

ON THE WARING-GOLDBACH PROBLEM FOR ONE SQUARE  
AND FIVE CUBES IN SHORT INTERVALS

FEI XUE, MIN ZHANG, JINJIANG LI, Beijing

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*Abstract.* Let  $N$  be a sufficiently large integer. We prove that almost all sufficiently large even integers  $n \in [N - 6U, N + 6U]$  can be represented as

$$\begin{cases} n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3, \\ \left| p_1^2 - \frac{N}{6} \right| \leq U, \quad \left| p_i^3 - \frac{N}{6} \right| \leq U, \quad i = 2, 3, 4, 5, 6, \end{cases}$$

where  $U = N^{1-\delta+\varepsilon}$  with  $\delta \leq 8/225$ .

*Keywords:* Waring-Goldbach problem; Hardy-Littlewood method; exponential sum; short interval

*MSC 2020:* 11P05, 11P32, 11P55

## 1. INTRODUCTION AND MAIN RESULT

Waring's problem of mixed powers concerns the representation of sufficiently large integer  $n$  in the form

$$n = x_1^{k_1} + x_2^{k_2} + \dots + x_s^{k_s}.$$

Among the most interesting cases of mixed powers is that of establishing the representations of sufficiently large integer as the sum of one square and  $s$  positive cubes for each  $s > 1$ , i.e.,

$$(1.1) \quad n = x^2 + y_1^3 + y_2^3 + \dots + y_s^3.$$

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In 1930, Stanley in [8] showed that (1.1) is solvable for  $s > 6$ . Afterwards, Stanley in [9] and Watson in [15] solved the cases  $s = 6$  and  $s = 5$ , respectively. It should be emphasized that Stanley in [8] obtained the asymptotic formula for  $s > 6$ , while Sinnadurai in [7] obtained the asymptotic formula for  $s = 6$ . But in [15], Watson only proved a quite weak lower bound for the number of representation (1.1) with  $s = 5$ . In 1986, Vaughan in [12] enhanced Watson's result and derived a lower bound with the expected order of magnitude. In 2002, Wooley in [16] illustrated that, although the expected asymptotic formula of (1.1) with  $s = 5$  can not be established by the technique currently available, the exceptional set is extremely sparse. To be specific, let  $E_1(N)$  denote the number of integers  $n \leq N$  which can not be represented as one square and five positive cubes with expected asymptotic formula. Then Wooley in [16] showed that  $E_1(N) \ll N^\varepsilon$ .

In view of the results of Vaughan (see [12]) and Wooley (see [16]), it is reasonable to conjecture that for every sufficiently large even integer  $N$ , the equation

$$(1.2) \quad N = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3$$

is solvable. Here and below, the letter  $p$ , with or without subscript, denotes a prime number. But this conjecture is perhaps out of reach at present. However, it is possible to replace a variable by an almost-prime. In 2014, Cai in [1] proved that for every sufficiently large even integer  $N$ , the equation

$$(1.3) \quad N = x^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3$$

is solvable with  $x$  being an almost-prime  $\mathcal{P}_{36}$  and  $p_j$  ( $j = 1, 2, 3, 4, 5$ ) being primes. Later, in 2018, Li and Zhang in [4] enhanced the result of Cai (see [1]) and showed that (1.3) is solvable with  $x$  being an almost-prime  $\mathcal{P}_6$  and  $p_j$  ( $j = 1, 2, 3, 4, 5$ ) being primes.

In this paper, we shall consider problem (1.2) with almost equal variables, i.e.,

$$(1.4) \quad \begin{cases} n = p_1^2 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3, & n \in [N - 6U, N + 6U], \\ \left| p_1^2 - \frac{N}{6} \right| \leq U, & \left| p_i^3 - \frac{N}{6} \right| \leq U, \quad i = 2, 3, 4, 5, 6, \end{cases}$$

where  $U = N^{1-\delta+\varepsilon}$  with  $\delta > 0$  hoped to be as large as possible. Let  $E(N, U)$  denote the number of all positive even integers  $n$  satisfying

$$N - 6U \leq n \leq N + 6U$$

which can not be represented as (1.4). One wants to show that there exists  $\delta \in (0, 1)$  such that

$$(1.5) \quad E(N, U) \ll U^{1-\varepsilon}, \quad U = N^{1-\delta+\varepsilon}.$$

In this paper, we establish the following result.

**Theorem 1.1.** *Let notations be defined as above. Then (1.5) holds for  $\delta \leq 8/225$ .*

We shall prove Theorem 1.1 by using Hardy-Littlewood circle method. For the treatment of minor arcs, we shall first give an estimate of exponential sum in short intervals for cubic cases in Section 5, then we also employ the estimates for exponential sum over primes in short intervals in [3]. On the other hand, for the major arcs, we deal with the integrals to devote to establishing the asymptotic formula for the number of solutions to the problem by using the iterative method in [5]. The explicit details will be demonstrated in the related sections.

**Notation.** Throughout this paper,  $\varepsilon$  and  $A$  always denote positive constants which are arbitrarily small and sufficiently large, respectively, which may not be the same at different occurrences. Let  $p$ , with or without subscripts, always denotes a prime number. As usual, we use  $\varphi(n)$ ,  $\Lambda(n)$  and  $d(n)$  to denote Euler's function, von Mangoldt's function and Dirichlet's divisor function, respectively. Moreover, we use  $d^j(n)$  to denote  $(d(n))^j$  for abbreviation. Also, we use  $\chi \bmod q$  to denote a Dirichlet character modulo  $q$ , and  $\chi^0 \bmod q$  the principal character. In addition, we use  $\sum^*$  to denote sums over all primitive characters. Let  $(a, b)$  and  $[a, b]$  be the greatest common divisor and the least common multiple of  $a$  and  $b$ , respectively. For  $e(x) = e^{2\pi i x}$ ;  $f(x) \ll g(x)$  means that  $f(x) = O(g(x))$ ;  $f(x) \asymp g(x)$  means that  $f(x) \ll g(x) \ll f(x)$ . The number  $N$  is a sufficiently large integer and  $n \asymp N$ , and thus we use  $L$  to denote both  $\log N$  and  $\log n$ . The letter  $c$ , with or without subscripts or superscripts, always denotes a positive constant.

## 2. OUTLINE OF THE PROOF OF THEOREM 1.1

Let  $N$  be a sufficiently large positive integer. Write

$$(2.1) \quad X = \frac{N}{6} + U, \quad Y = \frac{N}{6} - U, \quad U = N^{1-8/225+\varepsilon},$$

and

$$f_k(\alpha) = \sum_{Y < p^k \leq X} (\log p) e(p^k \alpha), \quad k = 2, 3.$$

Define

$$\mathcal{R}(n, U) = \sum_{\substack{n=p_1^2+p_2^3+p_3^3+p_4^3+p_5^3+p_6^3 \\ |p_1^2-N/6| \leq U, |p_i^3-N/6| \leq U \\ i=2,3,4,5,6}} (\log p_1)(\log p_2) \dots (\log p_6).$$

Then for any  $Q > 0$  we have

$$\mathcal{R}(n, U) = \int_0^1 f_2(\alpha) f_3^5(\alpha) e(-n\alpha) d\alpha = \int_{1/Q}^{1+1/Q} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) d\alpha.$$

In order to apply the circle method, we set

$$(2.2) \quad P = U^2 N^{-37/20}, \quad Q = N^{17/20+\varepsilon}.$$

By Dirichlet's lemma on rational approximation (for instance, see [6], Lemma 12, page 104), each  $\alpha \in [1/Q, 1 + 1/Q]$  can be written as

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{qQ},$$

for some integers  $a, q$  with  $1 \leq a \leq q \leq Q$  and  $(a, q) = 1$ . Then we define the major arcs  $\mathfrak{M}$  and minor arcs  $\mathfrak{m}$  as follows:

$$(2.3) \quad \mathfrak{M} = \bigcup_{q \leq P} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathfrak{M},$$

where

$$\mathfrak{M}(q, a) = \left[ \frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right].$$

Then one has

$$\mathcal{R}(n, U) = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) d\alpha.$$

In order to prove Theorem 1.1, we need the two following propositions, whose proofs will be given in Section 3 and Section 5, respectively.

**Proposition 2.1.** *Let the major arcs  $\mathfrak{M}$  be defined as in (2.3) with  $P$  and  $Q$  defined in (2.2). Then for  $N - 6U \leq n \leq N + 6U$  and any  $A > 0$ , there holds*

$$\int_{\mathfrak{M}} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) d\alpha = \frac{1}{486} \mathfrak{S}(n) \mathfrak{J}(n, U) + O(U^5 N^{-23/6} L^{-A}),$$

where  $\mathfrak{S}(n)$  is the singular series defined in (3.1), which is absolutely convergent and satisfies

$$(2.4) \quad 0 < c^* \leq \mathfrak{S}(n) \ll d(n)$$

for any even integer  $n$  and some fixed constant  $c^*$ ; while  $\mathfrak{J}(n, U)$  is defined by (3.10) and satisfies

$$\mathfrak{J}(n, U) \asymp U^5 N^{-23/6}.$$

For the properties (2.4) of singular series we shall give the proof in Section 4.

**Proposition 2.2.** *Let the minor arcs  $\mathfrak{m}$  be defined as in (2.3) with  $P$  and  $Q$  defined in (2.2). Define*

$$(2.5) \quad \mathcal{I}(t) := \int_{\mathfrak{m}} |f_2^2(\alpha) f_3^t(\alpha)| \, d\alpha, \quad t \geq 1.$$

Then we have

$$\mathcal{I}(10) \ll U^{8/3+\varepsilon} N^{10/27}.$$

The remaining part of this section is devoted to establishing Theorem 1.1 by using Propositions 2.1 and 2.2.

