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COMBINATORIAL INTERPRETATIONS FOR IDENTITIES USING CHROMATIC PARTITIONS

Mateus Alegri, Wagner Ferreira Santos,
Samuel Brito Silva, Itabaiana

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Abstract. We provide combinatorial interpretations for three new classes of partitions, the so-called chromatic partitions. Using only combinatorial arguments, we show that these partition identities resemble well-know ordinary partition identities.

Keywords: integer partition; chromatic partition; Ferrers graph; partition identity

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1. Introduction

We use the following standard notation:

$$(a,q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}) & \text{if } n>0,\\ 1 & \text{if } n=0, \end{cases}$$
$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n \text{ for a complex number } q, \text{where } |q|<1.$$

The theory of partition identities began with Euler (see [6]) and has been well-developed ever since. Bijective proofs, as described in [3] and [7], arise every year. The necessity for simple generalizations of the integer class of partitions generates new classes such as the overpartitions of integers instituted by Lovejoy in [5].

Our objective is to find combinatorial interpretations for hypergeometric series which involve terms such as $(1-kq^n)^{-1}$ and to obtain new partition identities.

To combinatorially interpret the terms $1-kq^n$ in the series as in the following:

$$\prod_{n=1}^{l}\frac{1}{1-kq^n},$$

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we will use the concept of the chromatic partition of an integer n. We will assume that if we change the color order of equal parts in a partition, this generates a different partition. For example, for the two colors, black and white, $1_{\mathrm{b}} + 1_{\mathrm{w}}$ is different from $1_{\mathrm{w}} + 1_{\mathrm{b}}$, where the subscripts b and w indicate that the part 1 is colored by black and white, respectively. In this case, the number of partitions of n in colored parts with two colors is given by the coefficient of q^n in $\prod_{n=1}^{\infty}(1-2q^n)^{-1}$. For example, the number of partitions of $n = 4$ in colored parts with two colors is given by the coefficient of q^4 in the product above, which is equal to 34.

In the last theorem, we work with a special case of a well-known result. For $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\alpha} y^n}{(q;q)_n} = (-y;q)_{\infty}, \text{ where } \alpha = \binom{n}{2}.$$

We consider $y = -xqk$ to obtain a demonstration of this identity in terms of super-chromatic partitions.

The definition of super-chromatic partitions is related to products such as $(kq;kq)_{\infty}^{-1}$. Using these concepts with a modified graph from the Ferrers graph, we obtain a generalization of an important identity involving the Durfee squares of a partition, as defined in Theorem 2 in [2].

We emphasize the fact that these demonstrations are not just made from variable changes. In fact, we get new proofs by using combinatorial arguments that generalize well-known identities.

2. Definitions and generating functions

Definition 1. A (n,k)-*chromatic partition* is an integer partition of n in which every part receives an index, ranging from 0 to $k-1$, referring to colours. The order of appearance of coloured parts should be considered, i.e., equal parts associated to different colours are considered distinct.

Definition 2. For positive integers n and k, an (n,k)-*super-chromatic partition* is a chromatic partition of n, where each part j can appear with up to k^j different colours.

Definition 3. An (n,k)-*semi-chromatic partition* is a partition of n in which the greatest part λ_1, receives a color in $\{0,\ldots,k^{\lambda_1}-1\}$ and the others appear non-colored.

Example 1.

▷ $5_8 + 5_3 + 3_1 + 3_7 + 2_5 + 1_8$ is a $(19,9)$-chromatic partition distinct from $5_3 + 5_8 + 3_1 + 3_7 + 2_5 + 1_8$.

▷ $5_{31} + 5_{30} + 4_{15} + 3_7 + 2_3 + 1_0$ is a $(20, 2)$-super-chromatic partition different from $5_{30} + 5_{31} + 4_{15} + 3_7 + 2_3 + 1_0$.

▷ For $n = k = 4$, 4_a, $3_a + 1_b$, $2_a + 2_b$,$2_a + 1_b + 1_c$, $1_a + 1_b + 1_c + 1_d$ are indexed partitions, and taking colours a, b, c and d, where $0 \leqslant a, b, c, d \leqslant 3$, there are 356 $(4, 4)$-chromatic partitions.

In this paper, we will denote the number of integer partitions of n by $p(n)$, the number of (n, k)-chromatic partitions by $p_k(n)$, the number of (n, k)-semi-chromatic partitions by $m_k(p)$, and the number of (n, k)-super-chromatic partitions by $sp_k(n)$.

