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# TRUDINGER'S INEQUALITY FOR DOUBLE PHASE FUNCTIONALS WITH VARIABLE EXPONENTS

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*Abstract.* Our aim in this paper is to establish Trudinger's inequality on Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(G)$ under conditions on $\Phi$ which are essentially weaker than those considered in a former paper. As an application and example, we show Trudinger's inequality for double phase functionals $\Phi(x,t)=t^{p(x)}+a(x)t^{q(x)}$, where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions and $a(\cdot)$ is nonnegative, bounded and Hölder continuous.

*Keywords*: Riesz potential; Trudinger's inequality; Musielak-Orlicz-Morrey space; double phase functional

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## 1. Introduction

Classical Trudinger's inequality for Riesz potentials of $L^p$-functions (see, e.g. [1], Theorem 3.1.4 (c)) has been also extended to various function spaces. The Trudinger type exponential integrability on variable exponent Lebesgue spaces $L^{p(\cdot)}$ was investigated in [7], [8] and [9]. See [15] for the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$, see [20] for Musielak-Orlicz spaces, see [14], [17], etc. for Morrey spaces of variable exponent. In [12], we established a Trudinger type inequality in Musielak-Orlicz-Morrey spaces $L^{\Phi,\kappa}(G)$ defined by general functions $\Phi(x,t)$ and $\kappa(x,r)$ satisfying certain conditions. In the present paper, we give the same result (see Theorem 3.4) with the help of relaxing the comparing condition ($\Phi5$) in [12] by ($\Phi5;\nu$) given below. We also give a Trudinger type inequality (see Theorem 3.7) in Musielak-Orlicz spaces $L^{\Phi}(G)$ as an improvement of [20].

Recently, regarding the regularity theory of differential equations, Baroni, Colombo and Mingione in [3], [4], [5], [6] studied a double phase functional $\Phi(x,t)=$

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$t^p + a(x)t^q$, $x \in \mathbb{R}^N$, $t \geqslant 0$, where $1 < p < q$, $a(\cdot)$ is a nonnegative, bounded and Hölder continuous function of order $\theta \in (0,1]$. In [4], regularity was studied under the assumption $q \leqslant (1+\theta/N)p$ and then Hästö in [10], Theorem 4.7 showed the boundedness of the maximal operator on $L^{\Phi}(G)$ for such functional $\Phi(x,t)$ under the same assumption $q \leqslant (1+\theta/N)p$. See also [13], Corollary 5.3.

In the final section, as applications of general theory, we give Trudinger type inequalities (see Theorems 4.4 and 4.10) for double phase functionals $\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)}$, where $p(\cdot)$ and $q(\cdot)$ satisfy log-Hölder conditions and $a(\cdot)$ is a nonnegative, bounded and Hölder continuous function of order $\theta \in (0,1]$. Our relaxed condition in Theorem 4.4 corresponds to $q \leqslant (1+\theta/N)p$, when $p(\cdot)$ and $q(\cdot)$ are constant, and it is shown to be sharp in Remark 4.7 below.

The remaining part of the present paper is organized as follows. In Section 2, we give the definitions of the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(G)$ and the Musielak-Orlicz space $L^{\Phi}(G)$. In Section 3, we prove Trudinger type inequalities for variable Riesz potentials in $L^{\Phi,\kappa}(G)$ and $L^{\Phi}(G)$, which are used to treat double phase functionals with variable exponents in Section 4. For this purpose, the condition ($\Phi$5) in [12] and [20] must be relaxed by ($\Phi5;\nu$) to cover the case of Hästö, see [10].

Throughout this paper, let $C$ denote various constants independent of the variables in question and $C(a,b,\ldots)$ be a constant that depends on $a,b,\ldots$

## 2. Preliminaries

Throughout this paper, let $G$ be a bounded open set in $\mathbb{R}^N$ and $d_G = \sup\{|x-y|\colon x,y \in G\}$ $(<\infty)$.

Let us begin with the assumptions on Musielak-Orlicz functions used in this paper.

We consider a function

$$\Phi(x,t)\colon\ G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Phi$1)–($\Phi$3):

($\Phi$1) $\Phi(\cdot,t)$ is measurable on $G$ for any $t \geqslant 0$ and $\Phi(x,\cdot)$ is continuous on $[0,\infty)$ for any $x \in G$;

($\Phi$2) there exists a constant $A_1 \geqslant 1$ such that

$$A_1^{-1} \leqslant \Phi(x,1) \leqslant A_1 \quad \text{for all } x \in G;$$

($\Phi$3) $t \mapsto \Phi(x,t)/t$ is uniformly almost increasing on $(0,\infty)$, namely there exists a constant $A_2 \geqslant 1$ such that

$$\Phi(x,t_1)/t_1 \leqslant A_2\Phi(x,t_2)/t_2 \quad \text{for all } x \in G \text{ whenever } 0 < t_1 < t_2.$$

Let $\bar{\varphi}(x,t) = \sup\limits_{0<s\leqslant t} \Phi(x,s)/s$ and

$$\bar{\Phi}(x,t) = \int_0^t \bar{\varphi}(x,r)\,\mathrm{d}r \quad \text{for all } x \in G,\ t \geqslant 0.$$

Then $\bar{\Phi}(x,\cdot)$ is convex and

$$\Phi(x,t/2) \leqslant \bar{\Phi}(x,t) \leqslant A_2\Phi(x,t) \quad \text{for all } x \in G,\ t \geqslant 0.$$

We also consider a function $\kappa(x,r)\colon G \times [0,d_G] \to [0,\infty)$ satisfying the following conditions: Let $0 < \sigma_0 \leqslant N$.

($\kappa$1) $\kappa(\cdot,t)$ is measurable on $G$ for any $0 \leqslant t \leqslant d_G$ and $\kappa(x,\cdot)$ is continuous on $[0,d_G]$ for any $x \in G$;

($\kappa$2) $r \mapsto \kappa(x,r)$ is uniformly almost increasing on $(0,d_G]$, namely there exists a constant $K_1 \geqslant 1$ such that

$$\kappa(x,r_1) \leqslant K_1\kappa(x,r_2) \quad \text{for all } x \in G \text{ whenever } 0 < r_1 < r_2 \leqslant d_G;$$

($\kappa3;\sigma_0$) there is a constant $K_2 \geqslant 1$ such that

$$K_2^{-1}\min\{1,r^{\sigma_0}\} \leqslant \kappa(x,r) \leqslant K_2 \quad \text{for all } x \in G,\ 0 < r \leqslant d_G.$$

Given $\Phi(x,t)$ and $\kappa(x,r)$ as above, the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(G)$ is defined by

$$L^{\Phi,\kappa}(G) \\ = \biggl\{f \in L^1_{\mathrm{loc}}(G)\colon \sup_{\substack{x\in G\\ 0<r\leqslant d_G}} \frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\Phi\Bigl(y,\frac{|f(y)|}{\lambda}\Bigr)\,\mathrm{d}y < \infty \text{ for some } \lambda > 0\biggr\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa;G} = \inf\biggl\{\lambda > 0\colon \sup_{\substack{x\in G\\ 0<r\leqslant d_G}} \frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\bar{\Phi}\Bigl(y,\frac{|f(y)|}{\lambda}\Bigr)\,\mathrm{d}y \leqslant 1\biggr\},$$

cf. [19].