**Proof of Theorem 1.1.** We only consider the integers  $n \in [N - 6U, N + 6U]$ . Let  $\mathcal{E}(N, U)$  denote the set of positive integers  $n \in [N - 6U, N + 6U]$  which can not be represented as (1.4). Then we have

$$(2.6) \quad \sum_{n \in (N, U)} \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \, d\alpha = \sum_{n \in \mathcal{E}(N, U)} \int_0^1 f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \, d\alpha = 0.$$

By Proposition 2.1, it follows that

$$(2.7) \quad \left| \sum_{n \in \mathcal{E}(N, U)} \int_{\mathfrak{M}} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \, d\alpha \right| \gg U^5 N^{-23/6} |\mathcal{E}(N, U)| = U^5 N^{-23/6} E(N, U).$$

Write

$$E(\alpha) = \sum_{n \in \mathcal{E}(N, U)} e(-n\alpha).$$

From Cauchy's inequality we deduce that

$$(2.8) \quad \left| \sum_{n \in \mathcal{E}(N, U)} \int_{\mathfrak{m}} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) \, d\alpha \right| = \left| \int_{\mathfrak{m}} f_2(\alpha) f_3^5(\alpha) E(\alpha) \, d\alpha \right| \leq \left( \int_{\mathfrak{m}} |f_2^2(\alpha) f_3^{10}(\alpha)| \, d\alpha \right)^{1/2} \left( \int_0^1 |E(\alpha)|^2 \, d\alpha \right)^{1/2} = (\mathcal{I}(10))^{1/2} (E(N, U))^{1/2}.$$

Therefore, from (2.6), (2.7) and (2.8) we get

$$(2.9) \quad E(N, U) \ll U^{-10} N^{23/3} \mathcal{I}(10).$$

It follows from Proposition 2.2 and the definition of  $U$  in (2.1) that

$$E(N, U) \ll U^{-22/3+\varepsilon} N^{217/27} \ll U^{1-\varepsilon}.$$

This completes the proof of Theorem 1.1. □

### 3. PROOF OF PROPOSITION 2.1

In this section, we shall concentrate on proving Proposition 2.1. We first introduce some notations. For a Dirichlet character  $\chi \pmod q$  and  $k = 2, 3$ , we define

$$C_k(\chi, a) = \sum_{h=1}^q \overline{\chi(h)} e\left(\frac{ah^k}{q}\right), \quad C_k(q, a) = C_k(\chi^0, a),$$

where  $\chi^0$  is the principal character modulo  $q$ , and  $C_k(q, a)$  is the Ramanujan sum. Let  $\chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}, \chi_3^{(4)}, \chi_3^{(5)}$  be Dirichlet characters modulo  $q$ . Define

$$B(n, q, \chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}, \chi_3^{(4)}, \chi_3^{(5)}) = \sum_{\substack{a=1 \\ (a, q)=1}}^q C_2(\chi_2, a) \left( \prod_{i=1}^5 C_3(\chi_3^{(i)}, a) \right) e\left(-\frac{an}{q}\right),$$

$$B(n, q) = B(n, q, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0, \chi^0),$$

and write

$$(3.1) \quad A(n, q) = \frac{B(n, q)}{\varphi^6(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q).$$

**Lemma 3.1.** *For  $(a, q) = 1$  and any Dirichlet character  $\chi \pmod q$ , it holds that*

$$|C_k(\chi, a)| \leq 2q^{1/2} d^{\beta_k}(q)$$

with  $\beta_k = \log k / \log 2$ .

*Proof.* See [14], Chapter VI, Problem 14. □

**Lemma 3.2.** *The singular series  $\mathfrak{S}(n)$  satisfies (2.4).*

The proof of Lemma 3.2 will be given in Section 4.

**Lemma 3.3.** *Let  $f(x)$  be a real differentiable function in the interval  $[a, b]$ . If  $f'(x)$  is monotonic and satisfies  $|f'(x)| \leq \theta < 1$ , then we have*

$$\sum_{a < n \leq b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} dx + O(1).$$

*Proof.* See [10], Lemma 4.8. □

**Lemma 3.4.** Let  $\chi_2 \pmod{r_2}$  and  $\chi_3^{(i)} \pmod{r_3^{(i)}}$  with  $i = 1, 2, 3, 4, 5$  be primitive characters,  $r_0 = [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}, r_3^{(5)}]$ , and  $\chi^0$  the principal character modulo  $q$ . Then there holds

$$(3.2) \quad \sum_{\substack{q \leq x \\ r_0 | q}} \frac{1}{\varphi^6(q)} |B(n, q, \chi_2 \chi^0, \chi_3^{(1)} \chi^0, \chi_3^{(2)} \chi^0, \chi_3^{(3)} \chi^0, \chi_3^{(4)} \chi^0, \chi_3^{(5)} \chi^0)| \ll r_0^{-2+\varepsilon} \log^c x.$$

*Proof.* By Lemma 3.1, we have

$$\begin{aligned} & |B(n, q, \chi_2 \chi^0, \chi_3^{(1)} \chi^0, \chi_3^{(2)} \chi^0, \chi_3^{(3)} \chi^0, \chi_3^{(4)} \chi^0, \chi_3^{(5)} \chi^0)| \\ & \ll \sum_{\substack{a=1 \\ (a, q)=1}}^q |C_2(\chi_2 \chi^0, a)| \prod_{i=1}^5 |C_3(\chi_3^{(i)} \chi^0, a)| \ll q^3 \varphi(q) d^9(q). \end{aligned}$$

Therefore, the left-hand side of (3.2) is

$$\ll \sum_{\substack{q \leq x \\ r_0 | q}} \frac{q^3 \varphi(q) d^9(q)}{\varphi^6(q)} = \sum_{t \leq x/r_0} \frac{r_0^3 t^3 d^9(r_0 t)}{\varphi^5(r_0 t)} \ll r_0^{-2+\varepsilon} (\log x) \sum_{t \leq x} \frac{d^9(t)}{t^2} \ll r_0^{-2+\varepsilon} \log^c x.$$

This completes the proof of Lemma 3.4. □

Let  $X, Y$  be as in (2.1) and write

$$(3.3) \quad \begin{aligned} V_k(\lambda) &= \sum_{Y < m^k \leq X} e(m^k \lambda), \\ W_k(\chi, \lambda) &= \sum_{Y < p^k \leq X} (\log p) \chi(p) e(p^k \lambda) - \delta_\chi \sum_{Y < m^k \leq X} e(m^k \lambda), \end{aligned}$$

where  $\delta_\chi = 1$  or  $0$  according to whether  $\chi$  is principal or not. Then by the orthogonality of Dirichlet characters for  $(a, q) = 1$  we have

$$f_k\left(\frac{a}{q} + \lambda\right) = \frac{C_k(q, a)}{\varphi(q)} V_k(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} C_k(\chi, a) W_k(\chi, \lambda).$$

For  $j = 1, 2, \dots, 12$ , we define the sets  $\mathcal{S}_j$  as follows:

$$\mathcal{S}_j = \begin{cases} \{2, 3, 3, 3, 3, 3\} & \text{if } j = 1; & \{2, 3, 3, 3, 3\} & \text{if } j = 2; & \{3, 3, 3, 3, 3\} & \text{if } j = 3; \\ \{2, 3, 3, 3\} & \text{if } j = 4; & \{3, 3, 3, 3\} & \text{if } j = 5; & \{2, 3, 3\} & \text{if } j = 6; \\ \{3, 3, 3\} & \text{if } j = 7; & \{2, 3\} & \text{if } j = 8; & \{3, 3\} & \text{if } j = 9; \\ \{2\} & \text{if } j = 10; & \{3\} & \text{if } j = 11; & \emptyset & \text{if } j = 12. \end{cases}$$

Also, we write  $\bar{\mathcal{S}}_j = \{2, 3, 3, 3, 3, 3\} \setminus \mathcal{S}_j$ . Then we have

$$(3.4) \quad \int_{\mathfrak{M}} f_2(\alpha) f_3^5(\alpha) e(-n\alpha) d\alpha = I_1 + 5I_2 + I_3 + 10I_4 + 5I_5 + 10I_6 + 10I_7 \\ + 5I_8 + 10I_9 + I_{10} + 5I_{11} + I_{12},$$

where

$$I_j = \sum_{q \leq P} \frac{1}{\varphi^6(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \prod_{k \in \mathcal{S}_j} C_k(q, a) \right) e\left(-\frac{an}{q}\right) \\ \times \int_{-1/qQ}^{1/qQ} \left( \prod_{k \in \mathcal{S}_j} V_k(\lambda) \right) \left( \prod_{k \in \bar{\mathcal{S}}_j} \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda) \right) e(-n\lambda) d\lambda.$$

In the following content of this section, we shall prove that  $I_1$  produces the main term, while the others contribute to the error term. For  $k = 2, 3$ , applying Lemma 3.3 to  $V_k(\lambda)$ , we have

$$(3.5) \quad V_k(\lambda) = \int_{Y^{1/k}}^{X^{1/k}} e(u^k \lambda) du + O(1) = \frac{1}{k} \int_Y^X e(v\lambda) v^{1/k-1} dv + O(1) \\ = \frac{1}{k} \sum_{Y < m \leq X} m^{1/k-1} e(m\lambda) + O(1).$$