For $|kq| < 1$, the infinite products $(kq, q)_\infty^{-1}$ and $(kq, kq)_\infty^{-1}$ are the generating functions for the number of chromatic partitions and super-chromatic partitions with k colours, respectively.

Definition 4. A *k-chromatic Ferrers graph* of a partition is a Ferrers graph of its partition in which every dot receives an index ranging from 0 to $k - 1$, referring to k distinct colors.

Example 2. The graph below is a 4-chromatic Ferrers graph of the partition $4 + 3 + 2 + 2 + 1$. See

$$
\begin{array}{llll}
\circ_0 & \circ_2 & \circ_1 & \circ_3 \\
\circ_3 & \circ_0 & \circ_1 & \\
\circ_2 & \circ_2 & & \\
\circ_1 & \circ_3 & & \\
\circ_2 & & &
\end{array}
$$

We will see in the next section that this representation is associated with $(12, 4)$-super-chromatic partitions.

Definition 5. For positive integers n, k and m, a *k-chromatic Gaussian polynomial* is obtained from a Gaussian polynomial by changing the variables q to kq. With the same notation,

$$
\begin{bmatrix} n \\ m \end{bmatrix}_{kq} = \frac{(1 - kq)(1 - k^2q^2) \dots (1 - k^nq^n)}{(1 - kq)(1 - k^2q^2) \dots (1 - k^{n-m}q^{n-m})(1 - kq)(1 - k^2q^2) \dots (1 - k^mq^m)}.
$$

Example 3. For $n = 5$ and $m = 2$,

$$
\begin{bmatrix} 5 \\ 2 \end{bmatrix}_{kq} = \frac{(1 - kq)(1 - k^2q^2) \dots (1 - k^5q^5)}{(1 - kq)(1 - k^2q^2)(1 - kq)(1 - k^2q^2)(1 - k^3q^3)}.
$$

Making simplifications,

$$
\begin{bmatrix} 5 \\ 2 \end{bmatrix}_{kq} = 1 + kq + 2k^2q^2 + 2k^3q^3 + 2k^4q^4 + k^5q^5 + k^6q^6.
$$

3. Results

The next result shows a bijection between (n, k)-chromatic and (n, k)-semi-chromatic partitions. The main tool used in this proof is a transformation that transposes the colored Ferrers graph representation of a (n, k)-chromatic and recolors in a way that obtains a (n, k)-semi-chromatic partition. This transformation can be understood through the following example.

Example 4. Let $4_1 + 3_2 + 2_2 + 2_0 + 1_1$ be a $(12, 3)$-chromatic partition, whose 3-colored Ferrers graph representation is

$$\begin{array}{llll} \circ_1 & \circ_1 & \circ_1 & \circ_1 \\ \circ_2 & \circ_2 & \circ_2 & \\ \circ_2 & \circ_2 & & \\ \circ_0 & \circ_0 & & \\ \circ_1 & & & \end{array}$$

If we take the transpose of this graph representation, we have

$$\begin{array}{lllll} \circ_1 & \circ_2 & \circ_2 & \circ_0 & \circ_1 \\ \circ_1 & \circ_2 & \circ_2 & \circ_0 & \\ \circ_1 & \circ_2 & & & \\ \circ_1 & & & & \end{array}$$

Note that this is not a $(12, 3)$-semi-chromatic partition, but if we take a suitable color to the major part and color by 0 the others, we will have

$$\begin{array}{lllll} \circ_\beta & \circ_\beta & \circ_\beta & \circ_\beta & \circ_\beta \\ \circ_0 & \circ_0 & \circ_0 & \circ_0 & \\ \circ_0 & \circ_0 & & & \\ \circ_0 & & & & \end{array}$$

If we take $\beta \in \{0, 1, \ldots, 3^5\}$, then $5_\beta + 4_0 + 2_0 + 1_0$ will be a $(12, 3)$-semi-chromatic partition. A way that we surely will take this is if we concatenate the colors of each part of the original $(12, 3)$-chromatic partition and interpret this as written in 3-base, that is, $(12201)_3 = 1.3^4 + 2.3^3 + 2.3^2 + 0.3^1 + 1.3^0 = 154$.