In case of $\kappa(x,r) = r^N$, $L^{\Phi,\kappa}(G)$ is the Musielak-Orlicz space $L^{\Phi}(G)$ (cf. [18]), namely

$$L^{\Phi}(G) = \biggl\{f \in L^1_{\mathrm{loc}}(G)\colon \int_G \Phi\Bigl(y,\frac{|f(y)|}{\lambda}\Bigr)\,\mathrm{d}y < \infty \text{ for some } \lambda > 0\biggr\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi;G} = \inf\biggl\{\lambda > 0\colon\ \int_G \bar{\Phi}\Bigl(y, \frac{|f(y)|}{\lambda}\Bigr)\,\mathrm{d}y \leqslant 1\biggr\}.$$

We also consider the following conditions for $\Phi(x,t)$: Let $q \geqslant 1$, $\nu > 0$ and $\omega > 0$.

($\Phi3;\infty;q$) $t \mapsto t^{-q}\Phi(x,t)$ is uniformly almost increasing on $[1,\infty)$, namely there exists a constant $A_{2,\infty,q} \geqslant 1$ such that

$$t_1^{-q}\Phi(x,t_1) \leqslant A_{2,\infty,q}t_2^{-q}\Phi(x,t_2) \quad \text{for all } x \in G \text{ whenever } 1 \leqslant t_1 < t_2,$$

($\Phi5;\nu$) for every $a > 0$, there exists a constant $B_{a,\nu} \geqslant 1$ such that

$$\Phi(x,t) \leqslant B_{a,\nu}\Phi(y,t) \quad \text{whenever } x,y \in G,\ |x-y| \leqslant at^{-\nu},\ t \geqslant 1.$$

In [12], we assumed the condition ($\Phi5$) = ($\Phi5;1/N$). For another condition corresponding to ($\Phi5;\nu$), we refer to [2], page 2544.

Set

$$\Phi^{-1}(x,t) = \sup\{s > 0\colon\ \Phi(x,s) < t\}.$$

Suitably modifying the proof of [12], Lemma 3.3 (cf. [13], Lemma 4.3), we can prove the following lemma:

**Lemma 2.1** ([12], Lemma 3.3). *Suppose $\Phi(x,t)$ satisfies ($\Phi3;\infty;q$) and ($\Phi5;\nu$) for $q \geqslant 1$ and $\nu > 0$ satisfying $\nu \leqslant q/\sigma_0$. Then there exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,\mathrm{d}y \leqslant C\Phi^{-1}(x,\kappa(x,r)^{-1})$$

*for all $x \in G$, $0 < r < d_G$ and nonnegative functions $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{\Phi,\kappa;G} \leqslant 1$.*

## 3. Trudinger's inequality

In this section, we establish two kinds of Trudinger type inequalities for Riesz potentials $I_{\alpha(\cdot)}f$ of order $\alpha(\cdot)$.

Let $\alpha(\cdot)$ be a measurable function on $G$ such that

$$0 < \alpha^- := \inf_{x\in G}\alpha(x) \leqslant \sup_{x\in G}\alpha(x) < N$$

and define

$$I_{\alpha(\cdot)}f(x) := \int_G |x-y|^{\alpha(x)-N}f(y)\,\mathrm{d}y$$

for a locally integrable function $f$ on $G$.

Let $E$ be a measurable subset of $G$. Set $s_0 = \min\{1, 1/d_G\}$. Before stating our Trudinger type inequalities, we consider a logarithmic type function

$$\Gamma(x,s)\colon\ E \times [s_0,\infty) \to (0,\infty)$$

and an exponential type function

$$\Psi(x,t)\colon\ E \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions:

($\Gamma 1$) $s \mapsto \Gamma(x,s)$ is uniformly almost increasing on $[s_0,\infty)$, that is, there exists a constant $c_{\Gamma 1} \geqslant 1$ such that

$$\Gamma(x,s_1) \leqslant c_{\Gamma 1}\Gamma(x,s_2) \quad \text{for all } x \in E,\ s_0 \leqslant s_1 < s_2;$$

($\Gamma 2$) there exists a constant $c_{\Gamma 2} \geqslant 1$ such that

$$\Gamma(x,2) \leqslant c_{\Gamma 2}\Gamma(x,s_0) \quad \text{for all } x \in E;$$

($\Gamma_{\log}$) there exists a constant $c_{\Gamma l} \geqslant 1$ such that

$$\Gamma(x,s^2) \leqslant c_{\Gamma l}\Gamma(x,s) \quad \text{for all } x \in E,\ s \geqslant 1;$$

($\Psi 1$) $\Psi(\cdot,t)$ is measurable on $E$ for any $t \in [0,\infty)$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for any $x \in E$;

($\Psi 2$) there is a constant $Q_1 \geqslant 1$ such that $\Psi(x,t_1) \leqslant \Psi(x,Q_1 t_2)$ for all $x \in E$ whenever $0 < t_1 < t_2$;

($\Psi\Gamma$) there are constants $Q_2, Q_3 \geqslant 1$ and $s_0^* \geqslant s_0$ such that $\Psi(x,\Gamma(x,s)/Q_2) \leqslant Q_3 s$ for all $x \in E$ and $s \geqslant s_0^*$.

We recall the following fundamental properties on $\Gamma$ which imply that $\Gamma$ behaves like logarithmic functions.

**Lemma 3.1** ([16], Lemmas 2.1 and 2.2)**.**

(1) $\Gamma(x,\cdot)$ *has the uniform doubling property on* $[s_0,\infty)$*; namely, there exists a constant* $C > 0$ *such that* $\Gamma(x,2s) \leqslant C\Gamma(x,s)$ *for all* $x \in E$ *and* $s \geqslant s_0$.

(2) *For* $a > 0$*, there exists a constant* $C \geqslant 1$ *such that*

$$C^{-1}\Gamma(x,s) \leqslant \Gamma(x,s^a) \leqslant C\Gamma(x,s) \quad \textit{for all } x \in E,\ s \geqslant 1.$$

(3) *There exists a constant* $C > 0$ *such that*

$$\Gamma(x,s) \leqslant Cs\Gamma(x,s_0) \quad \textit{for all } x \in E\ s \geqslant s_0.$$

P r o o f. (1) Let $x \in E$. In case of $s \geqslant 2$, we have

$$\Gamma(x, 2s) \leqslant c_{\Gamma 1}\Gamma(x, s^2) \leqslant c_{\Gamma 1}c_{\Gamma l}\Gamma(x, s).$$

On the other hand, if $s_0 \leqslant s \leqslant 2$, then we find

$$\Gamma(x, 2s) \leqslant c_{\Gamma 1}\Gamma(x, 4) \leqslant c_{\Gamma 1}c_{\Gamma l}\Gamma(x, 2) \leqslant c_{\Gamma 1}c_{\Gamma l}c_{\Gamma 2}\Gamma(x, s_0) \leqslant c_{\Gamma 1}^2 c_{\Gamma l}c_{\Gamma 2}\Gamma(x, s).$$