Putting (3.5) into  $I_1$ , we see that

$$(3.6) \quad I_1 = \frac{1}{486} \sum_{q \leq P} \frac{B(n, q)}{\varphi^6(q)} \int_{-1/qQ}^{1/qQ} \left( \sum_{Y < m \leq X} m^{-1/2} e(m\lambda) \right) \\ \times \left( \sum_{Y < m \leq X} m^{-2/3} e(m\lambda) \right)^5 e(-n\lambda) d\lambda \\ + O\left( \sum_{q \leq P} \frac{|B(n, q)|}{\varphi^6(q)} \int_{-1/qQ}^{1/qQ} \left| \sum_{Y < m \leq X} m^{-1/2} e(m\lambda) \right| \right. \\ \left. \times \left| \sum_{Y < m \leq X} m^{-2/3} e(m\lambda) \right|^4 d\lambda \right).$$

By using the elementary estimate

$$(3.7) \quad \sum_{Y < m \leq X} m^{1/k-1} e(m\lambda) \ll N^{1/k-1} \min\left(U, \frac{1}{|\lambda|}\right)$$



and Lemma 3.4 with  $r_0 = 1$ , the  $O$ -term in (3.6) can be estimated as

$$(3.8) \quad \ll \sum_{q \leq P} \frac{|B(n, q)|}{\varphi^6(q)} \left( \int_0^{1/U} U^5 N^{-19/6} d\lambda + \int_{1/U}^\infty N^{-19/6} \lambda^{-5} d\lambda \right) \\ \ll U^4 N^{-19/6} L^c \ll U^5 N^{-23/6} L^{-A}.$$

If we extend the interval of the integral in the main term of (3.6) to  $[-1/2, 1/2]$ , then from (2.2) we can see that the resulting error is

$$\ll L^c \int_{1/4Q}^{1/2} N^{-23/6} \lambda^{-6} d\lambda \ll N^{-23/6} q^5 Q^5 L^c \ll N^{-23/6} (PQ)^5 L^c \ll U^5 N^{-23/6-\varpi}$$

for some  $\varpi > 0$ . Therefore, by Lemma 3.2, (3.6) becomes

$$(3.9) \quad I_1 = \frac{1}{486} \mathfrak{S}(n) \mathfrak{J}(n, U) + O(U^5 N^{-23/6} L^{-A}),$$

where

$$(3.10) \quad \mathfrak{J}(n, U) := \sum_{\substack{m_1+m_2+\dots+m_6=n \\ Y < m_i \leq X \\ i=1,2,\dots,6}} m_1^{-1/2} (m_2 m_3 m_4 m_5 m_6)^{-2/3} \asymp U^5 N^{-23/6}.$$

In order to estimate the contribution of  $I_j$  for  $j = 2, 3, \dots, 12$ , we shall need the following three preliminary lemmas, i.e., Lemmas 3.5–3.7. In view of this, for  $k = 2, 3$ , we recall the definition of  $W_k(\chi, \lambda)$  in (3.3) and write

$$J_k(g) = \sum_{r \leq P} [g, r]^{-2+\varepsilon} \sum_{\chi \bmod r}^* \max_{|\lambda| \leq 1/rQ} |W_k(\chi, \lambda)|$$

and

$$K_k(g) = \sum_{r \leq P} [g, r]^{-2+\varepsilon} \sum_{\chi \bmod r}^* \left( \int_{-1/rQ}^{1/rQ} |W_k(\chi, \lambda)|^2 d\lambda \right)^{1/2}.$$

Here and below,  $\sum^*$  indicates that the summation is taken over all primitive characters.

**Lemma 3.5.** *Let  $P, Q$  be defined as in (2.2). Then we have*

$$J_k(g) \ll g^{-2+\varepsilon} U N^{1/k-1} L^c.$$

**Lemma 3.6.** *Let  $P, Q$  be defined as in (2.2). For  $g = 1$ , Lemma 3.5 can be improved to*

$$J_k(1) \ll UN^{1/k-1}L^{-A}.$$

**Lemma 3.7.** *Let  $P, Q$  be defined as in (2.2). Then we have*

$$K_k(g) \ll g^{-2+\varepsilon}U^{1/2}N^{1/k-1}L^c.$$

The proof of Lemmas 3.5–3.7 is exactly the same as that of Lemmas 3.5–3.7 in [17]. So we omit the details. Now, we concentrate on estimating the terms  $I_j$  for  $j = 2, 3, \dots, 12$ . We begin with the term  $I_{12}$ , which is the most complicated one. Reducing the Dirichlet characters in  $I_{12}$  into primitive characters, we have

$$\begin{aligned} |I_{12}| &= \left| \sum_{q \leq P} \frac{1}{\varphi^6(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^q e\left(-\frac{an}{q}\right) \int_{-1/qQ}^{1/qQ} \left( \sum_{\chi_2 \bmod q} C_2(\chi_2, a) W_2(\chi_2, \lambda) \right) \right. \\ &\quad \times \left. \left( \sum_{\chi_3 \bmod q} C_3(\chi_3, a) W_3(\chi_3, \lambda) \right)^5 e(-n\lambda) d\lambda \right| \\ &= \left| \sum_{q \leq P} \sum_{\chi_2} \sum_{\chi_3^{(1)}} \sum_{\chi_3^{(2)}} \sum_{\chi_3^{(3)}} \sum_{\chi_3^{(4)}} \sum_{\chi_3^{(5)}} \frac{1}{\varphi^6(q)} B(n, q, \chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}, \chi_3^{(4)}, \chi_3^{(5)}) \right. \\ &\quad \times \left. \int_{-1/qQ}^{1/qQ} W_2(\chi_2, \lambda) W_3(\chi_3^{(1)}, \lambda) \dots W_3(\chi_3^{(5)}, \lambda) e(-n\lambda) d\lambda \right| \\ &\leq \sum_{r_2 \leq P} \sum_{r_3^{(1)} \leq P} \dots \sum_{r_3^{(5)} \leq P} \\ &\quad \sum_{\chi_2 \bmod r_2}^* \sum_{\chi_3^{(1)} \bmod r_3^{(1)}}^* \dots \sum_{\chi_3^{(5)} \bmod r_3^{(5)}}^* \sum_{\substack{q \leq P \\ r_0 | q}} | \frac{B(n, q, \chi_2 \chi^0, \chi_3^{(1)} \chi^0, \dots, \chi_3^{(5)} \chi^0)}{\varphi^6(q)} | \\ &\quad \times \int_{-1/qQ}^{1/qQ} |W_2(\chi_2 \chi^0, \lambda)| \prod_{i=1}^5 |W_3(\chi_3^{(i)} \chi^0, \lambda)| d\lambda, \end{aligned}$$

where  $\chi^0$  is the principal character modulo  $q$  and  $r_0 = [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}, r_3^{(5)}]$ . For  $q \leq P$  and  $Y < p^k \leq X$  with  $k = 2, 3$ , we have  $(q, p) = 1$ . From this and the definition of  $W_k(\chi, \lambda)$ , we obtain  $W_2(\chi_2 \chi^0, \lambda) = W_2(\chi_2, \lambda)$  and  $W_3(\chi_3^{(i)} \chi^0, \lambda) = W_3(\chi_3^{(i)}, \lambda)$  for primitive characters  $\chi_2$  and  $\chi_3^{(i)}$  with  $i = 1, 2, 3, 4, 5$ . Therefore,

by Lemma 3.4, we obtain

$$\begin{aligned}
 |I_{12}| &\leq \sum_{r_2 \leq P} \sum_{r_3^{(1)} \leq P} \cdots \sum_{r_3^{(5)} \leq P} \sum_{\chi_2 \bmod r_2}^* \sum_{\chi_3^{(1)} \bmod r_3^{(1)}}^* \cdots \sum_{\chi_3^{(5)} \bmod r_3^{(5)}}^* \\
 &\int_{-1/r_0 Q}^{1/r_0 Q} |W_2(\chi_2, \lambda)| \prod_{i=1}^5 |W_3(\chi_3^{(i)}, \lambda)| d\lambda \\
 &\times \sum_{\substack{q \leq P \\ r_0 | q}} \frac{|B(n, q, \chi_2 \chi^0, \chi_3^{(1)} \chi^0, \chi_3^{(2)} \chi^0, \chi_3^{(3)} \chi^0, \chi_3^{(4)} \chi^0, \chi_3^{(5)} \chi^0)|}{\varphi^6(q)} \\
 &\ll L^c \sum_{r_2 \leq P} \sum_{r_3^{(1)} \leq P} \cdots \sum_{r_3^{(5)} \leq P} r_0^{-2+\varepsilon} \sum_{\chi_2 \bmod r_2}^* \sum_{\chi_3^{(1)} \bmod r_3^{(1)}}^* \cdots \sum_{\chi_3^{(5)} \bmod r_3^{(5)}}^* \\
 &\int_{-1/r_0 Q}^{1/r_0 Q} |W_2(\chi_2, \lambda)| \prod_{i=1}^5 |W_3(\chi_3^{(i)}, \lambda)| d\lambda.
 \end{aligned}$$