Theorem 1. *The number of (n, k)-chromatic partitions is equal to the number of (n, k)-semi-chromatic partitions.*

P r o o f. Let $(\lambda_1)_{\alpha_1} + \ldots + (\lambda_m)_{\alpha_m}$ with $\lambda_1 + \ldots + \lambda_m = n$, so that $\lambda_1 \geqslant \ldots \geqslant \lambda_m \geqslant 1$, and $\alpha_i \in \{0, \ldots, k-1\}$ is a (n, k)-chromatic partition. Taking the transpose of its colored Ferrers graph representation and recoloring its parts, we obtain $(\bar{\lambda}_1)_\beta + (\bar{\lambda}_2)_0 + \ldots + (\bar{\lambda}_{\lambda_1})_0$ with $\beta = \sum_{i=1}^{m} \alpha_i k^{m-i}$, and we have $\bar{\lambda}_1 = m$ and $\beta \in \{0, \ldots, k^m - 1\}$. So, $(\bar{\lambda}_1)_\beta + (\bar{\lambda}_2)_0 + \ldots + (\bar{\lambda}_{\lambda_1})_0$ is a (n,k)-semi-chromatic partition associated with the (n,k)-chromatic partition original $(\lambda_1)_{\alpha_1} + \ldots + (\lambda_m)_{\alpha_m}$. Because the representation in a k-base is unique, we obtained a 1–1 association between the (n,k)-semi-chromatic and (n,k)-chromatic partitions. □

For $n = 3$ and $k = 2$, the next table shows the chromatic partition and the correspondent semi-chromatic partition under the transformation described in Theorem 1.

chromatic	semi-chromatic	chromatic	semi-chromatic
3_0	$1_0 + 1_0 + 1_0$	3_1	$1_1 + 1_0 + 1_0$
$2_0 + 1_0$	$2_0 + 1_0$	$2_0 + 1_1$	$2_1 + 1_0$
$2_1 + 1_0$	$2_2 + 1_0$	$2_1 + 1_1$	$2_3 + 1_0$
$1_0 + 1_0 + 1_0$	3_0	$1_0 + 1_0 + 1_1$	3_1
$1_0 + 1_1 + 1_0$	3_2	$1_1 + 1_0 + 1_0$	3_4
$1_0 + 1_1 + 1_1$	3_3	$1_1 + 1_0 + 1_1$	3_5
$1_1 + 1_1 + 1_0$	3_6	$1_1 + 1_1 + 1_1$	3_7

Table 1. Bijective association between $(3,2)$-chromatic and semi-chromatic partitions

In the next theorem we provide a generalization of a partition identity that may be proved using the concept of Durfee squares. Since a part j of a (n,k)-super-chromatic partition can appear in k^j ways in a chromatic partition, the colored Ferrers graph may be a useful tool to prove results.

Lemma 1. *For positive integers M, N and k,*

$$\begin{bmatrix} N+M \\ N \end{bmatrix}_{kq}$$

is a generating function for (n,k)-super-chromatic partitions with at most M parts, with each part $j \leqslant N$.

Proof. Considering a part j of a partition of n, represented as a row of a k-colored Ferrers graph, there are k^j ways of coloring the points of this row.

Considering a part j of a generic k-colored Ferrers graph representation of a partition of n, there is k^j ways to color a row representing j. It is known that, without colors,

$$\begin{bmatrix} M+N \\ N \end{bmatrix}_{q}$$

is a generating function of partitions of $n \geqslant 1$, at most M, parts, where each part does not exceed N. Adding the fact that each dot can appear colored in k ways, a k-colored graph of at most N columns and M rows represents a (n,k)-super-chromatic partitions at most M parts and each part $j \leqslant N$. Thus, in the set of (n,k)-super-chromatic partitions of at most M parts, each part $j \leqslant N$ is enumerated by

$$\begin{bmatrix} N+M \\ N \end{bmatrix}_{kq}.$$

$\square$

Theorem 2. *For* $|kq| < 1$,

$$\frac{1}{(kq;kq)_\infty} = \sum_{n=0}^{\infty} \frac{(kq)^{n^2}}{(kq;kq)_n^2}. \tag{3.1}$$