(2) Let $x \in E$ and $s \geqslant 1$. It is enough to show the case $a > 1$ since the remaining case is treated by symmetry. Let $a > 1$ and take the nonnegative integer $m$ such that $2^m < a \leqslant 2^{m+1}$. Then we have

$$c_{\Gamma 1}^{-1}\Gamma(x, s) \leqslant \Gamma(x, s^a) \leqslant c_{\Gamma 1}\Gamma(x, s^{2^{m+1}}) \leqslant c_{\Gamma 1}c_{\Gamma l}^{m+1}\Gamma(x, s).$$

(3) If $s \geqslant c_{\Gamma l}$, then take a nonnegative integer $k$ such that $c_{\Gamma l}^{2^k} \leqslant s < c_{\Gamma l}^{2^{k+1}}$. Then

$$\Gamma(x, s) \leqslant c_{\Gamma 1}\Gamma(x, c_{\Gamma l}^{2^{k+1}}) \leqslant c_{\Gamma 1}c_{\Gamma l}^{k+1}\Gamma(x, c_{\Gamma l}) \leqslant c_{\Gamma 1}c_{\Gamma l}^{k+1}c_{\Gamma l}^{-2^k}s\Gamma(x, c_{\Gamma l}) \leqslant c_{\Gamma 1}s\Gamma(x, c_{\Gamma l}).$$

Since $\Gamma(x, c_{\Gamma l}) \leqslant C\Gamma(x, s_0)$ by (1) above, we have $\Gamma(x, s) \leqslant Cs\Gamma(x, s_0)$.

If $s_0 \leqslant s < c_{\Gamma l}$, then

$$\Gamma(x, s) \leqslant c_{\Gamma 1}\Gamma(x, c_{\Gamma l}) \leqslant (c_{\Gamma 1}/s_0)s\Gamma(x, c_{\Gamma l}) \leqslant Cs\Gamma(x, s_0).$$

□

**3.1. Musielak-Orlicz-Morrey case.** We consider the condition:

($\Gamma\Phi\kappa\alpha$) There exists a constant $c^* \geqslant 1$ such that

$$\int_{1/s}^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x, \kappa(x, \varrho)^{-1})\frac{\mathrm{d}\varrho}{\varrho} \leqslant c^*\Gamma(x, s) \quad \text{for all } x \in E,\ s \geqslant 2/d_G.$$

**Lemma 3.2** ([12], Lemma 3.6). *Suppose $\Phi(x, t)$ satisfies $(\Phi3; \infty; q)$ and $(\Phi5; \nu)$ for $q \geqslant 1$ and $\nu > 0$ fulfilling $\nu \leqslant q/\sigma_0$. Further assume that $(\Gamma\Phi\kappa\alpha)$ holds. Then there exists a constant $C > 0$ such that*

$$\int_{G\setminus B(x,\delta)} |x - y|^{\alpha(x)-N} f(y)\,\mathrm{d}y \leqslant C\Gamma\Big(x, \frac{1}{\delta}\Big)$$

*for all $x \in E$, $0 < \delta \leqslant d_G/2$ and nonnegative $f \in L^{\Phi,\kappa}(G)$ with $\|f\|_{\Phi,\kappa;G} \leqslant 1$.*

In the same way as Lemma 3.7 in [12], we have:

**Lemma 3.3.** *Suppose $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$ and $(\Phi5;\nu)$ for $q \geqslant 1$ and $\nu > 0$ fulfilling $\nu \leqslant q/\sigma_0$. Let $0<\varepsilon<N$ and define*

$$I_\varepsilon f(x) = \int_G |x-y|^{\varepsilon-N} f(y)\,\mathrm{d}y$$

*for a nonnegative measurable function $f$ on $G$ and*

$$\lambda_\varepsilon(z,r) = \frac{1}{1+\int_r^{d_G} \varrho^\varepsilon \Phi^{-1}(z,\kappa(z,\varrho)^{-1})\varrho^{-1}\,\mathrm{d}\varrho} \tag{3.1}$$

*for $z\in G$. Then there exists a constant $C_{I,\varepsilon}>0$ such that*

$$\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{G\cap B(z,r)} I_\varepsilon f(x)\,\mathrm{d}x \leqslant C_{I,\varepsilon}$$

*for all $z\in G$, $0<r\leqslant d_G$ and $f\geqslant 0$ satisfying $\|f\|_{\Phi,\kappa;G}\leqslant 1$.*

We first give a Trudinger type inequality in the Musielak-Orlicz-Morrey space $L^{\Phi,\kappa}(G)$ which is an improvement of [12], Theorem 4.4 in case of $J(x,r)=r^{\alpha(x)-N}$. In fact, $(\Phi5)=(\Phi5;1/N)$ in [12] is relaxed by $(\Phi5;\nu)$ with $\nu\leqslant q/\sigma_0$.

**Theorem 3.4.** *Suppose that $\Phi(x,t)$ satisfies $(\Phi3;\infty;q)$ and $(\Phi5;\nu)$ for $q\geqslant 1$ and $\nu>0$ fulfilling $\nu\leqslant q/\sigma_0$. Assume that $(\Gamma\Phi\kappa\alpha)$ holds for $\Gamma(x,s)$ satisfying $(\Gamma1)$, $(\Gamma2)$ and $(\Gamma_{\log})$. Further suppose that $\Psi(x,t)\colon E\times[0,\infty)\to[0,\infty)$ satisfies $(\Psi1)$, $(\Psi2)$ and $(\Psi\Gamma)$. Then, for $0<\varepsilon<\inf_{x\in E}\alpha(x)$, there exist constants $c_1=c_1(\varepsilon)>0$, $c_2=c_2(\varepsilon)>0$ such that*

$$\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{E\cap B(z,r)} \Psi\Big(x,\frac{|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)\,\mathrm{d}x \leqslant c_2$$

*for all $z\in G$, $0<r<d_G$ and $f\in L^{\Phi,\kappa}(G)$ with $\|f\|_{\Phi,\kappa;G}\leqslant 1$, where $\lambda_\varepsilon(z,r)$ is given by (3.1).*

**Remark 3.5.** Theorem 3.4 is used to obtain the Trudinger type inequality in Musielak-Orlicz-Morrey spaces defined by double phase functionals with variable exponents. To do so, condition $(\Phi5;\nu)$ is suitable, instead of $(\Phi5)=(\Phi5;1/N)$ in [12].

P r o o f of Theorem 3.4. Let $f\geqslant 0$ and $\|f\|_{\Phi,\kappa;G}\leqslant 1$. Let $x\in E$. For $0<\delta\leqslant d_G/2$, Lemma 3.2 implies

$$I_{\alpha(\cdot)}f(x)\leqslant \int_{B(x,\delta)\cap G}|x-y|^{\alpha(x)-N}f(y)\,\mathrm{d}y + C\Gamma\Big(x,\frac{1}{\delta}\Big)\leqslant \delta^{\alpha(x)-\varepsilon}I_\varepsilon f(x)+C\Gamma\Big(x,\frac{1}{\delta}\Big)$$

with a constant $C>0$ independent of $x$.