In the last integral, we pick out  $|W_2(\chi_2, \lambda)|, |W_3(\chi_3^{(1)}, \lambda)|, \dots, |W_3(\chi_3^{(3)}, \lambda)|$ , and then use Cauchy's inequality to derive that

$$\begin{aligned}
 (3.11) \quad |I_{12}| &\ll L^c \sum_{r_2 \leq P} \sum_{\chi_2 \bmod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} |W_2(\chi_2, \lambda)| \\
 &\times \prod_{i=1}^3 \left( \sum_{r_3^{(i)} \leq P} \sum_{\chi_3^{(i)} \bmod r_3^{(i)}}^* \max_{|\lambda| \leq 1/r_3^{(i)} Q} |W_3(\chi_3^{(i)}, \lambda)| \right) \\
 &\times \sum_{r_3^{(4)} \leq P} \sum_{\chi_3^{(4)} \bmod r_3^{(4)}}^* \left( \int_{-1/r_3^{(4)} Q}^{1/r_3^{(4)} Q} |W_3(\chi_3^{(4)}, \lambda)|^2 d\lambda \right)^{1/2} \\
 &\times \sum_{r_3^{(5)} \leq P} r_0^{-2+\varepsilon} \sum_{\chi_3^{(5)} \bmod r_3^{(5)}}^* \left( \int_{-1/r_3^{(5)} Q}^{1/r_3^{(5)} Q} |W_3(\chi_3^{(5)}, \lambda)|^2 d\lambda \right)^{1/2}.
 \end{aligned}$$

Now we introduce the iterative procedure to bound the sums over  $r_3^{(5)}, \dots, r_3^{(1)}, r_2$ , consecutively. We first estimate the above sum over  $r_3^{(5)}$  in (3.11) via Lemma 3.7. Since

$$r_0 = [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}, r_3^{(5)}] = [[r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}], r_3^{(5)}],$$

the sum over  $r_3^{(5)}$  is

$$\begin{aligned}
 (3.12) \quad &\sum_{r_3^{(5)} \leq P} [[r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}], r_3^{(5)}]^{-2+\varepsilon} \sum_{\chi_3^{(5)} \bmod r_3^{(5)}}^* \left( \int_{-1/r_3^{(5)} Q}^{1/r_3^{(5)} Q} |W_3(\chi_3^{(5)}, \lambda)|^2 d\lambda \right)^{1/2} \\
 &= K_3([r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}]) \ll [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}, r_3^{(4)}]^{-2+\varepsilon} U^{1/2} N^{-2/3} L^c.
 \end{aligned}$$

By Lemma 3.7 again, the contribution of the quantity on the right-hand side of (3.12) to the sum over  $r_3^{(4)}$  in (3.11) is

$$\begin{aligned}
(3.13) \quad &\ll U^{1/2} N^{-2/3} L^c \sum_{r_3^{(4)} \leq P} [[r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}], r_3^{(4)}]^{-2+\varepsilon} \\
&\quad \times \sum_{\chi_3^{(4)} \bmod r_3^{(4)}}^* \left( \int_{-1/r_3^{(4)} Q}^{1/r_3^{(4)} Q} |W_3(\chi_3^{(4)}, \lambda)|^2 d\lambda \right)^{1/2} \\
&= U^{1/2} N^{-2/3} L^c K_3([r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}]) \\
&\ll [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}]^{-2+\varepsilon} U N^{-4/3} L^c.
\end{aligned}$$

By Lemma 3.5, the contribution of the quantity on the right-hand side of (3.13) to the sum over  $r_3^{(3)}$  in (3.11) is

$$\begin{aligned}
(3.14) \quad &\ll U N^{-4/3} L^c \sum_{r_3^{(3)} \leq P} [[r_2, r_3^{(1)}, r_3^{(2)}], r_3^{(3)}]^{-2+\varepsilon} \\
&\quad \times \sum_{\chi_3^{(3)} \bmod r_3^{(3)}}^* \max_{|\lambda| \leq 1/r_3^{(3)} Q} |W_3(\chi_3^{(3)}, \lambda)| \\
&= U N^{-4/3} L^c J_3([r_2, r_3^{(1)}, r_3^{(2)}]) \ll [r_2, r_3^{(1)}, r_3^{(2)}]^{-2+\varepsilon} U^2 N^{-2} L^c.
\end{aligned}$$

The contribution of the quantity on the right-hand side of (3.14) to the sum over  $r_3^{(2)}$  in (3.11) is

$$\begin{aligned}
(3.15) \quad &\ll U^2 N^{-2} L^c \sum_{r_3^{(2)} \leq P} [[r_2, r_3^{(1)}], r_3^{(2)}]^{-2+\varepsilon} \sum_{\chi_3^{(2)} \bmod r_3^{(2)}}^* \max_{|\lambda| \leq 1/r_3^{(2)} Q} |W_3(\chi_3^{(2)}, \lambda)| \\
&= U^2 N^{-2} L^c J_3([r_2, r_3^{(1)}]) \ll [r_2, r_3^{(1)}]^{-2+\varepsilon} U^3 N^{-8/3} L^c.
\end{aligned}$$

The contribution of the quantity on the right-hand side of (3.15) to the sum over  $r_3^{(1)}$  in (3.11) is

$$\begin{aligned}
(3.16) \quad &\ll U^3 N^{-8/3} L^c \sum_{r_3^{(1)} \leq P} [r_2, r_3^{(1)}]^{-2+\varepsilon} \sum_{\chi_3^{(1)} \bmod r_3^{(1)}}^* \max_{|\lambda| \leq 1/r_3^{(1)} Q} |W_3(\chi_3^{(1)}, \lambda)| \\
&= U^3 N^{-8/3} L^c J_3(r_2) \ll (r_2)^{-2+\varepsilon} U^4 N^{-10/3} L^c.
\end{aligned}$$

At last, from Lemma 3.6, inserting the bound on the right-hand side of (3.16) to the sum over  $r_2$  in (3.11), we get

$$\begin{aligned}
(3.17) \quad &|I_{12}| \ll U^4 N^{-10/3} L^c \sum_{r_2 \leq P} [1, r_2]^{-2+\varepsilon} \sum_{\chi_2 \bmod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} |W_2(\chi_2, \lambda)| \\
&= U^4 N^{-10/3} L^c J_2(1) \ll U^5 N^{-23/6} L^{-A}.
\end{aligned}$$

For the estimation of the terms  $I_2, \dots, I_{11}$ , by noting (3.5) and (3.7), we obtain

$$\begin{aligned} \left( \int_{-1/Q}^{1/Q} |V_k(\lambda)|^2 d\lambda \right)^{1/2} &\ll \left( \int_{-1/Q}^{1/Q} N^{2/k-2} \min^2 \left( U, \frac{1}{|\lambda|} \right) d\lambda + \frac{1}{Q} \right)^{1/2} \\ &\ll N^{1/k-1} \left( \int_0^{1/U} U^2 d\lambda + \int_{1/U}^{1/Q} \lambda^{-2} d\lambda \right)^{1/2} \\ &\quad + \frac{1}{Q^{1/2}} \ll U^{1/2} N^{1/k-1}. \end{aligned}$$

Using this estimate and the upper bound of  $V_k(\lambda)$ , which derives from (3.5) and (3.7), that  $V_k(\lambda) \ll UN^{1/k-1}$ , we can argue similarly to the treatment of  $I_{12}$  and obtain

$$(3.18) \quad \sum_{j=2}^{11} I_j \ll U^5 N^{-23/6} L^{-A}.$$

Combining (3.4), (3.9), (3.17) and (3.18), we can get the conclusion of Proposition 2.1.

#### 4. THE SINGULAR SERIES

In this section, we shall investigate the properties of the singular series which appear in Proposition 2.1.

**Lemma 4.1.** *Let  $p$  be a prime and  $p^\alpha \parallel k$ . For  $(a, p) = 1$ , if  $l \geq \gamma(p)$ , we have  $C_k(p^l, a) = 0$ , where*

$$\gamma(p) = \begin{cases} \alpha + 2 & \text{if } p \neq 2 \text{ or } p = 2, \alpha = 0; \\ \alpha + 3 & \text{if } p = 2, \alpha > 0. \end{cases}$$

*Proof.* See [2], Lemma 8.3. □

For  $k \geq 1$  we define

$$S_k(q, a) = \sum_{m=1}^q e\left(\frac{am^k}{q}\right).$$

**Lemma 4.2.** *Suppose that  $(p, a) = 1$ . Then*

$$S_k(p, a) = \sum_{\chi \in \mathcal{A}_k} \overline{\chi(a)} \tau(\chi),$$

where  $\mathcal{A}_k$  denotes the set of non-principal characters  $\chi$  modulo  $p$  for which  $\chi^k$  is principal, and  $\tau(\chi)$  denotes the Gauss sum

$$\sum_{m=1}^p \chi(m) e\left(\frac{m}{p}\right).$$

Also, there hold  $|\tau(\chi)| = p^{1/2}$  and  $|\mathcal{A}_k| = (k, p-1) - 1$ .