P r o o f. Considering the Ferrers graph of a partition of n like in the diagram below, it is easy to understand that if the partition has a Durfee square $j \times j$ then the square is generated by the term q^{j^2}. The term $(kq)^{j^2}$ generates the Durfee square $j \times j$ with k colors. The upper right portion of a chromatic partition is generated by

$$\begin{bmatrix} n \\ j \end{bmatrix}_{kq} = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^{n-j})(1-q)(1-q^2)\dots(1-q^j)},$$

and the lower portion by

$$\begin{bmatrix} n \\ n-j \end{bmatrix}_{kq} = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)(1-q^2)\dots(1-q^{n-j})(1-q)(1-q^2)\dots(1-q^j)}.$$

See the graph below:

$$\begin{array}{cccccccccc}
\circ_{a_1} & \circ_{a_2} & \circ_{a_3} & \circ_{a_4} & \circ_{a_5} & \circ_{a_6} & \circ_{a_7} & \circ_{a_8} & \circ_{a_9} & \circ_{a_{10}} \\
\circ_{b_1} & \circ_{b_2} & \circ_{b_3} & \circ_{b_4} & \circ_{b_5} & \circ_{b_6} & \circ_{b_7} & \circ_{b_8} & & \\
\circ_{c_1} & \circ_{c_2} & \circ_{c_3} & \circ_{c_4} & \circ_{c_5} & \circ_{c_6} & \circ_{c_7} & & & \\
\circ_{d_1} & \circ_{d_2} & \circ_{d_3} & \circ_{d_4} & \circ_{d_5} & & & & & \\
\circ_{e_1} & \circ_{e_2} & \circ_{e_3} & \circ_{e_4} & & & & & & \\
\circ_{f_1} & \circ_{f_2} & \circ_{f_3} & \circ_{f_4} & & & & & & \\
\circ_{g_1} & \circ_{g_2} & \circ_{g_3} & & & & & & & \\
\circ_{h_1} & \circ_{h_2} & & & & & & & & \\
\circ_{i_1} & \circ_{i_2} & & & & & & & & \\
\circ_{j_1} & & & & & & & & & \\
\circ_{l_1} & & & & & & & & &
\end{array}$$

Here, we used the identity

$$\begin{bmatrix} n \\ j \end{bmatrix}_{kq} = \begin{bmatrix} n \\ n-j \end{bmatrix}_{kq}.$$

The generating function for chromatic partitions into at most n colored parts, each of them at most n of a $j \times j$ Durfee square is

$$q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_{kq} \begin{bmatrix} n \\ j \end{bmatrix}_{kq}.$$

Taking the limit as $n \to \infty$, we have:

$$\sum_{n=0}^{\infty} \frac{(kq)^{n^2}}{(kq;kq)_n^2},$$

which is the generating function for super-chromatic partitions with k colors. $\square$

In order to prove Theorem 3, we will consider the term

$$A_n(x,k,q) = \left(\frac{xkq}{1-q}\right)\left(\frac{xk^2q^2}{1-q^2}\right)\cdots\left(\frac{xk^nq^n}{1-q^n}\right).$$

The coefficient of a term $x^m k^r q^l$ in the expansion of $A_n(x,k,q)$ is the number of partitions of l in which:

- ▷ the partition has m parts,
- ▷ there is a part multiple of s for $1 \leqslant s \leqslant m$.

We denote the set of these partitions by $\mathcal{A}_{n,k}$, that is, a kind of non-gap chromatic partition. For example, if $n = 8$, we have that the partitions in $\mathcal{A}_{8,k}$ enumerated by $A_n(x,k,q)$, $1 \leqslant n \leqslant 3$ for $j_1, j_2, j_3 \in \{0,1,2,\ldots k-1\}$, are the following:

- ▷ $1_{j_1} + 1_{j_1} + 1_{j_1} + 1_{j_1} + 1_{j_1} + 1_{j_1} + 1_{j_1} + 1_{j_1}$,
- ▷ $2_{j_1} + (1_{j_2} + 1_{j_2} + 1_{j_2} + 1_{j_2} + 1_{j_2} + 1_{j_2})$,
- ▷ $(2_{j_1} + 2_{j_1}) + (1_{j_2} + 1_{j_2} + 1_{j_2} + 1_{j_2})$,
- ▷ $(2_{j_1} + 2_{j_1} + 2_{j_1}) + (1_{j_2} + 1_{j_2})$,
- ▷ $3_{j_1} + 2_{j_2} + (1_{j_3} + 1_{j_3} + 1_{j_3})$,
- ▷ $3_{j_1} + (2_{j_2} + 2_{j_2}) + 1_{j_3}$.