If $I_\varepsilon f(x) \leqslant 2/d_G$, then we take $\delta = d_G/2$. Then

$$I_{\alpha(\cdot)} f(x) \leqslant (d_G/2)^{\alpha(x)-\varepsilon-1} + C\Gamma\Big(x, \frac{2}{d_G}\Big) \leqslant C\Gamma\Big(x, \frac{2}{d_G}\Big).$$

By Lemma 3.1 (1), there exists a constant $C > 0$ independent of $x$ such that

$$I_{\alpha(\cdot)} f(x) \leqslant C\Gamma(x, s_0^*) \quad \text{if } I_\varepsilon f(x) \leqslant 2/d_G. \tag{3.2}$$

Next, suppose $2/d_G < I_\varepsilon f(x) < \infty$. By Lemma 3.1 (3) and (1), there exists a constant $m > 0$ such that $\Gamma(x,s)/s \leqslant m\Gamma(x, 2/d_G)$ for $s \geqslant 2/d_G$. Let

$$\delta = (d_G/2)\Big[\frac{\Gamma(x, I_\varepsilon f(x))}{m\Gamma(x, 2/d_G) I_\varepsilon f(x)}\Big]^{1/(\alpha(x)-\varepsilon)}.$$

Then

$$\delta^{\alpha(x)-\varepsilon} I_\varepsilon f(x) = (d_G/2)^{\alpha(x)-\varepsilon} \frac{\Gamma(x, I_\varepsilon f(x))}{m\Gamma(x, 2/d_G)} \leqslant C\Gamma(x, I_\varepsilon f(x)).$$

By the choice of $m$, $\delta \leqslant d_G/2$. Since $\Gamma(x, 2/d_G) \leqslant C\Gamma(x, I_\varepsilon f(x))$,

$$\frac{1}{\delta} \leqslant C(I_\varepsilon f(x))^{1/(\alpha(x)-\varepsilon)}.$$

Hence, using Lemma 3.1 (1) and (2), we obtain

$$\Gamma\Big(x, \frac{1}{\delta}\Big) \leqslant C\Gamma(x, I_\varepsilon f(x)).$$

Therefore, there exists a constant $C > 0$ independent of $x$ such that

$$I_{\alpha(\cdot)} f(x) \leqslant C\Gamma(x, I_\varepsilon f(x)) \quad \text{if } 2/d_G < I_\varepsilon f(x) < \infty. \tag{3.3}$$

By (3.2) and (3.3), there exists a constant $C^* > 0$ such that

$$I_{\alpha(\cdot)} f(x) \leqslant C^*\Gamma(x, \max\{s_0^*, I_\varepsilon f(x)\})$$

for a.e. $x \in E$.

Now, let $c_1 = Q_1 Q_2 C^*$. Then, by $(\Psi 2)$ and $(\Psi\Gamma)$, we have

$$\begin{aligned}\Psi\Big(x, \frac{I_{\alpha(\cdot)} f(x)}{c_1}\Big) &\leqslant \Psi(x, \Gamma(x, \max\{s_0^*, I_\varepsilon f(x)\})/Q_2) \\ &\leqslant Q_3 \max\{s_0^*, I_\varepsilon f(x)\} \leqslant Q_3(s_0^* + I_\varepsilon f(x))\end{aligned}$$

for a.e. $x \in E$. Thus, we have by Lemma 3.3

$$\begin{aligned}\frac{\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{E\cap B(z,r)} \Psi\Big(x, \frac{I_{\alpha(\cdot)} f(x)}{c_1}\Big)\,\mathrm{d}x &\leqslant Q_3 s_0^* + \frac{Q_3\lambda_\varepsilon(z,r)}{|B(z,r)|}\int_{E\cap B(z,r)} I_\varepsilon f(x)\,\mathrm{d}x \\ &\leqslant Q_3 s_0^* + Q_3 C_{I,\varepsilon} = c_2\end{aligned}$$

for all $z \in G$ and $0 < r < d_G$. $\square$

**3.2. Musielak-Orlicz case.** We consider a function

$$\gamma(x,\varrho)\colon\ E\times(0,d_G)\to[0,\infty)$$

satisfying the following conditions ($\gamma 1$) and ($\gamma 2$):

($\gamma 1$) $\gamma(\cdot,\varrho)$ is measurable on $E$ for any $0<\varrho<d_G$ and $\gamma(x,\cdot)$ is continuous on $(0,d_G)$ for any $x\in E$;

($\gamma 2$) there exists a constant $B_0\geqslant 1$ such that

$$B_0^{-1}\leqslant\gamma(x,\varrho)\leqslant B_0\varrho^{-N}\quad\text{for all } x\in E\quad\text{whenever } 0<\varrho<d_G.$$

In this subsection, we consider the condition:

($\Gamma\Phi\gamma\alpha$) There exist constants $c_1^{**}\geqslant 1$ and $c_2^{**}\geqslant 1$ such that

$$\varrho^{\alpha(x)-N}\gamma(x,\varrho)^{-1}\Phi^{-1}(x,\gamma(x,\varrho))\leqslant c_1^{**}\Gamma(x,\varrho^{-1})$$

for all $x\in E$ whenever $0<\varrho<d_G$ and

$$\int_\delta^{d_G}\varrho^{\alpha(x)}\Phi^{-1}(x,\gamma(x,\varrho))\frac{\mathrm{d}\varrho}{\varrho}\leqslant c_2^{**}\Gamma(x,\delta^{-1})$$

for all $x\in E$ whenever $0<\delta<d_G/2$.

**Lemma 3.6** ([20], Lemma 3.2)**.** *Suppose $\Phi(x,t)$ satisfies $(\Phi 3;\infty;q)$ and $(\Phi 5;\nu)$ for $q\geqslant 1$ and $\nu>0$ fulfilling $\nu\leqslant q/N$. Assume that $(\Gamma\Phi\gamma\alpha)$ holds. Then there exists a constant $C>0$ such that*

$$\int_{G\setminus B(x,\delta)}|x-y|^{\alpha(x)-N}f(y)\,\mathrm{d}y\leqslant C\Gamma(x,\delta^{-1})$$

*for all $x\in E$, $0<\delta\leqslant d_G/2$ and nonnegative functions $f\in L^\Phi(G)$ with $\|f\|_{\Phi;G}\leqslant 1$.*

P r o o f. Let $f$ be a nonnegative measurable function with $\|f\|_{\Phi;G}\leqslant\frac12$. By ($\Phi 3$) and $(\Phi 3;\infty;q)$,

$$\min\{1,(A_1A_2)^{-1}s\}\leqslant\Phi^{-1}(x,s)\leqslant\max\{1,(A_1A_{2,\infty,q}s)^{1/q}\};$$

see [11], Lemma 5.1 (5). Set

$$c_1=\max\{A_1A_2B_0,(A_1A_{2,\infty,q}B_0)^{-1}d_G^N\}.$$

Then we have by $(\gamma 2)$

$$\Phi^{-1}(x, c_1\gamma(x, |x-y|)) \geqslant \min\{1, (A_1A_2)^{-1}c_1B_0^{-1}\} = 1$$

and

$$\begin{aligned}\Phi^{-1}(x, c_1\gamma(x, |x-y|)) &\leqslant \max\{1, (A_1A_{2,\infty,q}c_1B_0|x-y|^{-N})^{1/q}\} \\ &= (A_1A_{2,\infty,q}c_1B_0d_G^{-N})^{1/q}(|x-y|/d_G)^{-N/q} \\ &\leqslant (A_1A_{2,\infty,q}c_1B_0d_G^{-N})^{1/q}(|x-y|/d_G)^{-1/\nu}\end{aligned}$$

for all $x \in E$ and $y \in G$. Hence

$$|x-y| \leqslant c_2(\Phi^{-1}(x, c_1\gamma(x, |x-y|)))^{-\nu} \tag{3.4}$$

for all $x \in E$ and $y \in G$, where $c_2 = d_G(A_1A_{2,\infty,q}c_1B_0d_G^{-N})^{\nu/q}$.