*Proof.* See [13], Lemma 4.3. □

**Lemma 4.3.** For  $(p, n) = 1$  we have

$$(4.1) \quad \left| \sum_{a=1}^{p-1} \frac{S_2(p, a) S_3^5(p, a)}{p^6} e\left(-\frac{an}{p}\right) \right| \leq 32p^{-5/2}.$$

*Proof.* We denote by  $\mathcal{S}$  the left-hand side of (4.1). By Lemma 4.2 we have

$$\mathcal{S} = \frac{1}{p^6} \sum_{a=1}^{p-1} \left( \sum_{\chi_2 \in \mathcal{A}_2} \overline{\chi_2(a)} \tau(\chi_2) \right) \left( \sum_{\chi_3 \in \mathcal{A}_3} \overline{\chi_3(a)} \tau(\chi_3) \right)^5 e\left(-\frac{an}{p}\right).$$

If  $|\mathcal{A}_k| = 0$  for some  $k \in \{2, 3\}$ , then  $\mathcal{S} = 0$ . If this is not the case, then

$$\begin{aligned} \mathcal{S} &= \frac{1}{p^6} \sum_{\chi_2 \in \mathcal{A}_2} \sum_{\chi_3^{(1)} \in \mathcal{A}_3} \sum_{\chi_3^{(2)} \in \mathcal{A}_3} \sum_{\chi_3^{(3)} \in \mathcal{A}_3} \sum_{\chi_3^{(4)} \in \mathcal{A}_3} \sum_{\chi_3^{(5)} \in \mathcal{A}_3} \\ &\quad \tau(\chi_2) \tau(\chi_3^{(1)}) \tau(\chi_3^{(2)}) \tau(\chi_3^{(3)}) \tau(\chi_3^{(4)}) \tau(\chi_3^{(5)}) \\ &\quad \times \sum_{a=1}^{p-1} \overline{\chi_2(a) \chi_3^{(1)}(a) \chi_3^{(2)}(a) \chi_3^{(3)}(a) \chi_3^{(4)}(a) \chi_3^{(5)}(a)} e\left(-\frac{an}{p}\right). \end{aligned}$$

From Lemma 4.2, the sextuple outer sums have not more than  $((2, p-1) - 1) \times ((3, p-1) - 1)^5 \leq 2^5 = 32$  terms. In each of these terms we have

$$|\tau(\chi_2) \tau(\chi_3^{(1)}) \tau(\chi_3^{(2)}) \tau(\chi_3^{(3)}) \tau(\chi_3^{(4)}) \tau(\chi_3^{(5)})| = p^3.$$

Since in any one of these terms

$$\overline{\chi_2(a) \chi_3^{(1)}(a) \chi_3^{(2)}(a) \chi_3^{(3)}(a) \chi_3^{(4)}(a) \chi_3^{(5)}(a)}$$

is a Dirichlet character  $\chi \pmod{p}$ , the inner sum is

$$\sum_{a=1}^{p-1} \chi(a) e\left(-\frac{an}{p}\right) = \overline{\chi(-n)} \sum_{a=1}^{p-1} \chi(-an) e\left(-\frac{an}{p}\right) = \overline{\chi(-n)} \tau(\chi).$$

From the fact that  $\tau(\chi^0) = -1$  for the principal character  $\chi^0 \pmod p$ , we have

$$|\overline{\chi(-n)}\tau(\chi)| \leq p^{1/2}.$$

By the above arguments, we obtain

$$|\mathcal{S}| \leq p^{-6}32p^3p^{1/2} = 32p^{-5/2}.$$

This completes the proof of Lemma 4.3. □

**Lemma 4.4.** *Let  $\mathcal{L}(p, n)$  denote the number of solutions of the congruence*

$$x_1^2 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 \equiv n \pmod p, \quad 1 \leq x_1, x_2, \dots, x_6 \leq p-1.$$

Then for  $p \geq 3$  we have  $\mathcal{L}(p, n) > 0$ . Moreover, we have  $\mathcal{L}(2, n) > 0$  for  $n \equiv 0 \pmod 2$ .

*Proof.* We have

$$p\mathcal{L}(p, n) = \sum_{a=1}^p C_2(p, a)C_3^5(p, a)e\left(-\frac{an}{p}\right) = (p-1)^6 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_2(p, a)C_3^5(p, a)e\left(-\frac{an}{p}\right).$$

By Lemma 4.2, we obtain

$$|E_p| \leq (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)^5.$$

It is easy to check that  $|E_p| < (p-1)^6$  for  $p \geq 13$ . Therefore we obtain  $\mathcal{L}(p, n) > 0$  for  $p \geq 13$ . If  $p = 3, 5, 7, 11$ , we can check  $\mathcal{L}(p, n) > 0$  directly. In addition, it is easy to see that  $\mathcal{L}(2, n) > 0$  for  $n \equiv 0 \pmod 2$ . This completes the proof of Lemma 4.4. □

**Lemma 4.5.**  *$A(n, q)$  is multiplicative in  $q$ .*

*Proof.* By the definition of  $A(n, q)$  in (3.1), we only need to show that  $B(n, q)$  is multiplicative in  $q$ . Suppose  $q = q_1q_2$  with  $(q_1, q_2) = 1$ . Then we have

$$\begin{aligned} (4.2) \quad B(n, q_1q_2) &= \sum_{\substack{a=1 \\ (a, q_1q_2)=1}}^{q_1q_2} C_2(q_1q_2, a)C_3^5(q_1q_2, a)e\left(-\frac{an}{q_1q_2}\right) \\ &= \sum_{\substack{a_1=1 \\ (a_1, q_1)=1}}^{q_1} \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} C_2(q_1q_2, a_1q_2 + a_2q_1) \\ &\quad \times C_3^5(q_1q_2, a_1q_2 + a_2q_1)e\left(-\frac{a_1n}{q_1}\right)e\left(-\frac{a_2n}{q_2}\right). \end{aligned}$$

For  $(q_1, q_2) = 1$  it holds that

$$\begin{aligned}
(4.3) \quad C_k(q_1 q_2, a_1 q_2 + a_2 q_1) &= \sum_{\substack{m=1 \\ (m, q_1 q_2)=1}}^{q_1 q_2} e\left(\frac{(a_1 q_2 + a_2 q_1) m^k}{q_1 q_2}\right) \\
&= \sum_{\substack{m_1=1 \\ (m_1, q_1)=1}}^{q_1} \sum_{\substack{m_2=1 \\ (m_2, q_2)=1}}^{q_2} e\left(\frac{(a_1 q_2 + a_2 q_1)(m_1 q_2 + m_2 q_1)^k}{q_1 q_2}\right) \\
&= \sum_{\substack{m_1=1 \\ (m_1, q_1)=1}}^{q_1} e\left(\frac{a_1 (m_1 q_2)^k}{q_1}\right) \sum_{\substack{m_2=1 \\ (m_2, q_2)=1}}^{q_2} e\left(\frac{a_2 (m_2 q_1)^k}{q_2}\right) \\
&= C_k(q_1, a_1) C_k(q_2, a_2).
\end{aligned}$$

Putting (4.3) into (4.2), we deduce that

$$\begin{aligned}
B(n, q_1 q_2) &= \sum_{\substack{a_1=1 \\ (a_1, q_1)=1}}^{q_1} C_2(q_1, a_1) C_3^5(q_1, a_1) e\left(-\frac{a_1 n}{q_1}\right) \\
&\quad \times \sum_{\substack{a_2=1 \\ (a_2, q_2)=1}}^{q_2} C_2(q_2, a_2) C_3^5(q_2, a_2) e\left(-\frac{a_2 n}{q_2}\right) \\
&= B(n, q_1) B(n, q_2).
\end{aligned}$$

This completes the proof of Lemma 4.5. □

**Lemma 4.6.** *Let  $A(n, q)$  be as defined in (3.1). Then:*

(i) *We have*

$$\sum_{q > Z} |A(n, q)| \ll Z^{-3/2+\varepsilon} d(n).$$

*Hence,  $\sum_{q=1}^{\infty} A(n, q)$  is absolutely convergent and satisfies  $\mathfrak{S}(n) \ll d(n)$ .*

(ii) *There exists an absolute positive constant  $c^* > 0$  such that for  $n \equiv 0 \pmod{2}$ ,*

$$\mathfrak{S}(n) \geq c^* > 0.$$

**P r o o f.** From Lemma 4.5 we know that  $B(n, q)$  is multiplicative in  $q$ . Therefore it holds that

$$(4.4) \quad B(n, q) = \prod_{p^t \parallel q} B(n, p^t) = \prod_{p^t \parallel q} \sum_{\substack{a=1 \\ (a, p)=1}}^{p^t} C_2(p^t, a) C_3^5(p^t, a) e\left(-\frac{an}{p^t}\right).$$



From (4.4) and Lemma 4.1 we deduce that  $B(n, q) = \prod_{p|q} B(n, p)$  or 0 according to whether  $q$  is square-free or not. Thus, one has

$$(4.5) \quad \sum_{q=1}^{\infty} A(n, q) = \sum_{\substack{q=1 \\ q \text{ square-free}}}^{\infty} A(n, q).$$

Write

$$\mathcal{R}(p, a) := C_2(p, a)C_3^5(p, a) - S_2(p, a)S_3^5(p, a).$$

Then

$$(4.6) \quad A(n, p) = \frac{1}{(p-1)^6} \sum_{a=1}^{p-1} S_2(p, a)S_3^5(p, a)e\left(-\frac{an}{p}\right) + \frac{1}{(p-1)^6} \sum_{a=1}^{p-1} \mathcal{R}(p, a)e\left(-\frac{an}{p}\right).$$

Applying Lemma 3.1 and noticing that  $S_k(p, a) = C_k(p, a) + 1$ , we get  $S_k(p, a) \ll p^{1/2}$ , and thus  $\mathcal{R}(p, a) \ll p^{5/2}$ . Therefore the second term in (4.6) is not greater than  $c_1 p^{-5/2}$ . On the other hand, by Lemma 4.3, we can see that the first term in (4.6) is not greater than  $2^6 \cdot 32p^{-5/2} = 2048p^{-5/2}$ . Let  $c_2 = \max(c_1, 2048)$ . Then we have proved that for  $p \nmid n$  it holds that

$$(4.7) \quad |A(n, p)| \leq c_2 p^{-5/2}.$$

Moreover, if we use Lemma 3.1 directly, it follows that

$$\begin{aligned} |B(n, p)| &= \left| \sum_{a=1}^{p-1} C_2(p, a)C_3^5(p, a)e\left(-\frac{an}{p}\right) \right| \leq \sum_{a=1}^{p-1} |C_2(p, a)| |C_3(p, a)|^5 \\ &\leq (p-1) \cdot 2^6 \cdot p^3 \cdot 486 = 31104p^3(p-1), \end{aligned}$$

and therefore

$$(4.8) \quad |A(n, p)| = \frac{|B(n, p)|}{\varphi^6(p)} \leq \frac{31104p^3}{(p-1)^5} \leq \frac{2^5 \cdot 31104p^3}{p^5} = \frac{995328}{p^2}.$$

