollary 4.4, we have $r^\# \varphi^* = \varphi r^\#$. Thus, we have

$$B(\varphi(x), y) = \langle \varphi(x), r^{\#-1}(y) \rangle = \langle x, \varphi^* r^{\#-1}(y) \rangle = \langle x, r^{\#-1} \varphi(y) \rangle = B(x, \varphi(y)).$$

Therefore, in view of Proposition 4.8, we have

$$B(x \cdot y, \varphi(z)) - B(\varphi(x), y \cdot z) - B(y \cdot x, \varphi(z)) + B(\varphi(y), x \cdot z) = 0,$$

that is, B is a 2-cocycle on A , see [4].

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