Therefore, we find by $(\Phi 3)$

$$\begin{aligned}&\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y)\,\mathrm{d}y \\ &\quad\leqslant \int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N}\Phi^{-1}(x, c_1\gamma(x, |x-y|))\,\mathrm{d}y \\ &\qquad + A_2\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N} f(y) \\ &\qquad \times \frac{f(y)^{-1}\Phi(y, f(y))}{\Phi^{-1}(x, c_1\gamma(x, |x-y|))^{-1}\Phi(y, \Phi^{-1}(x, c_1\gamma(x, |x-y|)))}\,\mathrm{d}y \\ &\quad= I_1 + I_2.\end{aligned}$$

By $(\Gamma\Phi\gamma\alpha)$,

$$I_1 \leqslant C\int_\delta^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x, \gamma(x, \varrho))\frac{\mathrm{d}\varrho}{\varrho} \leqslant C\Gamma(x, \delta^{-1}).$$

By $(\Phi 5;\nu)$ and (3.4),

$$\begin{aligned}\Phi(y, \Phi^{-1}(x, c_1\gamma(x, |x-y|))) &\geqslant B_{c_2,\nu}^{-1}\Phi(x, \Phi^{-1}(x, c_1\gamma(x, |x-y|))) \\ &= B_{c_2,\nu}^{-1}c_1\gamma(x, |x-y|).\end{aligned}$$

Hence, by $(\Gamma\Phi\gamma\alpha)$,

$$\begin{aligned}I_2 &\leqslant C\int_{G\setminus B(x,\delta)} |x-y|^{\alpha(x)-N}\gamma(x, |x-y|)^{-1}\Phi^{-1}(x, \gamma(x, |x-y|))\Phi(y, f(y))\,\mathrm{d}y \\ &\leqslant C\int_{G\setminus B(x,\delta)} \Gamma(x, |x-y|^{-1})\Phi(y, f(y))\,\mathrm{d}y \leqslant C\Gamma(x, \delta^{-1}).\end{aligned}$$

Thus we obtain the required result. □

As in the proof of Theorem 3.4, we can obtain the following result by Lemmas 3.6 and 3.3 with $r = d_G$, which is an improvement of [20], Theorem 4.1. In fact, $(\Phi5) = (\Phi5; 1/N)$ in [20] is relaxed by $(\Phi5; \nu)$ with $\nu \leqslant q/N$.

**Theorem 3.7.** *Suppose $\Phi(x,t)$ satisfies $(\Phi3; \infty; q)$ and $(\Phi5; \nu)$ for $q \geqslant 1$ and $\nu > 0$ fulfilling $\nu \leqslant q/N$. Assume that $(\Gamma\Phi\gamma\alpha)$ holds for $\gamma(x,\varrho)$ satisfying $(\gamma1)$ and $(\gamma2)$; and $\Gamma(x,s)$ satisfying $(\Gamma1)$, $(\Gamma2)$ and $(\Gamma_{\log})$. Further suppose that $\Psi(x,t)\colon E\times[0,\infty) \to [0,\infty)$ fulfills $(\Psi1)$, $(\Psi2)$ and $(\Psi\Gamma)$. Then there exist constants $c_1, c_2 > 0$ such that*

$$\int_E \Psi\Big(x, \frac{|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)\,\mathrm{d}x \leqslant c_2$$

*for all $f \in L^{\Phi}(G)$ with $\|f\|_{\Phi;G} \leqslant 1$.*

**Remark 3.8.** Theorem 3.7 is used to obtain the Trudinger type inequality in Musielak-Orlicz spaces defined by double phase functionals with variable exponents.

## 4. Double phase functionals

In this section, we give two kinds of Trudinger type inequalities when $\Phi$ is a double phase functional with variable exponents.

Let $p(\cdot)$ and $q(\cdot)$ be real valued measurable functions on $\mathbb{R}^N$ such that

(P1) $1 \leqslant p^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} p(x) \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} p(x) =: p^+ < \infty$,

(Q1) $1 \leqslant q^- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} q(x) \leqslant \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} q(x) =: q^+ < \infty$.

We assume that

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x)-p(y)| \leqslant \frac{C_p}{\log(\mathrm{e}+|x-y|^{-1})} \quad \text{for all } x,y \in \mathbb{R}^N \quad \text{with a constant } C_p \geqslant 0,$$

(Q2) $q(\cdot)$ is log-Hölder continuous, namely

$$|q(x)-q(y)| \leqslant \frac{C_q}{\log(\mathrm{e}+|x-y|^{-1})} \quad \text{for all } x,y \in \mathbb{R}^N \quad \text{with a constant } C_q \geqslant 0.$$

As an example and application, we consider the case, where $\Phi(x,t)$ is a double phase functional given by

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)} \quad (= t^{p(x)} + (b(x)t)^{q(x)}), \quad x \in G,\ t \geqslant 0,$$

where $p(x) < q(x)$ for $x \in G$, $a(\cdot)$ is nonnegative, bounded and Hölder continuous of order $\theta \in (0,1]$, and $b(x) = a(x)^{1/q(x)}$ (cf. [6]).

This $\Phi(x,t)$ satisfies $(\Phi1)$, $(\Phi2)$ and $(\Phi3;\infty;p^-)$. Set

$$G_0 = \{x \in G\colon\ a(x) = 0\} \quad \text{and} \quad G_+ = \{x \in G\colon\ a(x) > 0\}.$$

The functional $\Phi(x,t)$ also satisfies $(\Phi5;\nu)$ for $\nu \geqslant \sup_{x\in G_+}(q(x)-p(x))/\theta$; see [13], Lemma 5.1.

**4.1. Trudinger's inequality in Musielak-Orlicz-Morrey spaces.** In this subsection, let

$$\kappa(x,r) = r^{\sigma(x)}(\log(\mathrm{e}+r^{-1}))^{\beta(x)}$$

for $x \in G$ and $0 < r < d_G$ with measurable functions $\sigma(\cdot)$ and $\beta(\cdot)$ on $G$ satisfying the following conditions for $\sigma_0 \leqslant N$:

$$0 < \sigma^- := \inf_{x\in G}\sigma(x) \leqslant \sup_{x\in G}\sigma(x) \leqslant \sigma_0,$$
$$\sup_{x\in G}\beta(x) < \infty \quad \text{and} \quad \beta(x) \geqslant -c(\sigma_0 - \sigma(x)) \quad \text{for a constant } c > 0.$$

This $\kappa(x,r)$ satisfies $(\kappa1)$, $(\kappa2)$ and $(\kappa3;\sigma_0)$.