Let  $c_3 = \max(c_2, 995328)$ . Then for square-free  $q$  we have

$$\begin{aligned} |A(n, q)| &= \prod_{\substack{p|q \\ p \nmid n}} |A(n, p)| \prod_{\substack{p|q \\ p \nmid n}} |A(n, p)| \leq \prod_{\substack{p|q \\ p \nmid n}} (c_3 p^{-5/2}) \prod_{\substack{p|q \\ p \nmid n}} (c_3 p^{-2}) \\ &= c_3^{\omega(q)} \prod_{p|q} p^{-5/2} \prod_{p|(n, q)} p^{1/2} \ll q^{-5/2+\varepsilon} (n, q)^{1/2}. \end{aligned}$$

Hence, by (4.5), we obtain

$$\begin{aligned}
\sum_{q>Z} |A(n, q)| &\ll \sum_{q>Z} q^{-5/2+\varepsilon} (n, q)^{1/2} = \sum_{d|n} \sum_{q>Z/d} (dq)^{-5/2+\varepsilon} d^{1/2} \\
&= \sum_{d|n} d^{-2+\varepsilon} \sum_{q>Z/d} q^{-5/2+\varepsilon} \ll \sum_{d|n} d^{-2+\varepsilon} \left(\frac{Z}{d}\right)^{-3/2+\varepsilon} \\
&= Z^{-3/2+\varepsilon} \sum_{d|n} d^{-1/2+\varepsilon} \ll Z^{-3/2+\varepsilon} d(n).
\end{aligned}$$

This proves (i) of Lemma 4.6.

To prove (ii) of Lemma 4.6, by Lemma 4.5, we first note that

$$\begin{aligned}
(4.9) \quad \mathfrak{S}(n) &= \prod_p \left(1 + \sum_{t=1}^{\infty} A(n, p^t)\right) = \prod_p (1 + A(n, p)) \\
&= \prod_{p \leq c_3} (1 + A(n, p)) \prod_{\substack{p > c_3 \\ p \nmid n}} (1 + A(n, p)) \prod_{\substack{p > c_3 \\ p | n}} (1 + A(n, p)).
\end{aligned}$$

From (4.7) we have

$$(4.10) \quad \prod_{\substack{p > c_3 \\ p \nmid n}} (1 + A(n, p)) \geq \prod_{p > c_3} \left(1 - \frac{c_3}{p^{5/2}}\right) \geq c_4 > 0.$$

By (4.8) we obtain

$$(4.11) \quad \prod_{\substack{p > c_3 \\ p | n}} (1 + A(n, p)) \geq \prod_{p > c_3} \left(1 - \frac{c_3}{p^2}\right) \geq c_5 > 0.$$

On the other hand, it is easy to see that

$$(4.12) \quad 1 + A(n, p) = \frac{p\mathcal{L}(p, n)}{\varphi^6(p)}.$$

By Lemma 4.4 we know that for an even  $n$  we have  $\mathcal{L}(p, n) > 0$  for all  $p$  and thus  $1 + A(n, p) > 0$ . Therefore it holds that

$$(4.13) \quad \prod_{p \leq c_3} (1 + A(n, p)) \geq c_6 > 0.$$

Combining estimates (4.9)–(4.11) and (4.13), and taking  $c^* = c_4 c_5 c_6 > 0$ , we derive that

$$\mathfrak{S}(n) \geq c^* > 0.$$

This completes the proof Lemma 4.6. □

## 5. PROOF OF PROPOSITION 2.2

In this section, we shall present some lemmas that will be used to prove Proposition 2.2.

**Lemma 5.1.** *Let  $w_k(q)$  denote the multiplicative function defined by*

$$w(p^{uk+v}) = p^{-u-1} \quad \text{when } u \geq 0 \text{ and } 2 \leq v \leq k,$$

and

$$w(p^{uk+1}) = kp^{-u-1/2} \quad \text{when } u \geq 0.$$

Then

$$q^{-1}|S_k(q, a)| \ll w_k(q)$$

and

$$q^{-1/2} \leq w_k(q) \ll q^{-1/k}.$$

*Proof.* See [11], Lemma 3. □

**Lemma 5.2.** *Let  $c_k$  be a constant. For  $Q \geq 2$  one has*

$$\sum_{q \leq Q} d^{c_k}(q) w_k^2(q) \ll \log^{C_k} Q,$$

where  $C_k$  is a constant depending only on  $k$ .

*Proof.* See [18], Lemma 2.1. □

**Lemma 5.3.** *Suppose that  $x > 0$ ,  $y > 0$ ,  $x^{3/5} \ll y \ll x$ . Then either*

$$\sum_{x-y < m \leq x+y} e(m^3 \alpha) \ll y^{3/4+\varepsilon} + x^{1/2+\varepsilon} y^{1/6},$$

or there exist integers  $a \in \mathbb{Z}$  and  $q \in \mathbb{N}$  such that

$$1 \leq q \leq y^{10/3} x^{-2}, \quad (a, q) = 1, \quad |q\alpha - a| \leq y^{1/3} x^{-2},$$

and

$$\sum_{x-y < m \leq x+y} e(m^3 \alpha) \ll \frac{w_3(q)y}{1 + x^2 y |\alpha - a/q|} + x^{1/2+\varepsilon} y^{1/6}.$$

Proof. By Dirichlet's lemma on rational approximation, each real number  $\alpha$  can be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{qx^2y^{-1/3}}$$

for some integers  $a \in \mathbb{Z}$  and  $q \in \mathbb{N}$  with  $1 \leq a \leq q \leq x^2y^{-1/3}$  and  $(a, q) = 1$ . For  $y^{10/3}x^{-2} \leq q \leq x^2y^{-1/3}$ , by writing  $m = x - y + t$ , it follows from Weyl's inequality (see [13], Lemma 2.4) that

$$\begin{aligned} \sum_{x-y < m \leq x+y} e(m^3\alpha) &= \sum_{1 \leq t \leq 2y} e((x-y+t)^3y) \\ &= e((x-y)^3y) \sum_{1 \leq t \leq 2y} e(t^3\alpha + 3t^2(x-y)\alpha + 3t(x-y)^2\alpha) \\ &\ll y^{1+\varepsilon} \left( \frac{1}{q} + \frac{1}{y} + \frac{q}{y^3} \right)^{1/4} \ll y^{1+\varepsilon} \left( \frac{1}{y^{10/3}x^{-2}} + \frac{1}{y} + \frac{x^2y^{-1/3}}{y^3} \right)^{1/4} \\ &\ll x^{1/2+\varepsilon}y^{1/6} + y^{3/4+\varepsilon}. \end{aligned}$$

For  $1 \leq q \leq y^{10/3}x^{-2}$ , it follows from Theorem 4.1 of [13] that

$$\sum_{x-y < m \leq x+y} e(m^3\alpha) = q^{-1}S_3(q, a) \int_{x-y}^{x+y} e(u^3\lambda) du + O(q^{1/2+\varepsilon}(1+x^3|\lambda|)^{1/2}),$$

which combined with Lemma 5.1 and Lemma 4.8 of [10] yields

$$\begin{aligned} \sum_{x-y < m \leq x+y} e(m^3\alpha) &\ll w_3(q) \min\left(y, \frac{1}{x^2|\lambda|}\right) + (y^{10/3}x^{-2})^{1/2+\varepsilon} + x^{3/2+\varepsilon}(y^{1/3}x^{-2})^{1/2} \\ &\ll \frac{w_3(q)y}{1+x^2y|\alpha-a/q|} + x^{1/2+\varepsilon}y^{1/6}. \end{aligned}$$

This completes the proof of Lemma 5.3. □

For  $\mathcal{A} \subseteq [x_3 - y_3, x_3 + y_3] \cap \mathbb{N}$  we define

$$(5.1) \quad g(\alpha) = g_{\mathcal{A}}(\alpha) = \sum_{m \in \mathcal{A}} (\log m) e(m^3\alpha).$$

**Lemma 5.4.** *Let  $\mathcal{M}$  be defined as*

$$\mathcal{M} = \bigcup_{1 \leq q \leq y_3^{10/3}x_3^{-2}} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathcal{M}(q, a),$$

where

$$\mathcal{M}(q, a) = \{\alpha : |q\alpha - a| \leq x_3^{-2}y_3^{1/3}\}.$$

Suppose that  $G(\alpha)$  and  $h(\alpha)$  are integrable functions of period one. Let  $g(\alpha) = g_{\mathcal{A}}(\alpha)$  be given in (5.1), and let  $\mathfrak{n} \subseteq [0, 1)$  be a measurable set. Then we have

$$\int_{\mathfrak{n}} g(\alpha)G(\alpha)h(\alpha) \, d\alpha \ll y_3(\log x_3)\mathcal{J}_0^{1/4} \left( \int_{\mathfrak{n}} |G(\alpha)|^2 \, d\alpha \right)^{1/4} \mathcal{I}^{1/2}(\mathfrak{n}) \\ + (x_3^{1/4+\varepsilon} y_3^{7/12} + y_3^{7/8+\varepsilon}) \mathcal{I}(\mathfrak{n}),$$

where

$$\mathcal{I}(\mathfrak{n}) = \int_{\mathfrak{n}} |G(\alpha)h(\alpha)| \, d\alpha, \quad \mathcal{J}_0 = \sup_{\beta \in [0,1)} \int_{\mathcal{M}} \frac{w_3^2(q)|h(\alpha + \beta)|^2}{(1 + x_3^2 y_3 |\alpha - a/q|)^2} \, d\alpha.$$