The next theorem provides a identity between k-super-chromatic partitions generated by $\sum A_n(x,k,q)$ and super-chromatic partitions in distinct parts. We will use the k-colored Ferrers graph to obtain a bijective proof involving partitions in $\mathcal{A}_{n,k}$ and k-super-chromatic partitions in distinct parts.

From a partition of n in $\mathcal{A}_{n,k}$, we can describe its representation in a colored Ferrers graph. Using the conjugate operation in a Ferrers graph of its partition, we obtain a unique k-super-chromatic partition in distinct parts, due to the fact that the conjugation is a bijection. We consider that the transformation from a partition in $\mathcal{A}_{n,k}$ to where should take in a super chromatic partition in that the colors of this partition are written in k base. From this bijection we have the following theorem.

Theorem 3. *The number of (n, k)-super-chromatic partitions into distinct parts is equal to the number of $\mathcal{A}_{n,k}$ partitions.*

Example 5. The following diagram is the graphical representation of $4_{16} + 4_{29} + 3_7 + 2_5 + 2_6 + 1_1 + 1_0 + 1_2$ a 3-super-chromatic partition of 18. See

$$\begin{array}{llll}
\circ_0 & \circ_1 & \circ_2 & \circ_1 \\
\circ_1 & \circ_0 & \circ_0 & \circ_2 \\
\circ_0 & \circ_2 & \circ_1 & \\
\circ_1 & \circ_2 & & \\
\circ_2 & \circ_0 & & \\
\circ_1 & & & \\
\circ_0 & & & \\
\circ_2 & & &
\end{array}$$

The conjugate operation in the Ferrers graph takes this partition in a super-chromatic partition in distinct parts as following. Notice that we wrote the color numbers in base 3. See

$$\begin{array}{llllllll}
\circ_0 & \circ_1 & \circ_0 & \circ_1 & \circ_2 & \circ_1 & \circ_0 & \circ_2 \\
\circ_1 & \circ_0 & \circ_2 & \circ_2 & \circ_0 & & & \\
\circ_2 & \circ_0 & \circ_1 & & & & & \\
\circ_1 & \circ_2 & & & & & &
\end{array}$$

This graph represents the super-chromatic partition $8_{875} + 5_{105} + 3_{19} + 2_5$.

We conclude this article by citing some possible applications of some chromatic partition classes, as well as some contemporary results obtained with similar mathematical tools.

In [1], with similar ideas to those discussed here, for classes of chromatic overpartitions, the author found identities involving q-hypergeometric series such as the following:

$$\prod_{n=1}^{\infty}\frac{1+nq^n}{1-nq^n}=1+\sum_{n=0}^{\infty}\frac{2(1+q)(1+2q^2)\dots(1+(n-1)q^{(n-1)})nq^n}{(1-q)(1-2q^2)\dots(1-nq^n)},$$
$$\prod_{n=1}^{\infty}(1+2nq^n)=\prod_{j\text{ odd}}\Bigl(\sum_{n=0}^{\infty}c_j(n)q^n\Bigr),$$

where

$$c_1(r)=\begin{cases}1 & \text{if } r=0,\\ 2^{n_1+n_2+\ldots+n_s+1} & \text{if } r=2^{n_1}+2^{n_2}+\ldots+2^{n_s}\end{cases}$$

and for $j\geqslant 2$,

$$c_j(r)=\begin{cases}1 & \text{if } r=0,\\ j^s(2^{n_1+n_2+\ldots+n_s+1}) & \text{if } j\mid r,\ r/j=2^{n_1}+2^{n_2}+\ldots+2^{n_s},\\ 0 & \text{if } j\nmid r\end{cases}$$

so that terms like $1/(1-nq^n)$ can be easily interpreted in terms of the numbers of some chromatic partition classes.

We believe that due to the simple nature of the chromatic Ferrers diagrams, many identities, such as Pak in [7] and Bressoud in [4], can be obtained in a bijective way.

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Authors' address: Mateus Alegri (corresponding author), Wagner Ferreira Santos, Samuel Brito Sila, University of Sergipe, Av. Vereador Olímpio Grande, s/n, Itabaiana, Sergipe, Brazil, e-mail: `allegri.mateus@gmail.com`, `wagnermaths@gmail.com`, `s.complexo@hotmail.com`.