**Lemma 4.1.** *For $0 < \varepsilon' < \varepsilon < \inf_{x\in G}(\sigma(x)/q(x))$, there exists a constant $C > 0$ such that*

$$\lambda_\varepsilon(x,r) \geqslant C\max\{r^{\sigma(x)/p(x)-\varepsilon'}, b(x)r^{\sigma(x)/q(x)-\varepsilon'}\}$$

*for all $x \in G$ and $0 < r < d_G$.*

Proof. Since $\Phi^{-1}(x,s) \leqslant \min\{s^{1/p(x)}, b(x)^{-1}s^{1/q(x)}\}$, we have

$$\begin{aligned}\int_r^{d_G}\varrho^\varepsilon\Phi^{-1}(x,\kappa(x,\varrho)^{-1})\frac{\mathrm{d}\varrho}{\varrho} &\leqslant C\min\biggl\{\int_r^{d_G}\varrho^{\varepsilon-\sigma(x)/p(x)}(\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/p(x)}\frac{\mathrm{d}\varrho}{\varrho},\\ &\qquad b(x)^{-1}\int_r^{d_G}\varrho^{\varepsilon-\sigma(x)/q(x)}(\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/q(x)}\frac{\mathrm{d}\varrho}{\varrho}\biggr\}\\ &\leqslant C\min\{r^{\varepsilon'-\sigma(x)/p(x)}, b(x)^{-1}r^{\varepsilon'-\sigma(x)/q(x)}\}\end{aligned}$$

by our assumptions and the fact that $\varrho^{\varepsilon-\varepsilon'}(\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/p(x)}$ is almost increasing. Thus this lemma is proved. $\square$

In this subsection, let

$$E_1 = \{x \in G_0\colon\ \sigma(x) = \alpha(x)p(x)\}, \quad E_2 = \{x \in G_+\colon\ \sigma(x) = \alpha(x)q(x)\}$$

and $E = E_1 \cup E_2$;

$$\Gamma(x,s) = \begin{cases}(\log(\mathrm{e}+s))^{1-\beta(x)/p(x)}, & x \in E_1,\\ b(x)^{-1}(\log(\mathrm{e}+s))^{1-\beta(x)/q(x)}, & x \in E_2\end{cases}$$

for $s \geqslant s_0$. This $\Gamma(x,s)$ satisfies $(\Gamma1)$, $(\Gamma2)$ and $(\Gamma_{\log})$ if $\beta(x) \leqslant p(x)$ for $x \in E_1$ and $\beta(x) \leqslant q(x)$ for $x \in E_2$.

**Lemma 4.2.** *If* $\inf_{x\in E_1}(p(x)-\beta(x)) > 0$ *and* $\inf_{x\in E_2}(q(x)-\beta(x)) > 0$, *then* $(\Gamma\Phi\kappa\alpha)$ *holds.*

Proof. If $x \in E_1$, then $\Phi(x,t) = t^{p(x)}$, so that

$$\Phi^{-1}(x,\kappa(x,\varrho)^{-1}) = [\varrho^{\sigma(x)}(\log(\mathrm{e}+\varrho^{-1}))^{\beta(x)}]^{-1/p(x)} = \varrho^{-\alpha(x)}(\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/p(x)}.$$

Hence,

$$\begin{aligned}\int_{1/s}^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x,\kappa(x,\varrho)^{-1})\frac{\mathrm{d}\varrho}{\varrho} &= \int_{1/s}^{d_G} (\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/p(x)}\frac{\mathrm{d}\varrho}{\varrho} \\ &= \int_{1/d_G}^{s} (\log(\mathrm{e}+t))^{-\beta(x)/p(x)}\frac{\mathrm{d}t}{t} \leqslant C(\log(\mathrm{e}+s))^{1-\beta(x)/p(x)} = C\Gamma(x,s)\end{aligned}$$

for $s \geqslant 2/d_G$, since $\inf_{x\in E_1}(p(x)-\beta(x)) > 0$.

If $x \in E_2$, then $\Phi^{-1}(x,s) \leqslant b(x)^{-1}s^{1/q(x)}$, so that

$$\Phi^{-1}(x,\kappa(x,\varrho)^{-1}) \leqslant b(x)^{-1}\varrho^{-\alpha(x)}(\log(\mathrm{e}+\varrho^{-1}))^{-\beta(x)/q(x)}.$$

Hence, using the assumption $\inf_{x\in E_2}(q(x)-\beta(x)) > 0$, we see as above that

$$\int_{1/s}^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x,\kappa(x,\varrho)^{-1})\frac{d\varrho}{\varrho} \leqslant Cb(x)^{-1}(\log(\mathrm{e}+s))^{1-\beta(x)/q(x)} = C\Gamma(x,s)$$

for $s \geqslant 2/d_G$. □

Set

$$\Psi(x,t) = \begin{cases} \exp(t^{p(x)/(p(x)-\beta(x))}), & x \in \{y \in E_1\colon\ p(y)-\beta(y) > 0\}, \\ \exp((b(x)t)^{q(x)/(q(x)-\beta(x))}), & x \in \{y \in E_2\colon\ q(y)-\beta(y) > 0\}, \end{cases}$$

for $t > 0$. Then we can easily verify the following lemma.

**Lemma 4.3.** *If* $p(x)-\beta(x) > 0$ *for* $x \in E_1$ *and* $q(x)-\beta(x) > 0$ *for* $x \in E_2$, *then* $\Psi(x,t)$ *satisfies* $(\Psi1)$, $(\Psi2)$ *and* $(\Psi\Gamma)$ *with* $s_0^* = 2/d_G$.

By Lemmas 4.1, 4.2, 4.3 and Theorem 3.4, we obtain Trudinger's inequality for Musielak-Orlicz-Morrey spaces in the framework of double phase functionals.

**Theorem 4.4.** *Let*

$$E_1 = \{x \in G_0 \colon \sigma(x) = \alpha(x)p(x)\} \quad \textit{and} \quad E_2 = \{x \in G_+ \colon \sigma(x) = \alpha(x)q(x)\}.$$

*Suppose* $\sup_{x\in G_+}(q(x)-p(x))/\theta \leqslant p^-/\sigma_0$, $\inf_{x\in E_1}(p(x)-\beta(x)) > 0$ *and* $\inf_{x\in E_2}(q(x)-\beta(x)) > 0$. *Then, for* $0<\varepsilon<\inf_{x\in G}(\sigma(x)/q(x))$, *there exist constants* $c_1, c_2 > 0$ *such that*

$$\begin{aligned}(4.1)\qquad &\frac{\max\{r^{\sigma(z)/p(z)-\varepsilon}, b(z)r^{\sigma(z)/q(z)-\varepsilon}\}}{|B(z,r)|}\\ &\quad\times\bigg\{\int_{B(z,r)\cap E_1}\exp\Big(\Big(\frac{|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)^{p(x)/(p(x)-\beta(x))}\Big)\,\mathrm{d}x\\ &\quad+\int_{B(z,r)\cap E_2}\exp\Big(\Big(\frac{b(x)|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)^{q(x)/(q(x)-\beta(x))}\Big)\,\mathrm{d}x\bigg\}\leqslant c_2\end{aligned}$$

*for all* $z\in G$, $0<r<d_G$ *and* $f\in L^{\Phi,\kappa}(G)$ *with* $\|f\|_{\Phi,\kappa;G}\leqslant 1$.