*Proof.* By the definition of  $g(\alpha)$ , it follows from Cauchy's inequality that

$$(5.2) \quad \left| \int_{\mathfrak{n}} g(\alpha)G(\alpha)h(\alpha) \, d\alpha \right| = \left| \sum_{m \in \mathcal{A}} (\log m) \int_{\mathfrak{n}} e(m^3 \alpha)G(\alpha)h(\alpha) \, d\alpha \right| \\ \ll \sum_{m \in \mathcal{A}} (\log m) \left| \int_{\mathfrak{n}} e(m^3 \alpha)G(\alpha)h(\alpha) \, d\alpha \right| \\ \ll (\log x_3) \sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha)G(\alpha)h(\alpha) \, d\alpha \right| \\ \ll y_3^{1/2} (\log x_3) \left( \sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha)G(\alpha)h(\alpha) \, d\alpha \right|^2 \right)^{1/2}.$$

For the inner sum on the right-hand side of (5.2) we have

$$\sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha)G(\alpha)h(\alpha) \, d\alpha \right|^2 \\ = \sum_{x_3 - y_3 < m \leq x_3 + y_3} \int_{\mathfrak{n}} \int_{\mathfrak{n}} G(\alpha)h(\alpha)G(-\beta)h(-\beta)e(m^3(\alpha - \beta)) \, d\alpha \, d\beta \\ = \int_{\mathfrak{n}} \int_{\mathfrak{n}} G(\alpha)h(\alpha)G(-\beta)h(-\beta)\mathfrak{F}(\alpha - \beta) \, d\alpha \, d\beta,$$

where

$$\mathfrak{F}(\Xi) = \sum_{x_3 - y_3 < m \leq x_3 + y_3} e(m^3 \Xi).$$

For any  $\beta \in \mathbb{R}$  we define

$$\mathfrak{J}(\beta) = \int_{\mathfrak{n}} |G(\alpha)h(\alpha)\mathfrak{F}(\alpha - \beta)| \, d\alpha.$$

Then it is easy to see that

$$(5.3) \quad \sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha) G(\alpha) h(\alpha) d\alpha \right|^2 \leq \int_{\mathfrak{n}} |G(\beta) h(\beta)| \mathfrak{I}(\beta) d\beta.$$

Define

$$\mathcal{M}_\beta = \bigcup_{1 \leq q \leq y_3^{10/3} x_3^{-2}} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \{\alpha : |q\alpha - q\beta - a| \leq y_3^{1/3} x_3^{-2}\}.$$

By Lemma 5.3 we derive that

$$(5.4) \quad \begin{aligned} \mathfrak{I}(\beta) &= \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\alpha) h(\alpha)| \mathfrak{F}(\alpha - \beta) d\alpha + \int_{\mathfrak{n} \setminus \mathcal{M}_\beta} |G(\alpha) h(\alpha)| \mathfrak{F}(\alpha - \beta) d\alpha \\ &\ll y_3 \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\alpha) h(\alpha)| \frac{w_3(q)}{1 + x_3^2 y_3 |\alpha - \beta - a/q|} d\alpha \\ &\quad + x_3^{1/2 + \varepsilon} y_3^{1/6} \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\alpha) h(\alpha)| d\alpha \\ &\quad + x_3^{1/2 + \varepsilon} y_3^{1/6} \int_{\mathfrak{n} \setminus \mathcal{M}_\beta} |G(\alpha) h(\alpha)| d\alpha + y_3^{3/4 + \varepsilon} \int_{\mathfrak{n}} |G(\alpha) h(\alpha)| d\alpha \\ &\ll y_3 \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\alpha) h(\alpha)| \frac{w_3(q)}{1 + x_3^2 y_3 |\alpha - \beta - a/q|} d\alpha \\ &\quad + x_3^{1/2 + \varepsilon} y_3^{1/6} \int_{\mathfrak{n}} |G(\alpha) h(\alpha)| d\alpha + y_3^{3/4 + \varepsilon} \int_{\mathfrak{n}} |G(\alpha) h(\alpha)| d\alpha. \end{aligned}$$

From (5.3) and (5.4) we obtain

$$(5.5) \quad \begin{aligned} \sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha) G(\alpha) h(\alpha) d\alpha \right|^2 \\ \ll y_3 \int_{\mathfrak{n}} \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\beta) h(\beta)| |G(\alpha) h(\alpha)| \frac{w_3(q)}{1 + x_3^2 y_3 |\alpha - \beta - a/q|} d\alpha d\beta \\ + (x_3^{1/2 + \varepsilon} y_3^{1/6} + y_3^{3/4 + \varepsilon}) \left( \int_{\mathfrak{n}} |G(\alpha) h(\alpha)| d\alpha \right)^2. \end{aligned}$$

By noting the fact that  $\alpha \in \mathcal{M}_\beta$  is equivalent to  $\alpha - \beta \in \mathcal{M}$  for the inner integral of

the first term on the right-hand side of (5.5), it follows from Cauchy's inequality that

$$\begin{aligned}
 (5.6) \quad & \int_{\mathfrak{n} \cap \mathcal{M}_\beta} |G(\alpha)h(\alpha)| \frac{w_3(q)}{1 + x_3^2 y_3 |\alpha - \beta - a/q|} d\alpha \\
 & \ll \left( \int_{\mathfrak{n}} |G(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_{\mathcal{M}_\beta} \frac{w_3^2(q) |h(\alpha)|^2}{(1 + x_3^2 y_3 |\alpha - \beta - a/q|)^2} d\alpha \right)^{1/2} \\
 & \ll \left( \int_{\mathfrak{n}} |G(\alpha)|^2 d\alpha \right)^{1/2} \left( \sup_{\beta \in [0,1)} \int_{\mathcal{M}} \frac{w_3^2(q) |h(\alpha + \beta)|^2}{(1 + x_3^2 y_3 |\alpha - a/q|)^2} d\alpha \right)^{1/2} \\
 & \ll \left( \int_{\mathfrak{n}} |G(\alpha)|^2 d\alpha \right)^{1/2} \mathcal{J}_0^{1/2}.
 \end{aligned}$$

Putting (5.6) into (5.5), we deduce that

$$\begin{aligned}
 & \sum_{x_3 - y_3 < m \leq x_3 + y_3} \left| \int_{\mathfrak{n}} e(m^3 \alpha) G(\alpha) h(\alpha) d\alpha \right|^2 \\
 & \ll y_3 \mathcal{J}_0^{1/2} \left( \int_{\mathfrak{n}} |G(\alpha)|^2 d\alpha \right)^{1/2} \left( \int_{\mathfrak{n}} |G(\alpha)h(\alpha)| d\alpha \right) \\
 & \quad + (x_3^{1/2+\varepsilon} y_3^{1/6} + y_3^{3/4+\varepsilon}) \left( \int_{\mathfrak{n}} |G(\alpha)h(\alpha)| d\alpha \right)^2,
 \end{aligned}$$

which combined with (5.2) yields that

$$\begin{aligned}
 \left| \int_{\mathfrak{n}} g(\alpha) G(\alpha) h(\alpha) d\alpha \right| & \ll y_3 (\log x_3) \mathcal{J}_0^{1/4} \left( \int_{\mathfrak{n}} |G(\alpha)|^2 d\alpha \right)^{1/4} \mathcal{I}^{1/2}(\mathfrak{n}) \\
 & \quad + (x_3^{1/4+\varepsilon} y_3^{7/12} + y_3^{7/8+\varepsilon}) \mathcal{I}(\mathfrak{n}).
 \end{aligned}$$

This completes the proof of Lemma 5.3. □

**Lemma 5.5.** *Let  $\gamma \in \mathbb{R}$ ,  $D > 0$ ,  $\mathcal{Q} > 1$  and  $\mathcal{Q} < y_2 < x_2$ . We define*

$$\mathcal{L}(\gamma) := \sum_{q \leq \mathcal{Q}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{|\alpha - a/q| \leq 1} \frac{w_3^2(q) \left| \sum_{x_2 - y_2 < p \leq x_2 + y_2} (\log p) e(p^2(\alpha + \gamma)) \right|^2}{(1 + D|\alpha - a/q|)^2} d\alpha.$$

Then there exists a constant  $C_0 > 0$  such that

$$\mathcal{L}(\gamma) \ll y_2^2 D^{-1} (\log x_2)^{C_0}.$$

Proof. We have

$$(5.7) \quad \mathcal{L}(\gamma) \leq \sum_{q \leq \mathcal{Q}} w_3^2(q) \int_{|\beta| \leq 1} \frac{\sum_{1 \leq a \leq q} \left| \sum_{x_2 - y_2 < p \leq x_2 + y_2} (\log p) e(p^2(a/q + \beta + \gamma)) \right|^2}{(1 + D|\beta|)^2} d\beta.$$