**Remark 4.5.** In the setting of [12], we can show the following result: Suppose $\sigma(x) = \alpha(x)p(x)$ for $x\in G_0$, $\sigma(x)=\alpha(x)q(x)$ for $x\in G_+$, $\inf_{x\in G_0}(p(x)-\beta(x))>0$ and $\inf_{x\in G_+}(q(x)-\beta(x))>0$. If $\sup_{x\in G_+}(q(x)-p(x))/\theta\leqslant 1/N$, then, for $0<\varepsilon<\inf_{x\in G}(\sigma(x)/q(x))$, (4.1) with $E_1=G_0$ and $E_2=G_+$ holds for all $z\in G$, $0<r<d_G$ and $f\in L^{\Phi,\kappa}(G)$ with $\|f\|_{\Phi,\kappa;G}\leqslant 1$. This shows that Theorem 3.4 is an essential improvement of [12], Theorem 4.4.

By Theorem 4.4, we obtain the following result.

**Theorem 4.6.** *Suppose* $\sup_{x\in G}(q(x)-p(x))/\theta\leqslant p^-/N$. *Let*

$$E_1 = \{x \in G_0 \colon \alpha(x)p(x) = N\} \quad \textit{and} \quad E_2 = \{x \in G_+ \colon \alpha(x)q(x) = N\}.$$

*Then, for* $0<\varepsilon<N/q^+$, *there exist constants* $c_1,c_2>0$ *such that*

$$\begin{aligned}&\frac{\max\{r^{N/p(z)-\varepsilon}, b(z)r^{N/q(z)-\varepsilon}\}}{|B(z,r)|}\\ &\quad\times\bigg\{\int_{B(z,r)\cap E_1}\exp\Big(\frac{|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)\,\mathrm{d}x+\int_{B(z,r)\cap E_2}\exp\Big(\frac{b(x)|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)\,\mathrm{d}x\bigg\}\leqslant c_2\end{aligned}$$

*for all* $z\in G$, $0<r<d_G$ *and* $f\in L^{\Phi}(G)$ *with* $\|f\|_{\Phi;G}\leqslant 1$.

**Remark 4.7.** The condition $\sup_{x \in G_+} (q(x) - p(x))/\theta \leqslant p^-/\sigma_0$ in Theorem 4.4 is sharp as the following proposition shows.

**Proposition 4.8.** *Let* $\Phi(x,t) = t^p + (\max\{0, x_1\})^\theta t^q$ *and* $\kappa(x,r) = r^\sigma$ *with* $1 < p < q$, $0 < \theta \leqslant 1$ *and* $0 < \sigma \leqslant N$. *Suppose* $q = \sigma/\alpha$ *and* $(q-p)/\theta > p/\sigma$. *Then we can find* $f \in L^{\Phi,\kappa}(B(0,1))$ *for which*

$$\int_{B(0,r)} \exp((\max\{0, x_1\})^{\theta/q} |I_\alpha f(x)|) \, \mathrm{d}x = \infty$$

*for all* $0 < r \leqslant 1$.

Proof. By our assumption, we can take a number $a > 0$ such that $(\sigma + \theta)/q < a < \sigma/p$. Consider $f(y) = |y|^{-a} \chi_{B(0,1) \cap \{y_1 < 0\}}$, where $\chi_E$ is the characteristic function of $E$. Then

$$\frac{\kappa(x,r)}{|B(x,r)|} \int_{B(0,r)} \Phi(y, f(y)) \, \mathrm{d}y \leqslant \frac{r^\sigma}{|B(0,r)|} \int_{B(0,r)} |y|^{-ap} \, \mathrm{d}y \leqslant C < \infty$$

for $x \in B(0,1)$ and $0 < r \leqslant 1$. Hence, $f \in L^{\Phi,\kappa}(B(0,1))$. Moreover, we see that

$$I_\alpha f(x) \geqslant C|x|^{\alpha - a} \quad \text{for } x \in B(0,1),$$

so that

$$\int_{B(0,r)} ((\max\{0, x_1\})^{\theta/q} |I_\alpha f(x)|)^m \, \mathrm{d}x = \infty$$

for $0 < r \leqslant 1$ when $m(a - \alpha - \theta/q) > N$. $\square$

**4.2. Trudinger's inequality in Musielak-Orlicz spaces.** In this subsection, let

$$E_1 = \{x \in G_0 \colon \alpha(x)p(x) = N\}, \quad E_2 = \{x \in G_+ \colon \alpha(x)q(x) = N\}$$

and $E = E_1 \cup E_2$. Set

$$\gamma(x, \varrho) = \varrho^{-N} (\log(\mathrm{e} + \varrho^{-1}))^{-1}$$

for $x \in E$ and $0 < \varrho < d_G$ and

$$\Gamma(x,s) = \begin{cases} (\log(\mathrm{e} + s)^{(p(x)-1)/p(x)}, & x \in E_1, \\ b(x)^{-1} (\log(\mathrm{e} + s))^{(q(x)-1)/q(x)}, & x \in E_2 \end{cases}$$

for $s \geqslant s_0$.

This $\gamma(x, \varrho)$ satisfies ($\gamma$1) and ($\gamma$2); $\Gamma(x,s)$ satisfies ($\Gamma$1), ($\Gamma$2) and ($\Gamma_{\log}$).

**Lemma 4.9.** *The function* $\Gamma(x,t)$ *satisfies* ($\Gamma\Phi\gamma\alpha$).

P r o o f. If $x \in E_1$, then $\Phi^{-1}(x,s) = s^{1/p(x)}$, so that we have

$$\begin{aligned}
&\varrho^{\alpha(x)-N}\gamma(x,\varrho)^{-1}\Phi^{-1}(x,\gamma(x,\varrho)) \\
&\qquad = \varrho^{N/p(x)-N}\varrho^{N}\log(\mathrm{e}+\varrho^{-1})\{\varrho^{-N}(\log(\mathrm{e}+\varrho^{-1}))^{-1}\}^{1/p(x)} \\
&\qquad = (\log(\mathrm{e}+\varrho^{-1}))^{(p(x)-1)/p(x)} = \Gamma(x,\varrho^{-1})
\end{aligned}$$

for $0 < \varrho < d_G$ and

$$\begin{aligned}
\int_\delta^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x,\gamma(x,\varrho))\frac{\mathrm{d}\varrho}{\varrho} &= \int_\delta^{d_G} \varrho^{N/p(x)}\{\varrho^{-N}(\log(\mathrm{e}+\varrho^{-1}))^{-1}\}^{1/p(x)}\frac{\mathrm{d}\varrho}{\varrho} \\
&\leqslant C(\log(\mathrm{e}+\delta^{-1}))^{(p(x)-1)/p(x)} = C\Gamma(x,\delta^{-1})
\end{aligned}$$

for $0 < \delta < d_G$.