For the inner sum in (5.7) one has

$$\begin{aligned} & \sum_{a=1}^q \left| \sum_{x_2 - y_2 < p \leq x_2 + y_2} (\log p) e\left(p^2\left(\frac{a}{q} + \beta + \gamma\right)\right) \right|^2 \\ &= \sum_{a=1}^q \sum_{\substack{x_2 - y_2 < p_1 \\ p_2 \leq x_2 + y_2}} (\log p_1)(\log p_2) e\left((p_1^2 - p_2^2)\frac{a}{q} + (p_1^2 - p_2^2)(\beta + \gamma)\right) \\ &= \sum_{\substack{x_2 - y_2 < p_1 \\ p_2 \leq x_2 + y_2}} (\log p_1)(\log p_2) e((p_1^2 - p_2^2)(\beta + \gamma)) \sum_{a=1}^q e\left(\frac{(p_1^2 - p_2^2)a}{q}\right) \\ &= q \sum_{\substack{x_2 - y_2 < p_1 \\ p_2 \leq x_2 + y_2 \\ p_1^2 \equiv p_2^2 \pmod{q}}} (\log p_1)(\log p_2) e((p_1^2 - p_2^2)(\beta + \gamma)). \end{aligned}$$

Since  $q \leq \mathcal{Q} < y_2 < x_2$  and  $p_1, p_2 \asymp x_2$ , we have  $(p_1, q) = (p_2, q) = 1$ , and thus

$$(5.8) \quad \begin{aligned} \sum_{a=1}^q \left| \sum_{x_2 - y_2 < p \leq x_2 + y_2} (\log p) e\left(p^2\left(\frac{a}{q} + \beta + \gamma\right)\right) \right|^2 &\ll \frac{y_2^2}{q} (\log x_2)^2 \sum_{\substack{1 \leq n_1, n_2 < q \\ (n_1 n_2, q) = 1 \\ n_1^2 \equiv n_2^2 \pmod{q}}} 1 \\ &\ll \frac{y_2^2 (\log x_2)^2}{q} \sum_{1 \leq n_1 < q} \sum_{\substack{1 \leq n_2 < q \\ n_1^2 \equiv n_2^2 \pmod{q}}} 1 \ll y_2^2 d^c(q) (\log x_2)^2. \end{aligned}$$

Therefore from (5.7), (5.8) and Lemma 5.2 we deduce that

$$\mathcal{L}(\gamma) \ll y_2^2 (\log x_2)^2 \left( \sum_{q \leq \mathcal{Q}} w_3^2(q) d^c(q) \right) \int_{|\beta| \leq 1} \frac{d\beta}{(1 + D|\beta|)^2} \ll y_2^2 D^{-1} (\log x_2)^{C_0}.$$

This completes the proof of Lemma 5.5.  $\square$

**Lemma 5.6.** Let  $\mathfrak{m}$  be defined as in (2.3) and  $\vartheta_3$  be a real number with  $8/9 < \vartheta_3 \leq 1$ . Suppose that  $0 < \varrho \leq \varrho_3(\vartheta_3)$ , where

$$\varrho_3(\vartheta_3) = \min\left(\frac{2\vartheta_3 - 1}{14}, \frac{9\vartheta_3 - 8}{6}\right).$$



Then for any fixed  $\varepsilon > 0$  it holds that

$$\sup_{\alpha \in \mathfrak{m}} \left| \sum_{x_3 < m \leq x_3 + x_3^{\vartheta_3}} \Lambda(m) e(m^3 \alpha) \right| \ll x_3^{\vartheta_3 - \varrho + \varepsilon} + x_3^{\vartheta_3 + \varepsilon} P^{-1/2}.$$

*Proof.* See [3], Theorem 2. □

*Proof of Proposition 2.2.* Taking

$$\begin{aligned} h(\alpha) &= f_2(\alpha), \quad g(\alpha) = f_3(\alpha), \quad x_3 = N^{1/3}, \quad y_3 = UN^{-2/3}, \\ G(\alpha) &= f_2(-\alpha) |f_3(\alpha)|^8 f_3(-\alpha), \end{aligned}$$

in Lemma 5.4 we obtain

$$(5.9) \quad \mathcal{I}(10) \ll (UN^{-2/3})(\log N) \mathcal{J}_0^{1/4} \left( \int_{\mathfrak{m}} |f_2^2(\alpha) f_3^{18}(\alpha)| d\alpha \right)^{1/4} \mathcal{I}^{1/2}(9) \\ + (N^{1/12 + \varepsilon} (UN^{-2/3})^{7/12} + (UN^{-2/3})^{7/8 + \varepsilon}) \mathcal{I}(9),$$

where

$$\mathcal{J}_0 = \sup_{\beta \in [0,1]} \sum_{1 \leq q \leq U^{10/3} N^{-26/9}} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\mathcal{M}(q,a)} \frac{w_3^2(q) |h(\alpha + \beta)|^2}{(1 + U|\alpha - a/q|)^2} d\alpha$$

with

$$\mathcal{M}(q, a) = \{\alpha : |q\alpha - a| \leq U^{1/3} N^{-8/9}\}.$$

By Lemma 5.5 with parameters  $\mathcal{Q} = U^{10/3} N^{-26/9}$ ,  $y_2 = UN^{-1/2}$ ,  $x_2 = N^{1/2}$ ,  $D = U$ , we deduce that

$$(5.10) \quad \mathcal{J}_0 \ll UN^{-1} (\log N)^{C_0}.$$

For  $\alpha \in \mathfrak{m}$ , according to Lemma 5.6 by taking  $x_3 = N^{1/3}$  and  $y_3 = x_3^{\vartheta_3} = UN^{-2/3}$  with  $U$  defined in (2.1), we know that  $\vartheta_3 = 201/225 + \varepsilon$ , and thus  $\varrho_3(\vartheta_3) = (9\vartheta_3 - 8)/6$ . Therefore, taking  $\varrho = \varrho_3(\vartheta_3) = (9\vartheta_3 - 8)/6$ , we obtain

$$(5.11) \quad \sup_{\alpha \in \mathfrak{m}} |f_3(\alpha)| \ll x_3^{\vartheta_3 - \varrho + \varepsilon} + x_3^{\vartheta_3 + \varepsilon} P^{-1/2} = x_3^{4/3 - \vartheta_3/2 + \varepsilon} + x_3^{\vartheta_3 + \varepsilon} P^{-1/2} \\ \ll (N^{1/3})^{4/3 + \varepsilon} (UN^{-2/3})^{-1/2} + (UN^{-2/3})^{1 + \varepsilon} (U^2 N^{-37/20})^{-1/2} \\ \ll U^{-1/2 + \varepsilon} N^{7/9} + N^{31/120 - \varepsilon} \ll U^{-1/2 + \varepsilon} N^{7/9}.$$

Hence, we deduce that

$$(5.12) \quad \mathcal{I}(18) \ll \sup_{\alpha \in \mathfrak{m}} |f_3(\alpha)|^8 \mathcal{I}(10) \ll (U^{-1/2 + \varepsilon} N^{7/9})^8 \mathcal{I}(10) = U^{-4 + \varepsilon} N^{56/9} \mathcal{I}(10).$$

It follows from Hölder's inequality that

$$\begin{aligned}
 (5.13) \quad \mathcal{I}(9) &= \int_{\mathfrak{m}} |(f_2^{3/2}(\alpha)f_3^{15/2}(\alpha))(f_2^{1/2}(\alpha)f_3^{3/2}(\alpha))| \, d\alpha \\
 &\ll \left( \int_{\mathfrak{m}} |f_2^2(\alpha)f_3^{10}(\alpha)| \, d\alpha \right)^{3/4} \left( \int_{\mathfrak{m}} |f_2^2(\alpha)f_3^6(\alpha)| \, d\alpha \right)^{1/4} \\
 &\ll (\mathcal{I}(10))^{3/4} (\mathcal{I}(6))^{1/4}.
 \end{aligned}$$

From Lemma 6.6 of Zhang and Li [17] we know that

$$(5.14) \quad \mathcal{I}(6) \ll \int_0^1 |f_2^2(\alpha)f_3^6(\alpha)| \, d\alpha \ll U^{6+\varepsilon} N^{-4}.$$

By (5.13) and (5.14), we obtain

$$(5.15) \quad \mathcal{I}(9) \ll (\mathcal{I}(10))^{3/4} U^{3/2+\varepsilon} N^{-1}.$$

Combining (5.9), (5.10), (5.12) and (5.15), we derive that

$$\begin{aligned}
 \mathcal{I}(10) &\ll U^{1+\varepsilon} N^{5/36} (\mathcal{I}(10))^{5/8} + U^{25/12+\varepsilon} N^{-47/36} (\mathcal{I}(10))^{3/4} \\
 &\quad + U^{19/8+\varepsilon} N^{-19/12} \mathcal{I}(10)^{3/4}
 \end{aligned}$$

which implies

$$\mathcal{I}(10) \ll U^{8/3+\varepsilon} N^{10/27} + U^{25/3+\varepsilon} N^{-47/9} + U^{19/2+\varepsilon} N^{-19/3} \ll U^{8/3+\varepsilon} N^{10/27}.$$

This completes the proof of Proposition 2.2. □

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*Authors’ addresses:* Fei Xue, Department of Mathematics, China University of Mining and Technology, Beijing 100083, P.R. China, e-mail: fei.xue.math@gmail.com; Min Zhang (corresponding author), School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, P.R. China, e-mail: min.zhang.math@gmail.com; Jinjiang Li, Department of Mathematics, China University of Mining and Technology, Beijing 100083, P.R. China, e-mail: jinjiang.li.math@gmail.com.