If $x \in E_2$, then $\Phi^{-1}(x,s) \leqslant b(x)^{-1}s^{1/q(x)}$. Hence, we similarly have

$$\varrho^{\alpha(x)-N}\gamma(x,\varrho)^{-1}\Phi^{-1}(x,\gamma(x,\varrho)) \leqslant b(x)^{-1}(\log(\mathrm{e}+\varrho^{-1}))^{(q(x)-1)/q(x)} = \Gamma(x,\varrho^{-1})$$

for $0 < \varrho < d_G$ and

$$\int_\delta^{d_G} \varrho^{\alpha(x)}\Phi^{-1}(x,\gamma(x,\varrho))\frac{\mathrm{d}\varrho}{\varrho} \leqslant Cb(x)^{-1}(\log(\mathrm{e}+\delta^{-1}))^{(q(x)-1)/q(x)} = C\Gamma(x,\delta^{-1})$$

for $0 < \delta < d_G$.

Thus $\Gamma(x,t)$ satisfies $(\Gamma\Phi\gamma\alpha)$. □

Set

$$\Psi(x,t) = \begin{cases} \exp(t^{p(x)/(p(x)-1)}), & x \in \{y \in E_1 \colon p(y) > 1\}, \\ \exp((b(x)t)^{q(x)/(q(x)-1)}), & x \in \{y \in E_2 \colon q(y) > 1\} \end{cases}$$

for $t > 0$, where $E_1 = \{x \in G_0 \colon \alpha(x)p(x) = N\}$ and $E_2 = \{x \in G_+ \colon \alpha(x)q(x) = N\}$ as before.

The function $\Psi(x,t)$ satisfies $(\Psi1)$, $(\Psi2)$ and $(\Psi\Gamma)$ with $s_0^* = 2/d_G$.

By Lemma 4.9 and Theorem 3.7, we obtain the following Trudinger's inequality for Musielak-Orlicz spaces in the framework of double phase functionals.

**Theorem 4.10.** *Suppose* $\sup_{x\in G_+}(q(x)-p(x))/\theta \leqslant p^-/N$. *Then there exist constants* $c_1, c_2 > 0$ *such that*

$$\tag{4.2} \int_{E_1} \exp\Big(\Big(\frac{|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)^{p(x)/(p(x)-1)}\Big)\,\mathrm{d}x + \int_{E_2} \exp\Big(\Big(\frac{b(x)|I_{\alpha(\cdot)}f(x)|}{c_1}\Big)^{q(x)/(q(x)-1)}\Big)\,\mathrm{d}x \leqslant c_2$$

*for all* $f \in L^\Phi(G)$ *with* $\|f\|_{\Phi;G} \leqslant 1$.

**Remark 4.11.** In Theorems 4.4 and 4.10, compare the exponent to $I_{\alpha(\cdot)}f$; see also the case $q_{j_0}(x) = \beta_{j_0}(x) = 0$ of Corollary 4.6 (1) in [12] and the case $q_{j_0}(x) = 0$ of Corollary 4.2 (1) in [20].

*References*


[1] *D. R. Adams, L. I. Hedberg*: Function Spaces and Potential Theory. Grundlehren der Mathematischen Wissenschaften 314. Springer, Berlin, 1996. zbl MR doi

[2] *Y. Ahmida, I. Chlebicka, P. Gwiazda, A. Youssfi*: Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces. J. Funct. Anal. *275* (2018), 2538–2571. zbl MR doi

[3] *P. Baroni, M. Colombo, G. Mingione*: Non-autonomous functionals, borderline cases and related function classes. St. Petersbg. Math. J. *27* (2016), 347–379. zbl MR doi

[4] *P. Baroni, M. Colombo, G. Mingione*: Regularity for general functionals with double phase. Calc. Var. Partial Differ. Equ. *57* (2018), Article ID 62, 48 pages. zbl MR doi

[5] *M. Colombo, G. Mingione*: Bounded minimisers of double phase variational integrals. Arch. Ration. Mech. Anal. *218* (2015), 219–273. zbl MR doi

[6] *M. Colombo, G. Mingione*: Regularity for double phase variational problems. Arch. Ration. Mech. Anal. *215* (2015), 443–496. zbl MR doi

[7] *T. Futamura, Y. Mizuta*: Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent. Math. Inequal. Appl. *8* (2005), 619–631. zbl MR doi

[8] *T. Futamura, Y. Mizuta, T. Shimomura*: Sobolev embedding for variable exponent Riesz potentials on metric spaces. Ann. Acad. Sci. Fenn., Math. *31* (2006), 495–522. zbl MR

[9] *T. Futamura, Y. Mizuta, T. Shimomura*: Integrability of maximal functions and Riesz potentials in Orlicz spaces of variable exponent. J. Math. Anal. Appl. *366* (2010), 391–417. zbl MR doi

[10] *P. Hästö*: The maximal operator on generalized Orlicz spaces. J. Funct. Anal. *269* (2015), 4038–4048; corrigendum ibid. *271* (2016), 240–243. zbl MR doi

[11] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces. Bull. Sci. Math. *137* (2013), 76–96. zbl MR doi

[12] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Trudinger's inequality and continuity of potentials on Musielak-Orlicz-Morrey spaces. Potential Anal. *38* (2013), 515–535. zbl MR doi

[13] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Sobolev's inequality for double phase functionals with variable exponents. Forum Math. *31* (2019), 517–527. zbl MR doi

[14] *Y. Mizuta, E. Nakai, T. Ohno, T. Shimomura*: Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponents. Complex Var. Elliptic Equ. *56* (2011), 671–695. zbl MR doi

[15] *Y. Mizuta, T. Ohno, T. Shimomura*: Sobolev embeddings for Riesz potential spaces of variable exponents near 1 and Sobolev's exponent. Bull. Sci. Math. *134* (2010), 12–36. zbl MR doi

[16] *Y. Mizuta, T. Shimomura*: Differentiability and Hölder continuity of Riesz potentials of Orlicz functions. Analysis, München *20* (2000), 201–223. zbl MR doi

[17] *Y. Mizuta, T. Shimomura*: Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent. J. Math. Soc. Japan *60* (2008), 583–602. zbl MR doi

[18] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics 1034. Springer, Berlin, 1983. zbl MR doi

[19] *E. Nakai*: Generalized fractional integrals on Orlicz-Morrey spaces. Banach and Function Spaces. Yokohama Publishers, Yokohama, 2004, pp. 323–333. zbl MR

[20] *T. Ohno, T. Shimomura*: Trudinger's inequality for Riesz potentials of functions in Musielak-Orlicz spaces. Bull. Sci. Math. *138* (2014), 225–235. zbl MR doi

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